STEADY-STATE VIBRATION RESPONSE OF A PLATE SIMPLY-SUPPORTED ON ALL SIDES SUBJECTED TO A PRESSURE FIELD WITH SPATIAL VARIATION

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The simply-supported plate in Figure 1 is subjected to a pressure field with spatial variation.

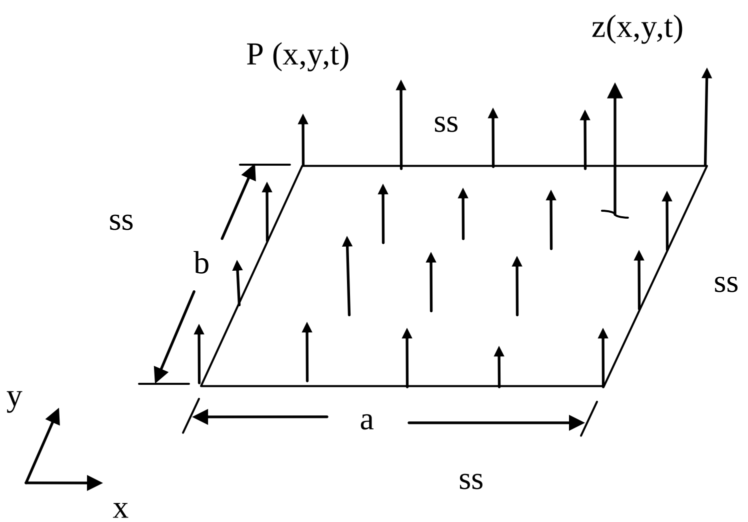


Figure 1.

The following equations are taken from Reference 1.

The governing differential equation is

$$D\left(\frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4}\right) + \rho h \frac{\partial^2 z}{\partial t^2} = P(x, y, t) \tag{1}$$

The plate stiffness factor D is given by

$$D = \frac{Eh^3}{12\left(1-\mu^2\right)} \tag{2}$$

where

E	is the modulus of elasticity
μ	Poisson’s ratio
h	is the thickness
ρ	is the mass density (mass/area)
P	is the applied pressure

Now assume that the pressure field is separable such that

$$P(x, y, t) = \hat{P}(x, y) f(t) \tag{3}$$

where $\hat{P}(x, y)$ is dimensionless

Assume that the displacement field can be represented as

$$z(x, y, t) = Z(x, y) T(t) \tag{4}$$

By substitution,

$$D\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) Z(x, y) T(t) + \rho h \frac{\partial^2}{\partial t^2} Z(x, y) T(t) = \hat{P}(x, y) f(t) \tag{5}$$

$$D\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)Z(x,y)T(t)+\rho h\,Z(x,y)\ddot{T}(t)=\hat{P}(x,y)f(t) \tag{6}$$

The homogeneous equation is

$$D\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)Z(x,y)T(t)+\rho h\,Z(x,y)\ddot{T}(t)=0 \tag{7}$$

$$\rho h\,Z(x,y)\ddot{T}(t)=-D\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)Z(x,y)T(t) \tag{8}$$

$$\frac{\ddot{T}(t)}{T(t)}=-\frac{D}{\rho h\,Z(x,y)}\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)Z(x,y)=-c^2 \tag{9}$$

Thus

$$\ddot{T}(t)+c^2T(t)=0 \tag{10}$$

$$-\frac{D}{\rho h\,Z(x,y)}\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)Z(x,y)=-c^2 \tag{11}$$

A mass-normalized mode shape solution for equation (11) from Reference 1 is

$$Z_{mn}=\frac{2}{\sqrt{\rho a\,b\,h}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{12}$$

The derivatives are

$$\frac{\partial}{\partial x} Z_{mn} = \frac{2}{\sqrt{\rho a\, b\, h}} \left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right) \tag{13}$$

$$\frac{\partial^2}{\partial x^2} Z_{mn} = -\frac{2}{\sqrt{\rho a\, b\, h}} \left(\frac{m\pi}{a}\right)^2 \sin\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right) \tag{14}$$

$$\frac{\partial}{\partial y} Z_{mn} = \frac{2}{\sqrt{\rho a\, b\, h}} \left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi\, x}{a}\right) \cos\left(\frac{n\pi\, y}{b}\right) \tag{15}$$

$$\frac{\partial^2}{\partial y^2} Z_{mn} = -\frac{2}{\sqrt{\rho a\, b\, h}} \left(\frac{n\pi}{b}\right)^2 \sin\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right) \tag{16}$$

The natural frequencies are

$$\omega_{mn} = \sqrt{\frac{D}{\rho h}} \left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right) \tag{17}$$

Recall

$$D\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) Z(x,y)T(t) + \rho h\, Z(x,y)\ddot{T}(t) = \hat{P}(x,y) f(t) \tag{18}$$

Let

$$Z(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Z_{mn}(x,y) T_{mn}(t) \tag{19}$$

By substitution,

$$D\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Z_{mn}(x,y)T_{mn}(t)$$

$$+\rho h\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Z_{mn}(x,y)\ddot{T}_{mn}(t)=\hat{P}(x,y)f(t) \tag{20}$$

$$D\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{mn}(t)\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)Z_{mn}(x,y)$$

$$+\rho h\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Z_{mn}(x,y)\ddot{T}_{mn}(t)=\hat{P}(x,y)f(t) \tag{21}$$

From Reference 1,

$$D\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)Z_{mn}(x,y)=\rho h\omega_{mn}{}^2\, Z_{mn}(x,y) \tag{22}$$

By substitution,

$$\rho h\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\omega_{mn}{}^2\, T_{mn}(t)\; Z_{mn}(x,y)+\rho h\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Z_{mn}(x,y)\ddot{T}_{mn}(t)=\hat{P}(x,y)f(t) \tag{23}$$

Multiply each term by $Z_{pq}(x, y)$.

$$\rho h \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn}{}^2 T_{mn}(t)\ Z_{mn}(x, y) Z_{pq}(x, y)$$

$$+ \rho h \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Z_{mn}(x, y) Z_{pq}(x, y) \ddot{T}_{mn}(t) = \hat{P}(x, y) Z_{pq}(x, y) f(t) \tag{24}$$

Integrate with respect to area.

$$\int_0^b \int_0^a \rho h \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn}{}^2 T_{mn}(t)\ Z_{mn}(x, y) Z_{pq}(x, y)\, dxdy$$

$$+ \int_0^b \int_0^a \rho h \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Z_{mn}(x, y) Z_{pq}(x, y) \ddot{T}_{mn}(t) dxdy = f(t) \int_0^b \int_0^a \hat{P}(x, y) Z_{pq}(x, y) dxdy \tag{25}$$

$$\rho h \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn}{}^2 T_{mn}(t) \int_0^b \int_0^a Z_{mn}(x, y) Z_{pq}(x, y) dxdy$$

$$+ \rho h \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \ddot{T}_{mn}(t) \int_0^b \int_0^a Z_{mn}(x, y) Z_{pq}(x, y)\, dxdy = f(t) \int_0^b \int_0^a \hat{P}(x, y) Z_{pq}(x, y) dxdy \tag{26}$$

The eigenvectors are orthogonal such that

$$\rho h \int_0^b \int_0^a Z_{mn} Z_{pq} dxdy = 0 \qquad \text{for } m \neq p \text{ or } n \neq q \tag{27}$$

$$\rho h \int_0^b \int_0^a Z_{mn} Z_{pq} dxdy = 0 \qquad \text{for } m = p \text{ and } n = q \tag{28}$$

$$\ddot{T}_{mn}(t) + \omega_{mn}{}^2\, T_{mn}(t) = f(t)\int_0^b\int_0^a \hat{P}(x,y) Z_{mn}(x,y)\,dx\,dy \tag{29}$$

Add a damping term.

$$\ddot{T}_{mn}(t) + 2\,\xi_{mn}\omega_{mn}\dot{T}_{mn}(t) + \omega_{mn}{}^2\, T_{mn}(t) = f(t)\int_0^b\int_0^a \hat{P}(x,y) Z_{mn}(x,y)\,dx\,dy \tag{30}$$

<u>Steady-State Solution</u>

Take a Fourier transform of both sides of (30).

$$\int_{-\infty}^{\infty}\left\{\ddot{T}_{mn}(t) + 2\,\xi_{mn}\omega_{mn}\dot{T}_{mn}(t) + \omega_{mn}{}^2\, T_{mn}(t)\right\}\exp(-j\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} f(t)\int_0^b\int_0^a \hat{P}(x,y) Z_{mn}(x,y)\,dx\,dy \tag{31}$$

$$\int_{-\infty}^{\infty}\ddot{T}_{mn}(t)\exp(-j\omega t)\,dt + 2\,\xi_{mn}\omega_{mn}\int_{-\infty}^{\infty}\dot{T}_{mn}(t)\exp(-j\omega t)\,dt + \omega_n{}^2\int_{-\infty}^{\infty} T_{mn}(t)\exp(-j\omega t)\,dt$$

$$= \left\{\int_0^b\int_0^a \hat{P}(x,y) Z_{mn}(x,y)\,dx\,dy\right\}\left\{\int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\,dt\right\} \tag{32}$$

Note that

$$\int_{-\infty}^{\infty}\dot{T}_{mn}(t)\,\exp(-j\omega t)\,dt = j\omega\int_{-\infty}^{\infty} T_{mn}(t)\,\exp(-j\omega t)\,dt \tag{33}$$

$$\int_{-\infty}^{\infty}\ddot{T}_{mn}(t)\,\exp(-j\omega t)\,dt = -\omega^2\int_{-\infty}^{\infty} T_{mn}(t)\,\exp(-j\omega t)\,dt \tag{34}$$

$$-\omega^2\int_{-\infty}^{\infty} T_{mn}(t)\exp(-j\omega t)dt + j2\xi_{mn}\omega_{mn}\omega\int_{-\infty}^{\infty} T_{mn}(t)\exp(-j\omega t)dt$$

$$+\omega_{mn}{}^2\int_{-\infty}^{\infty} T_{mn}(t)\exp(-j\omega t)dt = \left\{\int_0^b\int_0^a \hat{P}(x,y)Z_{mn}(x,y)dxdy\right\}\left\{\int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt\right\}$$

(35)

$$\left[\left(\omega_{mn}{}^2-\omega^2\right)+j2\xi_{mn}\omega_{mn}\omega\right]\int_{-\infty}^{\infty} T_{mn}(t)\exp(-j\omega t)dt$$

$$=\left\{\int_0^b\int_0^a \hat{P}(x,y)Z_{mn}(x,y)dxdy\right\}\left\{\int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt\right\}$$

(36)

$$\hat{T}_{mn}(\omega)=\int_{-\infty}^{\infty} T_{mn}(t)\exp(-j\omega t)dt \quad (37)$$

$$F(\omega)=\int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt \quad (38)$$

$$\left[\left(\omega_{mn}{}^2-\omega^2\right)+j2\xi_{mn}\omega_{mn}\omega\right]\hat{T}_{mn}(\omega)=\left\{\int_0^b\int_0^a \hat{P}(x,y)Z_{mn}(x,y)dxdy\right\}F(\omega) \quad (39)$$

$$\hat{T}_{mn}(\omega)=F(\omega)\frac{\int_0^b\int_0^a \hat{P}(x,y)Z_{mn}(x,y)dxdy}{\left(\omega_{mn}{}^2-\omega^2\right)+j2\xi_{mn}\omega_{mn}\omega} \quad (40)$$

$$Z(x,y,\omega) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Z_{mn}(x,y)\hat{T}_{mn}(\omega) \tag{41}$$

The Fourier transform of the displacement is

$$Z(x,y,\omega) = F(\omega)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{Z_{mn}(x,y)\int_0^b\int_0^a \hat{P}(x,y)Z_{mn}(x,y)dxdy}{\left(\omega_{mn}{}^2 - \omega^2\right) + j2\xi_{mn}\omega_{mn}\omega} \tag{42}$$

The frequency response function relating the displacement to the pressure field is

$$\frac{Z(x,y,\omega)}{F(\omega)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{Z_{mn}(x,y)\int_0^b\int_0^a \hat{P}(x,y)Z_{mn}(x,y)dxdy}{\left(\omega_{mn}{}^2 - \omega^2\right) + j2\xi_{mn}\omega_{mn}\omega} \tag{43}$$

$$\frac{Z(x,y,\omega)}{F(\omega)}$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\frac{2}{\sqrt{\rho a b h}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\int_0^b\int_0^a \hat{P}(x,y)\frac{2}{\sqrt{\rho a b h}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy}{\left(\omega_{mn}{}^2 - \omega^2\right) + j2\xi_{mn}\omega_{mn}\omega}$$

(44)

$$\frac{Z(x,y,\omega)}{F(\omega)}$$

$$= \frac{4}{\rho a\, b\, h} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\int_0^b\int_0^a \hat{P}(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy}{\left(\omega_{mn}{}^2 - \omega^2\right) + j\,2\,\xi_{mn}\omega_{mn}\omega} \tag{45}$$

The bending moments are

$$M_{xx}(x,y,\omega) = -D\left(\frac{\partial^2}{\partial x^2} + \mu\frac{\partial^2}{\partial y^2}\right)Z(x,y,\omega) \tag{46}$$

$$M_{yy}(x,y,\omega) = -D\left(\frac{\partial^2}{\partial y^2} + \mu\frac{\partial^2}{\partial x^2}\right)Z(x,y,\omega) \tag{47}$$

The bending stresses from Reference 2 are

$$\sigma_{xx}(x,y,\omega) = -\frac{E\,\hat{z}}{1-\mu^2}\left(\frac{\partial^2}{\partial x^2} + \mu\frac{\partial^2}{\partial y^2}\right)Z(x,y,\omega) \tag{48}$$

$$\sigma_{yy}(x,y,\omega) = -\frac{E\,\hat{z}}{1-\mu^2}\left(\frac{\partial^2}{\partial y^2} + \mu\frac{\partial^2}{\partial x^2}\right)Z(x,y,\omega) \tag{49}$$

$$\tau_{xy}(x,y,\omega) = -\frac{E\,\hat{z}}{1+\mu}\left(\frac{\partial^2}{\partial x \partial y}Z(x,y,\omega)\right) \tag{50}$$

$\hat{z}$ is the distance from the centerline in the vertical axis

References

1. T. Irvine, Natural Frequencies of Rectangular Plate Bending Modes, Revision B, Vibrationdata, 2011.
2. J.S. Rao, Dynamics of Plates, Narosa, New Delhi, 1999.

APPENDIX A

Special Case

$$\frac{Z(x,y,\omega)}{F(\omega)} = \frac{4}{\rho a b h}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\int_0^b\int_0^a \hat{P}(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy}{\left(\omega_{mn}^2-\omega^2\right)+j2\xi_{mn}\omega_{mn}\omega}$$

(A-1)

Now assume that

$$\hat{P}(x,y) = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) \qquad \text{(A-2)}$$

By substitution,

$$\frac{Z(x,y,\omega)}{F(\omega)} = \frac{4}{\rho a b h}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\int_0^b\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy}{\left(\omega_{mn}^2-\omega^2\right)+j2\xi_{mn}\omega_{mn}\omega}$$

(A-3)

$$\frac{Z(x,y,\omega)}{F(\omega)} = \frac{4}{\rho a b h}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx\int_0^b \sin\left(\frac{\pi y}{b}\right)\sin\left(\frac{n\pi y}{b}\right)dy}{\left(\omega_{mn}^2-\omega^2\right)+j2\xi_{mn}\omega_{mn}\omega}$$

(A-4)

$$\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \begin{cases} a/2 & \text{for } m = 1 \\ 0 & \text{for } m = 2,3,4,\ldots \end{cases} \tag{A-5}$$

$$\int_0^b \sin\left(\frac{\pi y}{b}\right)\sin\left(\frac{n\pi y}{b}\right)dx = \begin{cases} b/2 & \text{for } n = 1 \\ 0 & \text{for } n = 2,3,4,\ldots \end{cases} \tag{A-6}$$

The transfer function for m=n=1 is

$$\frac{Z(x,y,\omega)}{F(\omega)} = \frac{4}{\rho a\,b\,h}\left(\frac{a\,b}{4}\right)\frac{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2 - \omega^2\right) + j\,2\,\xi_{11}\omega_{11}\omega} \tag{A-7}$$

$$\frac{Z(x,y,\omega)}{F(\omega)} = \frac{1}{\rho\,h}\frac{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2 - \omega^2\right) + j\,2\,\xi_{11}\omega_{11}\omega} \tag{A-8}$$

Again,

$$\omega_{11} = \sqrt{\frac{D}{\rho h}}\left(\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right) \tag{A-9}$$

The bending stresses transfer functions are

$$\frac{\sigma_{xx}(x,y,\omega)}{F(\omega)} = -\frac{E\,\hat{z}}{1-\mu^2}\left(\frac{\partial^2}{\partial x^2} + \mu\frac{\partial^2}{\partial y^2}\right)\left\{\frac{1}{\rho\,h}\frac{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2 - \omega^2\right) + j\,2\,\xi_{11}\omega_{11}\omega}\right\} \tag{A-10}$$

$$\frac{\sigma_{xx}(x,y,\omega)}{F(\omega)}$$

$$= -\frac{E\hat{z}}{1-\mu^2}\left(\frac{1}{\rho h}\right)\left(\frac{\partial^2}{\partial x^2}\right)\left\{\frac{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2-\omega^2\right)+j2\xi_{11}\omega_{11}\omega}\right\}$$

$$-\frac{E\hat{z}}{1-\mu^2}\left(\frac{1}{\rho h}\right)\left(\mu\frac{\partial^2}{\partial y^2}\right)\left\{\frac{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2-\omega^2\right)+j2\xi_{11}\omega_{11}\omega}\right\}$$

(A-11)

$$\frac{\sigma_{xx}(x,y,\omega)}{F(\omega)}$$

$$= \frac{E\hat{z}}{1-\mu^2}\left(\frac{1}{\rho h}\right)\left(\frac{\pi}{a}\right)^2\left\{\frac{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2-\omega^2\right)+j2\xi_{11}\omega_{11}\omega}\right\}$$

$$+\frac{\mu E\hat{z}}{1-\mu^2}\left(\frac{1}{\rho h}\right)\left(\frac{\pi}{b}\right)^2\left\{\frac{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2-\omega^2\right)+j2\xi_{11}\omega_{11}\omega}\right\}$$

(A-12)

$$\frac{\sigma_{xx}(x,y,\omega)}{F(\omega)} = \frac{E\hat{z}}{1-\mu^2}\left(\frac{1}{\rho h}\right)\left[\left(\frac{\pi}{a}\right)^2+\mu\left(\frac{\pi}{b}\right)^2\right]\left\{\frac{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2-\omega^2\right)+j2\xi_{11}\omega_{11}\omega}\right\}$$

(A-13)

$$\frac{\sigma_{xx}(x,y,\omega)}{F(\omega)} = \frac{E\hat{z}}{1-\mu^2}\left(\frac{\pi^2}{\rho h}\right)\left[\frac{1}{a^2}+\frac{\mu}{b^2}\right]\left\{\frac{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2-\omega^2\right)+j2\xi_{11}\omega_{11}\omega}\right\}$$

(A-14)

$$\sigma_{yy}(x,y,\omega) = -\frac{E\hat{z}}{1-\mu^2}\left(\frac{\partial^2}{\partial y^2}+\mu\frac{\partial^2}{\partial x^2}\right)Z(x,y,\omega) \quad \text{(A-15)}$$

$$\sigma_{yy}(x,y,\omega) = \frac{E\hat{z}}{1-\mu^2}\left(\frac{\pi^2}{\rho h}\right)\left[\frac{\mu}{a^2}+\frac{1}{b^2}\right]\left\{\frac{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2-\omega^2\right)+j2\xi_{11}\omega_{11}\omega}\right\} \quad \text{(A-16)}$$

$$\tau_{xy}(x,y,\omega) = -\frac{E\hat{z}}{1+\mu}\left(\frac{\partial^2}{\partial x\partial y}Z(x,y,\omega)\right) \quad \text{(A-17)}$$

$$\tau_{xy}(x,y,\omega) = -\frac{E\hat{z}}{1+\mu}\left(\frac{\partial^2}{\partial x\partial y}\right)\left\{\frac{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2-\omega^2\right)+j2\xi_{11}\omega_{11}\omega}\right\} \quad \text{(A-18)}$$

$$\tau_{xy}(x,y,\omega) = -\frac{E\hat{z}}{1+\mu}\left(\frac{\pi}{a}\right)\left(\frac{\pi}{b}\right)\left\{\frac{\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2-\omega^2\right)+j2\xi_{11}\omega_{11}\omega}\right\} \quad \text{(A-19)}$$

$$\tau_{xy}(x,y,\omega) = -\frac{E\hat{z}}{1+\mu}\left(\frac{\pi^2}{ab}\right)\left\{\frac{\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)}{\left(\omega_{11}{}^2-\omega^2\right)+j2\xi_{11}\omega_{11}\omega}\right\} \quad \text{(A-20)}$$

APPENDIX B

Example

Consider a rectangular plate with the following properties:

Boundary Conditions	Simply Supported on All Sides
Material	Aluminum

Thickness	h	=	0.125 inch
Length	a	=	10 inch
Width	b	=	8 inch
Elastic Modulus	E	=	10E+06 lbf/in^2
Mass per Volume	ρ_v	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)
Mass per Area	ρ	=	0.0125 lbm / in^2 (3.24E-05 lbf sec^2/in^3)
Viscous Damping Ratio	ξ	=	0.05

The normal modes and frequency response function analysis are performed via a Matlab script.

The normal modes results are:

Table B-1. Natural Frequency Results, Plate Simply-Supported on all Sides			
fn (Hz)	m	n	Participation Factor
302	1	1	0.04126
656	2	1	0
855	1	2	0
1209	2	2	0
1246	3	1	0.01375
1777	1	3	0.01375
1799	3	2	0
2072	4	1	0
2131	2	3	0
2625	4	2	0
2721	3	3	0.004584
3067	1	4	0
3134	5	1	0.008251
3421	2	4	0

Note that the mode shape and participation factors are considered as dimensionless, but they must be consistent with respect to one another.

The resulting displacement and transfer functions magnitudes are shown in Figures B-1 and B-2, respectively.

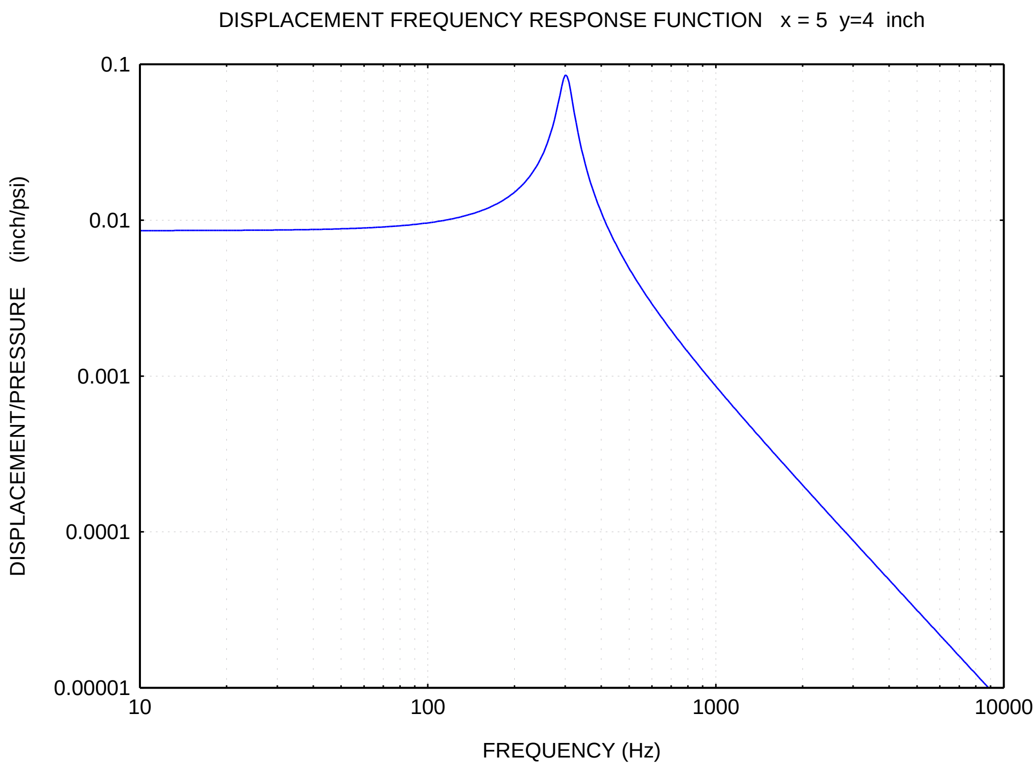


Figure B-1.

The maximum displacement response is: max = 0.0853 in/psi at 300.1 Hz

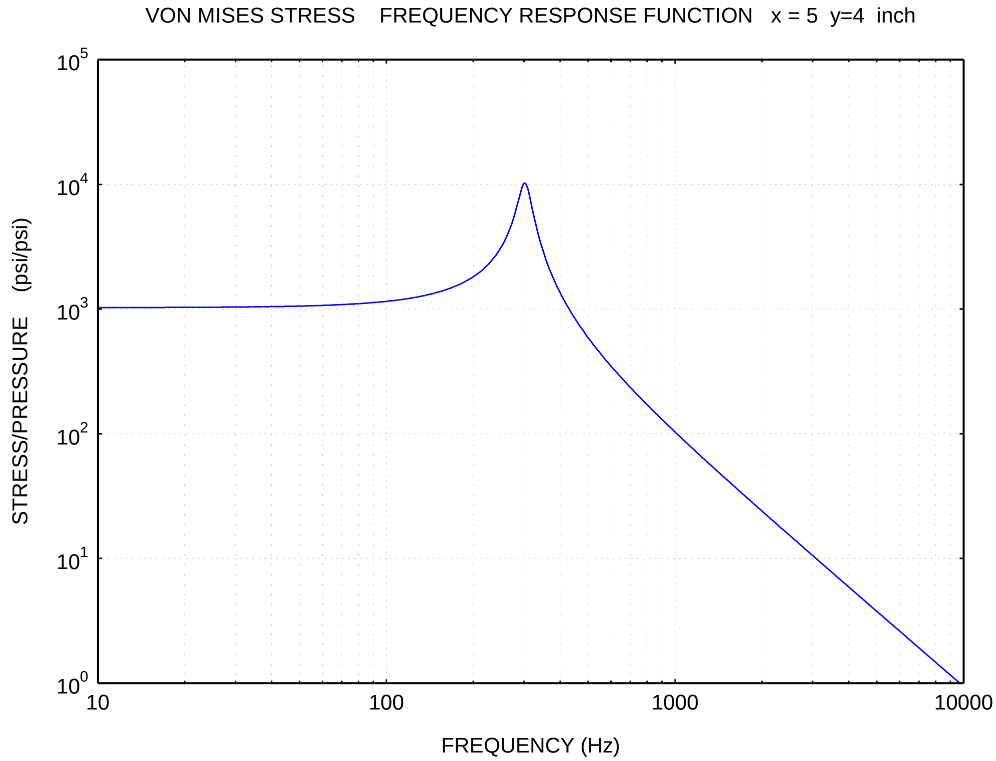
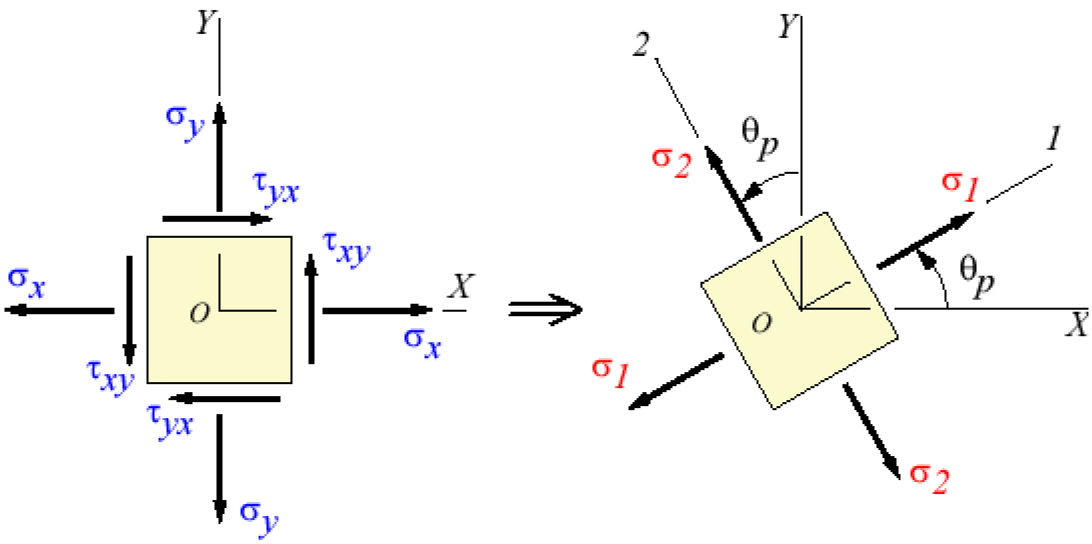


Figure B-2.

The maximum von Mises stress response is: max = 1.025e+04 (psi/psi) at 300.1 Hz

APPENDIX C

<u>Principal Stress</u>



Stresses in given coordinate system

Principal Stresses

The principle stresses are

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \qquad \text{(C-1)}$$

The angle at which the shear stress becomes zero is

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{(C-2)}$$

The von Mises stress σ_e is

$$\sigma_e = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} \qquad \text{(C-3)}$$

The von Mises stress is used to predict yielding of materials under any loading condition from results of simple uniaxial tensile tests. The von Mises stress satisfies the property that two stress states with equal distortion energy have equal von Mises stress.

An alternate formula from Reference 2 is

$$\sigma_e = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2} \qquad \text{(C-4)}$$

<u>References</u>

1. http://www.efunda.com/formulae/solid_mechanics/mat_mechanics/plane_stress_principal.cfm

2. D. Segalman, C. Flucher, G. Reese, R Field; An Efficient Method for Calculating RMS von Mises Stress in a Random Vibration Environment, Sandia Report: SAND98-0260, UC-705, 1998.