STEADY-STATE VIBRATION RESPONSE OF A FIXED-FIXED BEAM SUBJECTED TO A UNIFORM DISTRIBUTED APPLIED FORCE
Revision G

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Normal Modes

The fixed-fixed beam in Figure 1 is subjected to a uniform, distributed applied force.

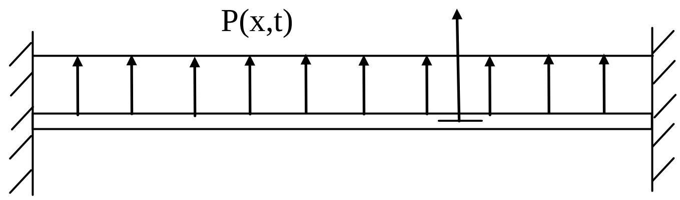

Figure 1.

The following equations are taken from References 1 and 2.

The governing differential equation is

$$EI\frac{\partial^4 y}{\partial x^4}+\rho\frac{\partial^2 y}{\partial t^2}=P(x,t) \tag{1}$$

where

E	is the modulus of elasticity
I	is the area moment of inertia
L	is the length
ρ	is the mass density (mass/length)
P	is the applied force per length

Now assume that the distributed force is uniform such that

$$W(t) = P(x,t) \tag{2}$$

The differential equation becomes

$$EI\frac{\partial^4 y}{\partial x^4} + \rho\frac{\partial^2 y}{\partial t^2} = W(t) \tag{3}$$

The eigenvalues are

N	$\beta_n L$
1	4.73004
2	7.85321
3	10.9956
4	14.13717
5	17.27876

For n> 5

$$\beta_n L \approx \pi\left[\frac{1}{2} + n\right] \tag{4}$$

The natural frequencies are

$$\omega_n = \beta_n{}^2\sqrt{EI/\rho} \tag{5}$$

The mass-normalized mode shapes are

$$Y_n(x) = \frac{1}{\sqrt{\rho L}}\left\{\left[\cosh(\beta_n x) - \cos(\beta_n x)\right] - \sigma_n \left[\sinh(\beta_n x) - \sin(\beta_n x)\right]\right\} \tag{6}$$

where

$$\sigma_n = \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)}\right]$$

The derivatives are

$$Y_n'(x) = \frac{\beta_n}{\sqrt{\rho L}}\left\{\left[\sinh(\beta_n x) + \sin(\beta_n x)\right] - \sigma_n \left[\cosh(\beta_n x) - \cos(\beta_n x)\right]\right\} \tag{7}$$

$$Y_n''(x) = \frac{\beta_n{}^2}{\sqrt{\rho L}}\left\{\left[\cosh(\beta_n x) + \cos(\beta_n x)\right] - \sigma_n \left[\sinh(\beta_n x) + \sin(\beta_n x)\right]\right\} \tag{8}$$

<u>Participation Factors</u>

The participation factors for constant mass density are

$$\Gamma_n = \rho \int_0^L Y_n(x)\,dx \tag{9}$$

The participation factors from a numerical calculation are

$$\Gamma_1 = 0.8309\sqrt{\rho L} \tag{10}$$

$$\Gamma_2 = 0 \tag{11}$$

$$\Gamma_3 = 0.3638\sqrt{\rho L} \tag{12}$$

$$\Gamma_4 = 0 \tag{13}$$

$$\Gamma_5 = 0.2315\sqrt{\rho L} \tag{14}$$

The participation factors are non-dimensional.

Displacement Response

The displacement response Y(x, ω) to the applied force is

$$Y(x,\omega) = \frac{1}{\rho} W(\omega) \sum_{n=1}^{\infty} \left\{ \frac{\Gamma_n Y_n(x)}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n \omega\omega_n} \right\} \tag{15}$$

The displacement transfer function is

$$\frac{Y(x,\omega)}{W(\omega)} = \frac{1}{\rho} \sum_{n=1}^{\infty} \left\{ \frac{\Gamma_n Y_n(x)}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n \omega\omega_n} \right\} \tag{16}$$

The bending moment transfer function is

$$\frac{M(x,\omega)}{W(\omega)} = \frac{EI\ Y''(x,\omega)}{W(\omega)} = \frac{EI}{\rho} \sum_{n=1}^{\infty} \left\{ \frac{\Gamma_n Y_n''(x)}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n \omega\omega_n} \right\} \tag{17}$$

The bending stress transfer function is

$$\frac{\sigma(x,\omega)}{W(\omega)} = \left(\frac{c}{I}\right) \frac{M(x,\omega)}{W(\omega)} \tag{18}$$

where c is the distance from the neutral axis

The shear force $V(x, \omega)$ is

$$V(x, \omega) = EI \frac{\partial^3}{\partial x^3} Y(x, \omega) \tag{19}$$

$$V(x, \omega) = \frac{EI}{\rho} W(\omega) \sum_{n=1}^{m} \left\{ \frac{\Gamma_n Y_n{}'''(x)}{\left(\omega_n^2 - \omega^2\right) + j 2 \xi_n \omega \omega_n} \right\} \tag{20}$$

The average shear stress τ_{ave} is

$$\tau_{ave} = \frac{V(x, \omega)}{A} \tag{21}$$

where A is the cross-section area

The maximum shear stress τ_{max} occurs at the neutral axis and is

$$\tau_{max} = \alpha \, \tau_{ave} \tag{22}$$

where

Cross-section	α
Solid Rectangle	3/2
Solid Cylinder	4/3
Pipe	2
I-beam	A / A_{web}

References

1. T. Irvine, Bending Frequencies of Beams, Rod, and Pipes, Revision P, Vibrationdata, 2011.

2. T. Irvine, Steady-State Vibration Response of a Cantilever Beam Subjected to Base Excitation, Rev A, Vibrationdata, 2009.

APPENDIX A

Example

Consider a beam with the following properties:

Cross-Section	Rectangular
Boundary Conditions	Fixed at Each End
Material	Aluminum

Thickness	T	=	0.125 inch
Width	W	=	1.0 inch
Length	L	=	27.5 inch
Cross-Section Area	A	=	0.125 in^2
Area Moment of Inertia	I	=	0.000163 in^4
Elastic Modulus	E	=	10E+06 lbf/in^2
Stiffness	EI	=	1628 lbf in^2
Mass per Volume	ρ_V	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)
Mass per Length	ρ	=	0.0125 lbm / in (0.00003237 lbf sec^2/in^4)
Mass	ρL	=	0.3438 lbm (0.0008906 lbf sec^2/in)
Viscous Damping Ratio	ξ	=	0.05

The normal modes and frequency response function analysis are performed via a Matlab script.

The normal modes results are:

Table 1. Natural Frequency Results, Beam Fixed at Each End		
Mode	fn (Hz)	Participation Factor
1	33.38	0.02479
2	92.02	0
3	180.4	0.01086
4	298.2	0
5	445.4	0.006908

Note that the mode shape and participation factors are considered as dimensionless, but they must be consistent with respect to one another.

The resulting displacement and stress transfer function magnitudes are shown in Figures A-1 and A-2, respectively.

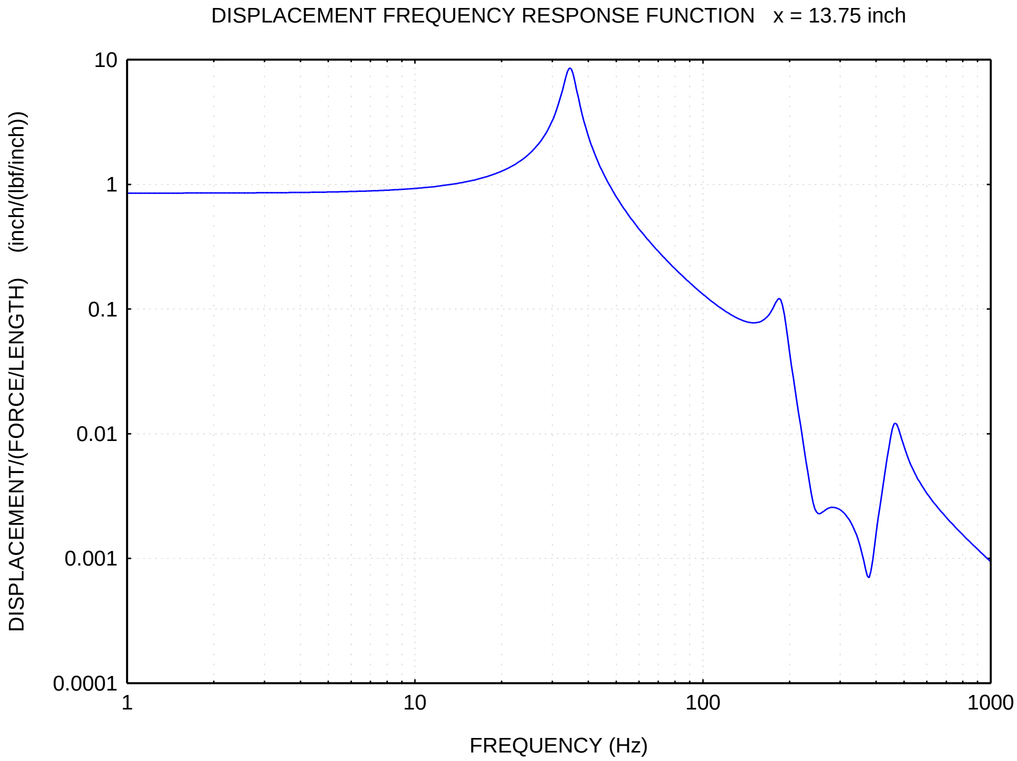

Figure A-1.

The maximum displacement response is 9.251 [in/(lbf/in)] at 33.4 Hz

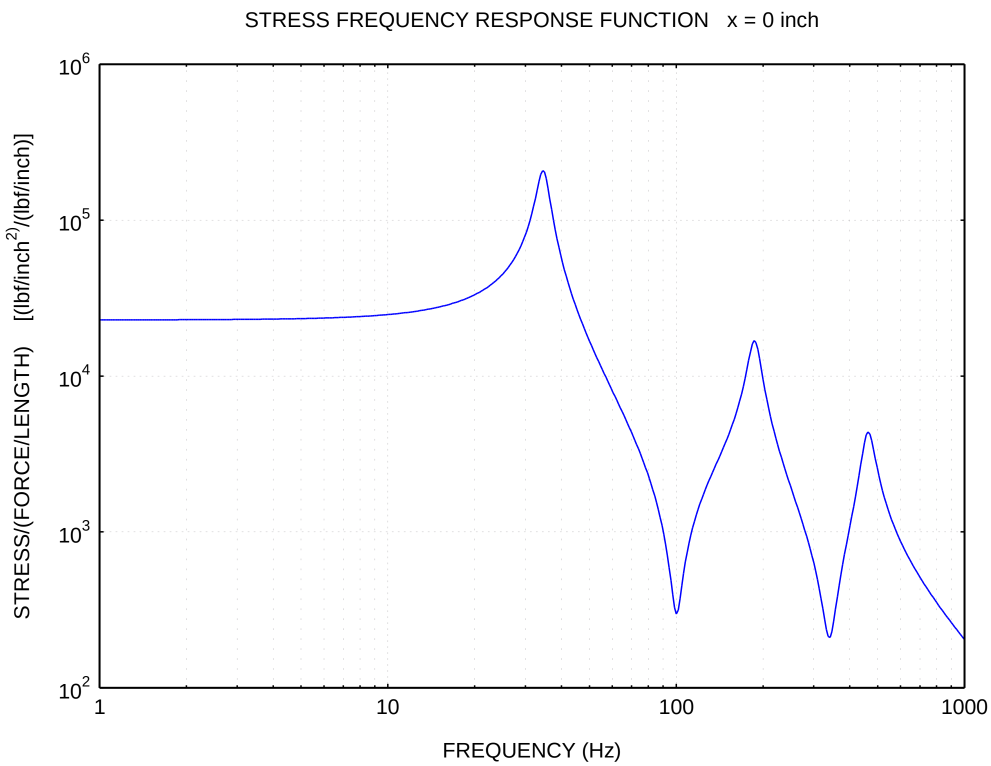

Figure A-2.

The maximum bending stress response is 2.078e+05 [(lbf/in^2) /(lbf/in)] at 34.4 Hz