STEADY-STATE VIBRATION RESPONSE OF A PLATE
SIMPLY-SUPPORTED ON ALL SIDES SUBJECTED TO A UNIFORM PRESSURE
Revision C

By Tom Irvine
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Email: tom@vibrationdata.com

The simply-supported plate in Figure 1 is subjected to a uniform pressure.

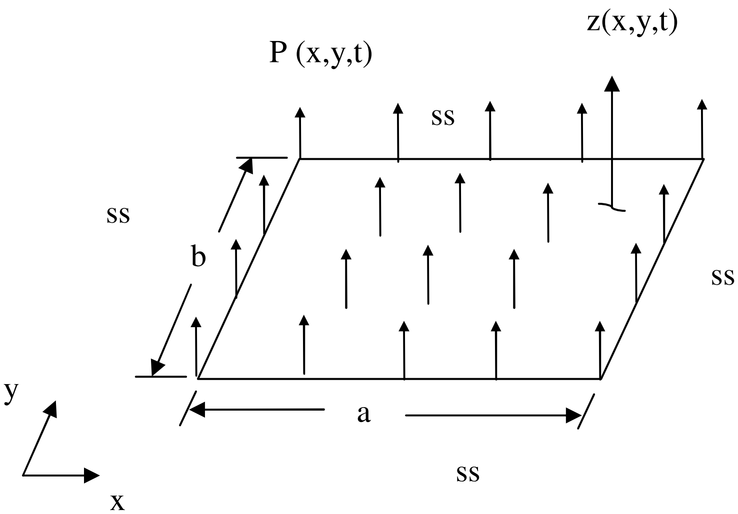


Figure 1.

The following equations are taken from Reference 1.

The governing differential equation is

$$D\left(\frac{\partial^4 z}{\partial x^4}+2\frac{\partial^4 z}{\partial x^2 \partial y^2}+\frac{\partial^4 z}{\partial y^4}\right)+\rho h\frac{\partial^2 z}{\partial t^2}=P(x,y,t) \quad (1)$$

The plate stiffness factor D is given by

$$D = \frac{Eh^3}{12\left(1-\mu^2\right)} \tag{2}$$

where

E	is the modulus of elasticity
μ	Poisson's ratio
h	is the thickness
ρ	is the mass density (mass/volume)
P	is the applied pressure

Now assume that the pressure field is uniform such that

$$W(t) = P(x, y, t) \tag{2}$$

The differential equation becomes

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \rho h \frac{\partial^2 w}{\partial t^2} = W(t) \tag{3}$$

The mass-normalized mode shapes are

$$Z_{mn} = \frac{2}{\sqrt{\rho a\, b\, h}} \sin\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right) \tag{4}$$

$$\frac{\partial}{\partial x} Z_{mn} = \frac{2}{\sqrt{\rho a\, b\, h}} \left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right) \tag{5}$$

$$\frac{\partial^2}{\partial x^2} Z_{mn} = -\frac{2}{\sqrt{\rho a b h}} \left(\frac{m\pi}{a}\right)^2 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{6}$$

$$\frac{\partial}{\partial y} Z_{mn} = \frac{2}{\sqrt{\rho a b h}} \left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \tag{7}$$

$$\frac{\partial^2}{\partial y^2} Z_{mn} = -\frac{2}{\sqrt{\rho a b h}} \left(\frac{n\pi}{b}\right)^2 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{8}$$

The natural frequencies are

$$\omega_{mn} = \sqrt{\frac{D}{\rho h}} \left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right) \tag{9}$$

The participation factors for constant mass density are

$$\Gamma_{mn} = \rho h \int_0^b \int_0^a Z_{mn}(x, y)\, dx dy \tag{10}$$

$$\Gamma_{mn} = \left(\frac{2\sqrt{\rho a b h}}{m n \pi^2} \right) [\cos(n\pi) - 1][\cos(m\pi) - 1] \tag{11}$$

The displacement response Z(x, y, ω) to the applied force is

$$Z(x,y,\omega)=\frac{1}{\rho h}W(\omega)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\Gamma_{mn}Z_{mn}(x,y)}{\left(\omega_{mn}^{2}-\omega^{2}\right)+j2\xi_{mn}\omega\omega_{mn}}\right\} \tag{12}$$

$$Z(x,y,\omega)=$$
$$\frac{1}{\rho h}W(\omega)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\left(\frac{2\sqrt{\rho\, a\, b\, h}}{m\, n\, \pi^{2}}\right)[\cos(n\pi)-1][\cos(m\pi)-1]\frac{2}{\sqrt{\rho a\, b\, h}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}{\left(\omega_{mn}^{2}-\omega^{2}\right)+j2\xi_{mn}\omega\omega_{mn}}\right\} \tag{13}$$

$$Z(x,y,\omega)=\frac{4}{\rho h\,\pi^{2}}W(\omega)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\left(\frac{1}{m\,n}\right)\frac{[\cos(n\pi)-1][\cos(m\pi)-1]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}{\left(\omega_{mn}^{2}-\omega^{2}\right)+j2\xi_{mn}\omega\omega_{mn}}\right\} \tag{14}$$

The bending moments are

$$M_{xx}(x,y,\omega)=-D\left(\frac{\partial^{2}}{\partial x^{2}}+\mu\frac{\partial^{2}}{\partial y^{2}}\right)Z(x,y,\omega) \tag{15}$$

$$M_{yy}(x,y,\omega)=-D\left(\frac{\partial^{2}}{\partial y^{2}}+\mu\frac{\partial^{2}}{\partial x^{2}}\right)Z(x,y,\omega) \tag{16}$$

The bending stresses from Reference 2 are

$$\sigma_{xx}(x, y, \omega) = -\frac{E\,\hat{z}}{1-\mu^2}\left(\frac{\partial^2}{\partial x^2} + \mu\frac{\partial^2}{\partial y^2}\right)Z(x, y, \omega) \tag{17}$$

$$\sigma_{yy}(x, y, \omega) = -\frac{E\,\hat{z}}{1-\mu^2}\left(\frac{\partial^2}{\partial y^2} + \mu\frac{\partial^2}{\partial x^2}\right)Z(x, y, \omega) \tag{18}$$

$$\tau_{xy}(x, y, \omega) = -\frac{E\,\hat{z}}{1+\mu}\left(\frac{\partial^2}{\partial x \partial y} Z(x, y, \omega)\right) \tag{19}$$

$\hat{z}$ is the distance from the centerline in the vertical axis

References

1. T. Irvine, Natural Frequencies of Rectangular Plate Bending Modes, Revision B, Vibrationdata, 2011.

2. J.S. Rao, Dynamics of Plates, Narosa, New Delhi, 1999.

3. http://www.efunda.com/formulae/solid_mechanics/mat_mechanics/plane_stress_principal.cfm

4. D. Segalman, C. Flucher, G. Reese, R Field; An Efficient Method for Calculating RMS von Mises Stress in a Random Vibration Environment, Sandia Report: SAND98-0260, UC-705, 1998.

APPENDIX A

Example

Consider a rectangular plate with the following properties:

Boundary Conditions	Simply Supported on All Sides
Material	Aluminum

Thickness	h	=	0.125 inch
Length	a	=	10 inch
Width	b	=	8 inch
Elastic Modulus	E	=	10E+06 lbf/in^2
Mass per Volume	ρ_V	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)
Mass per Area	ρ	=	0.0125 lbm / in^2 (3.24E-05 lbf sec^2/in^3)
Viscous Damping Ratio	ξ	=	0.05

The normal modes and frequency response function analysis are performed via a Matlab script.

The normal modes results are:

Table A-1. Natural Frequency Results, Plate Simply-Supported on all Sides			
fn (Hz)	m	n	Participation Factor
302	1	1	0.04126
656	2	1	0
855	1	2	0
1209	2	2	0
1246	3	1	0.01375
1777	1	3	0.01375
1799	3	2	0
2072	4	1	0
2131	2	3	0
2625	4	2	0
2721	3	3	0.004584
3067	1	4	0
3134	5	1	0.008251
3421	2	4	0

Note that the mode shape and participation factors are considered as dimensionless, but they must be consistent with respect to one another.

The resulting displacement and transfer functions magnitudes are shown in Figures A-1 and A-2, respectively.

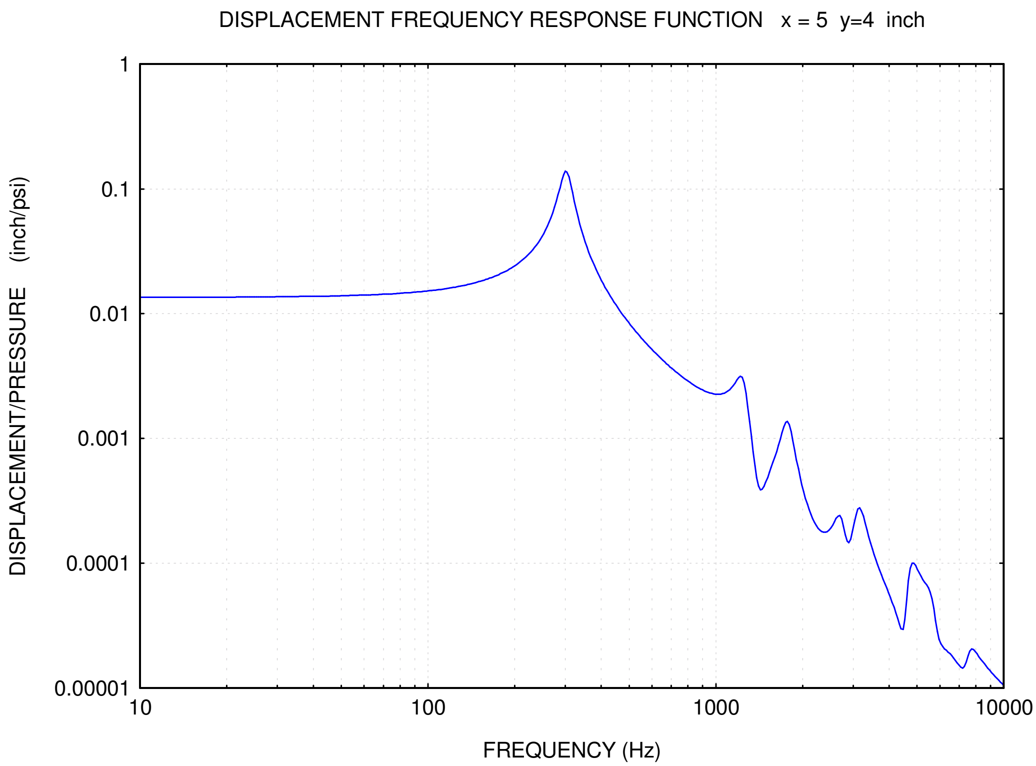


Figure A-1.

The maximum displacement response is: max = 0.138 in/psi at 300.1 Hz

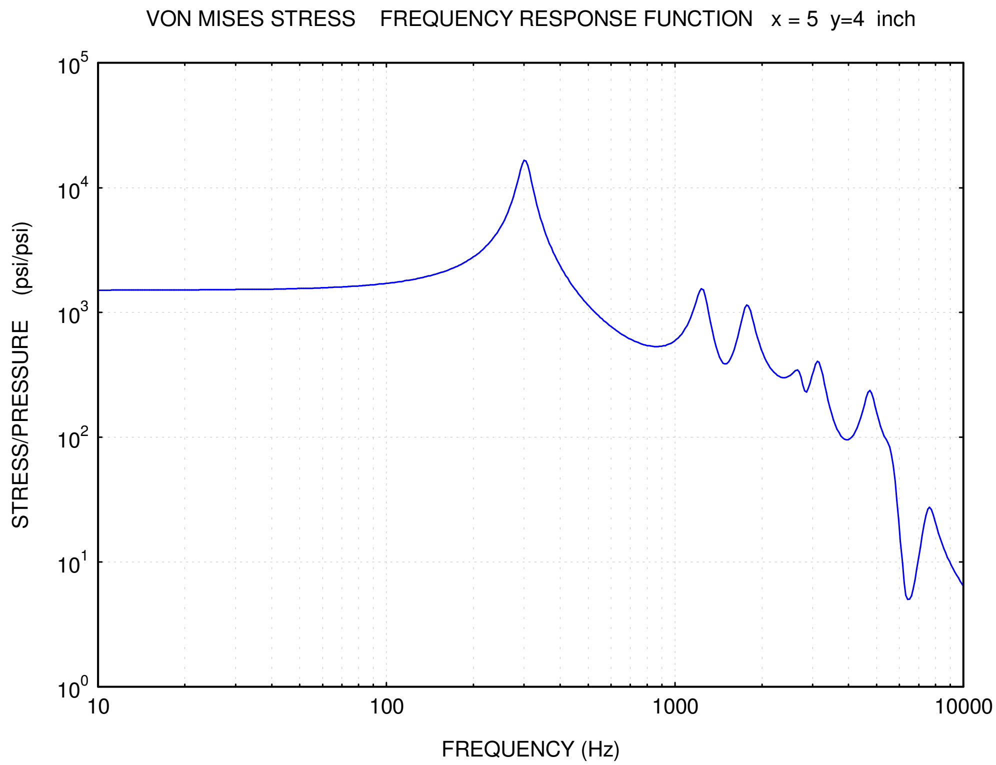
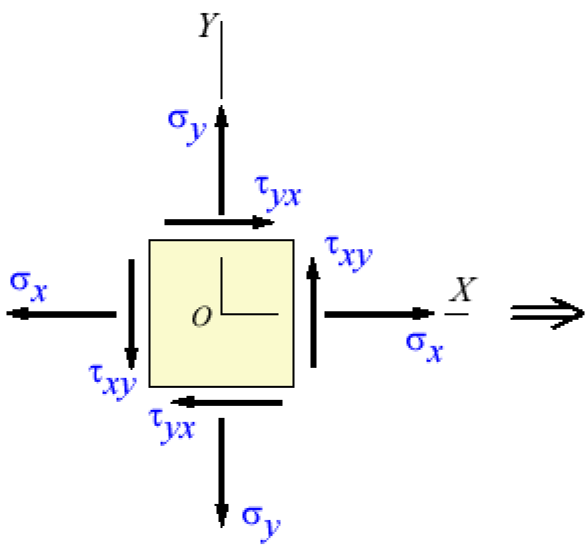


Figure A-2.

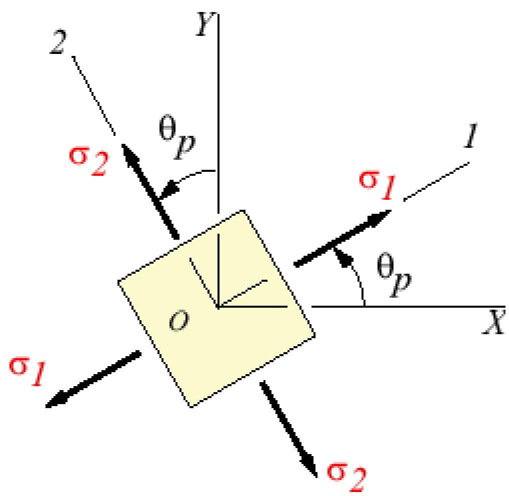
The maximum von Mises stress response is: max = 1.658e+04 (psi/psi) at 300.1 Hz

APPENDIX B

Principal Stress



Stresses in Given Coordinate System



Principal Stresses

The diagrams are taken from Reference 3.

The principle stresses are

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \qquad \text{(B-1)}$$

The angle at which the shear stress becomes zero is

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{(B-2)}$$

The von Mises stress σ_e is

$$\sigma_e = \sqrt{\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2} \quad \text{(B-3)}$$

The von Mises stress is used to predict yielding of materials under any loading condition from results of simple uniaxial tensile tests. The von Mises stress satisfies the property that two stress states with equal distortion energy have equal von Mises stress.

An alternate formula from Reference 4 is

$$\sigma_e = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2} \quad \text{(B-4)}$$