STEADY-STATE VIBRATION RESPONSE OF A
FIXED-FIXED BEAM SUBJECTED TO BASE EXCITATION

By Tom Irvine
May 4, 2012

Email: tomirvine@aol.com

The fixed-fixed beam in Figure 1 is subject to base excitation.

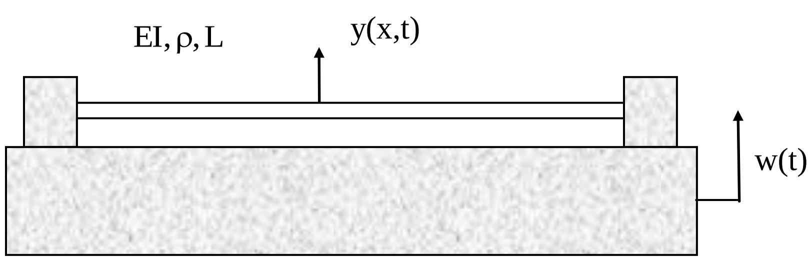

Figure 1.

The following equations are taken from References 1 and 2.

The governing differential equation is

$$EI\frac{\partial^4 y}{\partial x^4}+\rho\frac{\partial^2 y}{\partial t^2}=-\rho\frac{\partial^2 w}{\partial t^2} \tag{1}$$

where

- E is the modulus of elasticity
- I is the area moment of inertia
- L is the length
- ρ is the mass density (mass/length)

The mass-normalized mode shapes are

$$Y_n(x) = \frac{1}{\sqrt{\rho L}} \left\{ \left[\cosh(\beta_n x) - \cos(\beta_n x) \right] - \sigma_n \left[\sinh(\beta_n x) - \sin(\beta_n x) \right] \right\} \tag{2}$$

where

$$\sigma_n = \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)} \right]$$

$$\frac{d^2}{dx^2} Y_n(x) = \frac{\beta_n^2}{\sqrt{\rho L}} \left\{ \left[\cosh(\beta_n x) + \cos(\beta_n x) \right] - \sigma_n \left[\sinh(\beta_n x) + \sin(\beta_n x) \right] \right\} \tag{3}$$

The eigenvalues are

n	$\beta_n L$
1	4.73004
2	7.85321
3	10.9956
4	14.13717
5	17.27876

For n> 5

$$\beta_n L \approx \pi \left[\frac{1}{2} + n \right] \tag{4}$$

The natural frequencies are

$$\omega_n = \beta_n^2 \sqrt{EI / \rho} \tag{5}$$

The relative displacement response Y(x, ω) to base acceleration is

$$Y(x,\omega)=\ddot{W}(\omega)\sum_{n=1}^{m}\left\{\frac{-\Gamma_n Y_n(x)}{\left(\omega_n^2-\omega^2\right)+j\,2\xi_n\omega\omega_n}\right\} \tag{6}$$

The participation factors for constant mass density are

$$\Gamma_n=\rho\int_0^L Y_n(x)\,dx \tag{7}$$

$$\Gamma_n=\frac{1}{\beta_n}\sqrt{\frac{\rho}{L}}\left\{\left[\sinh(\beta_n L)-\sin(\beta_n L)\right]+\sigma_n\left[2-\cosh(\beta_n L)-\cos(\beta_n L)\right]\right\} \tag{8}$$

The participation factors from a numerical calculation are

$$\Gamma_1=0.8309\sqrt{\rho L} \tag{9}$$

$$\Gamma_2=0 \tag{10}$$

$$\Gamma_3=0.3638\sqrt{\rho L} \tag{11}$$

$$\Gamma_4=0 \tag{12}$$

$$\Gamma_5=0.2315\sqrt{\rho L} \tag{13}$$

The participation factors are non-dimensional

Effective Modal Mass

The effective modal mass is

$$m_{eff,n} = \frac{\left[\int_0^L m(x)\, Y_n(x) dx\right]^2}{\int_0^L m(x) [Y_n(x)]^2 dx} \tag{14}$$

The eigenvectors are already normalized such that

$$\int_0^L m(x) [Y_n(x)]^2 dx = 1 \tag{15}$$

Thus,

$$m_{eff,n} = [\Gamma_n]^2 = \left[\int_0^L m(x)\, Y_n(x) dx\right]^2 \tag{16}$$

Example

Consider a beam with the following properties:

Cross-Section	Circular
Boundary Conditions	Fixed-Fixed
Material	Aluminum

Diameter	D	=	0.5 inch
Cross-Section Area	A	=	0.1963 in^2
Length	L	=	32 inch
Area Moment of Inertia	I	=	0.003068 in^4
Elastic Modulus	E	=	1.0e+07 lbf/in^2
Stiffness	EI	=	30680 lbf in^2
Mass per Volume	ρ_v	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)

Mass per Length	ρ	=	0.01963 lbm/in (5.08e-05 lbf sec^2/in^2)
Mass	ρL	=	0.628 lbm (0.00163 lbf sec^2/in)
Viscous Damping Ratio	ξ	=	0.05

The normal modes and frequency response function analysis are performed via Matlab script: ss_beam_stress.m. The normal modes results are:

Table 1.
Natural Frequency Results, Fixed-Fixed Beam

Mode	fn (Hz)	Participation Factor	Effective Modal Mass (lbf sec^2/in)	Effective Modal Mass (lbm)
1	85.4	0.0335	0.001124	0.4337
2	235.4	0	0	0
3	461.5	0.015	0.000215	0.083
4	762.9	0	0	0
5	1140	0.0093	8.72E-05	0.0337

Note that the mode shape and participation factors are considered as dimensionless, but they must be consistent with respect to one another.

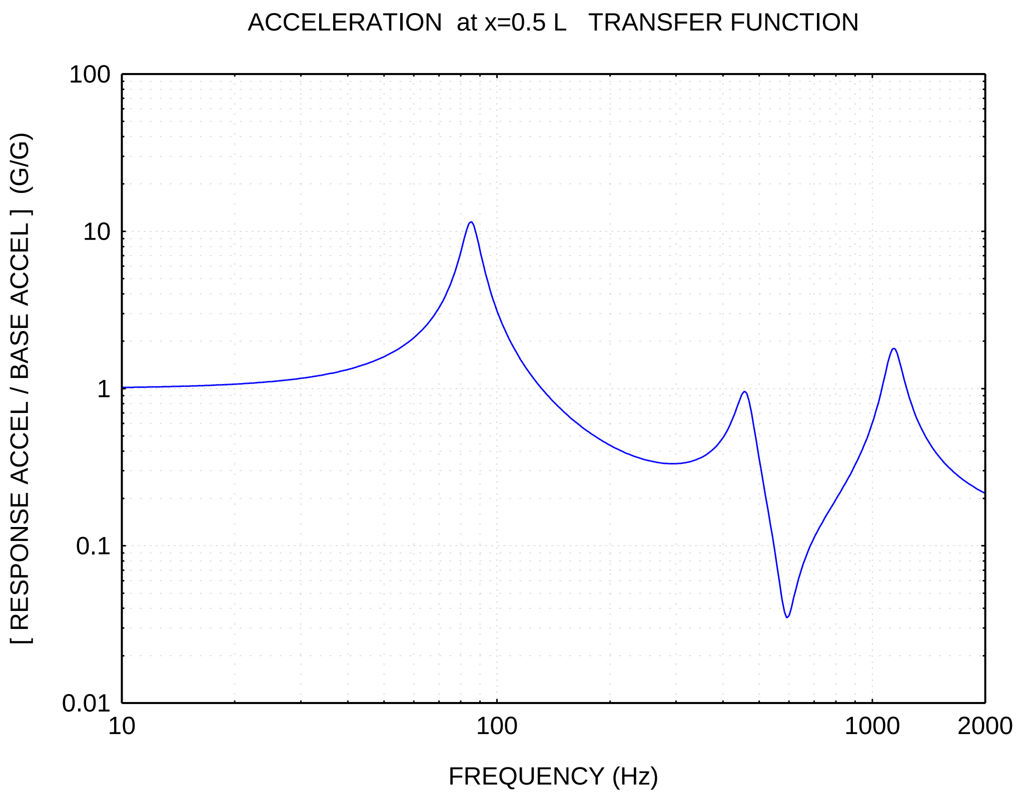
ACCELERATION at x=0.5 L TRANSFER FUNCTION
[RESPONSE ACCEL / BASE ACCEL] (G/G)
100
10
1
0.1
0.01
10
100
1000
2000
FREQUENCY (Hz)

Figure 2.

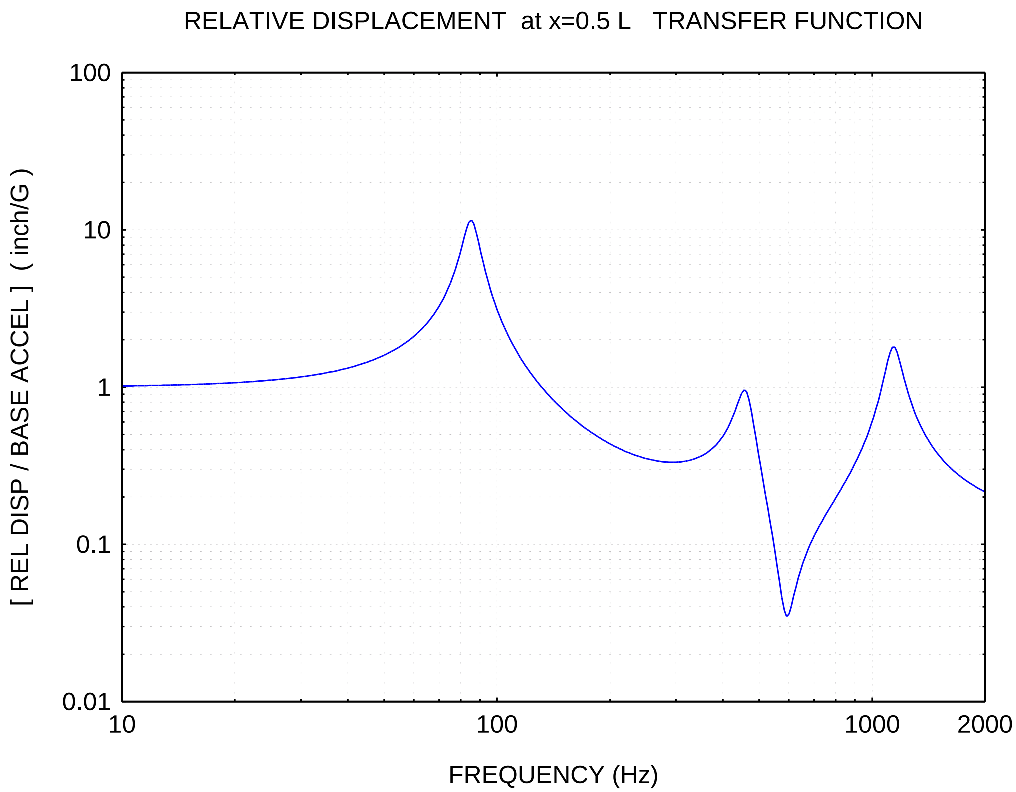
RELATIVE DISPLACEMENT at x=0.5 L TRANSFER FUNCTION
[REL DISP / BASE ACCEL] (inch/G)
100
10
1
0.1
0.01
10
100
1000
2000
FREQUENCY (Hz)

Figure 3.

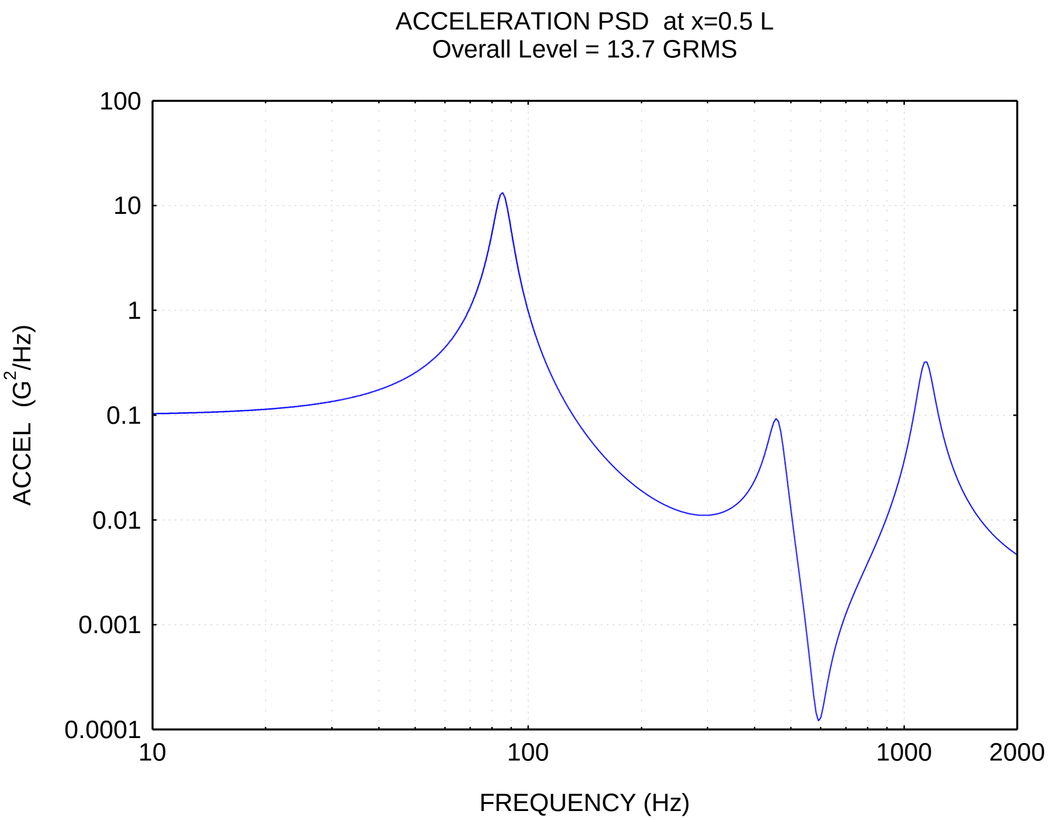

Figure 4.

Table 2. PSD, Base Input, 14.1 GRMS	
Freq (Hz)	Accel (G^2/Hz)
10	0.1
2000	0.1

The fixed-fixed beam is subjected to the base input PSD in Table 2. The resulting response PSD curves are shown in Figures 5 & 6 for the midpoint.

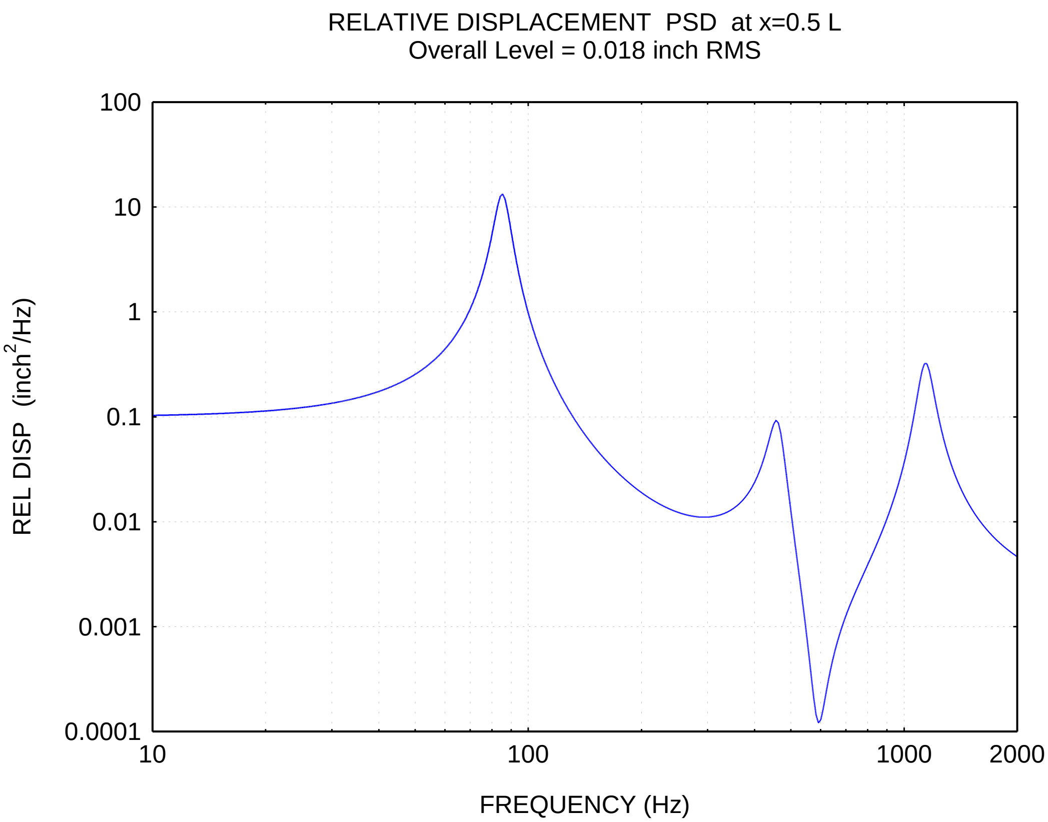

Figure 5.

References

1. T. Irvine, Bending Frequencies of Beams, Rod, and Pipes, Revision M, Vibrationdata, 2010.
2. T. Irvine, Steady-State Vibration Response of a Cantilever Beam Subjected to Base Excitation, Rev A, Vibrationdata, 2009.