STEADY-STATE VIBRATION RESPONSE OF A FIXED-FIXED BEAM SUBJECTED TO AN APPLIED FORCE WITH SPATIAL VARIATION

Revision C

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The fixed-fixed beam in Figure 1 is subjected to an applied force with spatial variation.

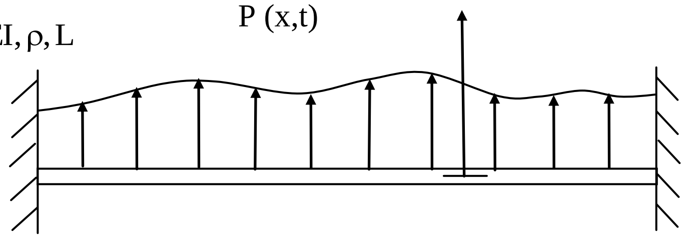

Figure 1.

The following equations are taken from References 1 and 2.

The governing differential equation is

$$EI\frac{\partial^4 y}{\partial x^4} + \rho\frac{\partial^2 y}{\partial t^2} = P(x,t) \tag{1}$$

where

E	is the modulus of elasticity
I	is the area moment of inertia
L	is the length
ρ	is the mass density (mass/length)
P	is the applied force per length

Assume that the pressure field can be represented as

$$P(x,t) = \hat{P}(x) f(t) \tag{2}$$

where $\hat{P}(x)$ is dimensionless

Assume that the displacement field can be represented as

$$y(x,t) = Y(x)T(t) \tag{3}$$

$$EI \frac{\partial^4}{\partial x^4} Y(x)T(t) + \rho \frac{\partial^2}{\partial t^2} Y(x)T(t) = \hat{P}(x) f(t) \tag{4}$$

$$EI \frac{d^4}{dx^4} Y(x)T(t) + \rho\, Y(x)\ddot{T}(t) = \hat{P}(x) f(t) \tag{5}$$

The homogeneous equation is

$$EI \frac{d^4}{dx^4} Y(x)T(t) + \rho\, Y(x)\ddot{T}(t) = 0 \tag{6}$$

$$\rho\, Y(x)\ddot{T}(t) = -EI \frac{d^4}{dx^4} Y(x)T(t) \tag{7}$$

$$\frac{\ddot{T}(t)}{T(t)} = -\frac{EI}{\rho\, Y(x)} \frac{d^4}{dx^4} Y(x) = -c^2 \tag{8}$$

Thus

$$\frac{d^2}{dt^2} T(t) + c^2\, T(t) = 0 \tag{9}$$

$$\left\{\frac{-EI}{\rho}\right\}\frac{\left\{\frac{d^4}{dx^4}Y(x)\right\}}{Y(x)} = -c^2 \tag{10}$$

$$\frac{d^4}{dx^4}Y(x) - c^2\left\{\frac{\rho}{EI}\right\}Y(x) = 0 \tag{11}$$

A solution for equation (11) is

$$Y(x) = a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x) \tag{12}$$

Intermediate steps are given in Reference 1.

The eigenvalues are

N	$\beta_n L$
1	4.73004
2	7.85321
3	10.9956
4	14.13717
5	17.27876

For n> 5

$$\beta_n L \approx \pi\left[\frac{1}{2} + n\right] \tag{13}$$

The mass-normalized mode shapes are

$$Y_n(x) = \frac{1}{\sqrt{\rho L}}\left\{\left[\cosh(\beta_n x) - \cos(\beta_n x)\right] - \sigma_n\left[\sinh(\beta_n x) - \sin(\beta_n x)\right]\right\} \tag{14}$$

where

$$\sigma_n = \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)}\right]$$

The derivatives are

$$Y_n{}'(x) = \frac{\beta_n}{\sqrt{\rho L}}\left\{\left[\sinh(\beta_n x) + \sin(\beta_n x)\right] - \sigma_n\left[\cosh(\beta_n x) - \cos(\beta_n x)\right]\right\} \tag{15}$$

$$Y_n{}''(x) = \frac{\beta_n{}^2}{\sqrt{\rho L}}\left\{\left[\cosh(\beta_n x) + \cos(\beta_n x)\right] - \sigma_n\left[\sinh(\beta_n x) + \sin(\beta_n x)\right]\right\} \tag{16}$$

Recall

$$EI\frac{\partial^4}{\partial x^4}y(x,t) + \rho\frac{\partial^2}{\partial t^2}y(x,t) = \hat{P}(x)f(t) \tag{17}$$

Let

$$y(x,t) = \sum_{n=1}^{m} Y_n(x)T_n(t) \tag{18}$$

$$EI\frac{\partial^4}{\partial x^4}\left[\sum_{n=1}^{m} Y_n(x)T_n(t)\right] + \rho\frac{\partial^2}{\partial t^2}\left[\sum_{n=1}^{m} Y_n(x)T_n(t)\right] = \hat{P}(x)f(t) \tag{19}$$

$$EI\left[\sum_{n=1}^{m} T_n(t)\frac{\partial^4}{\partial x^4} Y_n(x)\right] + \rho\left[\sum_{n=1}^{m} Y_n(x)\frac{\partial^2}{\partial t^2} T_n(t)\right] = \hat{P}(x)f(t) \tag{20}$$

$$EI\left[\sum_{n=1}^{m} T_n(t)\frac{d^4}{dx^4} Y_n(x)\right] + \rho\left[\sum_{n=1}^{m} Y_n(x)\frac{d^2}{dt^2} T_n(t)\right] = \hat{P}(x)f(t) \tag{21}$$

Note that

$$\frac{d^4}{dx^4} Y_n(x) = \beta_n{}^4 Y_n(x) \tag{22}$$

By substitution,

$$EI\left[\sum_{n=1}^{m} \beta_n{}^4 T_n(t) Y_n(x)\right] + \rho\left[\sum_{n=1}^{m} Y_n(x)\frac{d^2}{dt^2} T_n(t)\right] = \hat{P}(x)f(t) \tag{23}$$

$$EI\beta_n{}^4\left[\sum_{n=1}^{m} T_n(t) Y_n(x)\right] + \rho\left[\sum_{n=1}^{m} Y_n(x)\frac{d^2}{dt^2} T_n(t)\right] = -\hat{P}(x)f(t) \tag{24}$$

Multiply each term by $Y_p(x)$.

$$EI\beta_n{}^4\left[\sum_{n=1}^{m} T_n(t) Y_n(x) Y_p(x)\right] + \rho\left[\sum_{n=1}^{m} Y_n(x) Y_p(x)\frac{d^2}{dt^2} T_n(t)\right] = Y_p(x)\hat{P}(x)f(t) \tag{25}$$

Integrate with respect to length.

$$\int_0^L \left\{ EI\beta_n{}^4 \left[\sum_{n=1}^{m} T_n(t) Y_n(x) Y_p(x) \right] + \rho \left[\sum_{n=1}^{m} Y_n(x) Y_p(x) \frac{d^2}{dt^2} T_n(t) \right] \right\} dx$$

$$= \int_0^L Y_p(x) \hat{P}(x) f(t)\, dx$$

(26)

$$EI\beta_n{}^4 \int_0^L \left[\sum_{n=1}^{m} T_n(t) Y_n(x) Y_p(x) \right] dx + \rho \int_0^L \left[\sum_{n=1}^{m} Y_n(x) Y_p(x) \frac{d^2}{dt^2} T_n(t) \right] dx$$

$$= f(t) \int_0^L Y_p(x) \hat{P}(x)\, dx$$

(27)

$$EI\beta_n{}^4 \sum_{n=1}^{m} \left\{ T_n(t) \int_0^L Y_n(x) Y_p(x)\, dx \right\} + \rho \sum_{n=1}^{m} \left\{ \frac{d^2}{dt^2} T_n(t) \int_0^L Y_n(x) Y_p(x) dx \right\}$$

$$= f(t) \int_0^L Y_p(x) \hat{P}(x)\, dx$$

(28)

$$\frac{EI\beta_n{}^4}{\rho} \sum_{n=1}^{m} \left\{ T_n(t) \int_0^L \rho Y_n(x) Y_p(x)\, dx \right\} + \sum_{n=1}^{m} \left\{ \frac{d^2}{dt^2} T_n(t) \int_0^L \rho Y_n(x) Y_p(x) dx \right\}$$

$$= f(t) \int_0^L Y_p(x) \hat{P}(x)\, dx$$

(29)

The eigenvectors are orthogonal such that

$$\int_0^L \rho Y_n(x) Y_p(x)\, dx = 0 \qquad \text{for } n \neq p \qquad (30)$$

$$\int_0^L \rho Y_n(x) Y_p(x)\, dx = 1 \qquad \text{for } n = p \qquad (31)$$

$$\frac{d^2}{dt^2}T_n(t) + \frac{EI\beta_n^{\,4}}{\rho}T_n(t) = f(t)\int_0^L Y_n(x)\hat{P}(x)\,dx \tag{32}$$

$$\omega_n = \beta_n^{\,2}\sqrt{\frac{EI}{\rho}} \tag{33}$$

$$\omega_n^{\,2} = \frac{EI}{\rho}\beta_n^{\,4} \tag{34}$$

$$\frac{d^2}{dt^2}T_n(t) + \omega_n^{\,2}\,T_n(t) = f(t)\int_0^L Y_n(x)\hat{P}(x)\,dx \tag{35}$$

Add a damping term.

$$\ddot{T}_n(t) + 2\xi_n\omega_n\dot{T}_n(t) + \omega_n^{\,2}\,T_n(t) = f(t)\int_0^L Y_n(x)\hat{P}(x)\,dx \tag{36}$$

Steady-State Solution

Take a Fourier transform of both sides of (36).

$$\int_{-\infty}^{\infty}\left\{\ddot{T}_n(t) + 2\xi_n\omega_n\dot{T}(t) + \omega_n^{\,2}\,T_n(t)\right\}\exp(-j\omega t)\,dt = \int_{-\infty}^{\infty} f(t)\int_0^L Y_n(x)\hat{P}(x)\,dx\,dt \tag{37}$$

$$\int_{-\infty}^{\infty}\ddot{T}_n(t)\exp(-j\omega t)\,dt + 2\xi_n\omega_n\int_{-\infty}^{\infty}\dot{T}(t)\exp(-j\omega t)\,dt + \omega_n^{\,2}\int_{-\infty}^{\infty}T_n(t)\exp(-j\omega t)\,dt$$

$$= \left\{\int_0^L Y_n(x)\hat{P}(x)\,dx\right\}\left\{\int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\,dt\right\}$$

(38)

Note that

$$\int_{-\infty}^{\infty} \dot{T}_n(t)\exp(-j\omega t)dt = j\omega\int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt \tag{39}$$

$$\int_{-\infty}^{\infty} \ddot{T}_n(t)\exp(-j\omega t)dt = -\omega^2\int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt \tag{40}$$

$$-\omega^2\int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt + j2\xi_n\omega_n\omega\int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt + \omega_n{}^2\int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt$$
$$= \left\{\int_0^L Y_n(x)\hat{P}(x)\,dx\right\}\left\{\int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt\right\} \tag{41}$$

$$\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega\right]\int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt = \left\{\int_0^L Y_n(x)\hat{P}(x)\,dx\right\}\left\{\int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt\right\} \tag{42}$$

$$\hat{T}_n(\omega) = \int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt \tag{43}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt \tag{44}$$

$$\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega\right]\hat{T}_n(\omega) = \left\{\int_0^L Y_n(x)\hat{P}(x)\,dx\right\}F(\omega) \tag{45}$$

$$\hat{T}_n(\omega) = F(\omega)\frac{\int_0^L Y_n(x)\hat{P}(x)\,dx}{\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega} \tag{46}$$

$$Y(x,\omega) = \sum_{n=1}^{m} Y_n(x)\hat{T}_n(\omega) \tag{47}$$

The Fourier transform of the displacement is

$$Y(x,\omega) = F(\omega)\sum_{n=1}^{m} \frac{Y_n(x)\int_0^L Y_n(x)\hat{P}(x)\,dx}{\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega} \tag{48}$$

The frequency response function relating the displacement to the distributed force is

$$\frac{Y(x,\omega)}{F(\omega)} = \sum_{n=1}^{m} \frac{Y_n(x)\int_0^L Y_n(x)\hat{P}(x)\,dx}{\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega} \tag{49}$$

The bending moment transfer function is

$$\frac{M(x,\omega)}{F(\omega)} = \frac{EI\ Y''(x,\omega)}{F(\omega)} = \frac{EI}{\rho}\sum_{n=1}^{m}\left\{\frac{\Gamma_n Y_n''(x)}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n\omega\omega_n}\right\} \tag{50}$$

The bending stress transfer function is

$$\frac{\sigma(x,\omega)}{F(\omega)} = \left(\frac{c}{I}\right)\frac{M(x,\omega)}{F(\omega)} \tag{51}$$

where c is the distance from the neutral axis

References

1. T. Irvine, Bending Frequencies of Beams, Rod, and Pipes, Revision P, Vibrationdata, 2011.

2. T. Irvine, Steady-State Vibration Response of a Cantilever Beam Subjected to Base Excitation, Rev A, Vibrationdata, 2009.

APPENDIX A

Special Case

$$\frac{Y(x,\omega)}{F(\omega)} = \sum_{n=1}^{m} \frac{Y_n(x)\int_0^L Y_n(x)\hat{P}(x)\,dx}{\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega} \tag{A-1}$$

$$Y_n(x) = \frac{1}{\sqrt{\rho L}}\left\{\left[\cosh(\beta_n x) - \cos(\beta_n x)\right] - \sigma_n\left[\sinh(\beta_n x) - \sin(\beta_n x)\right]\right\} \tag{A-2}$$

Now assume that

$$\hat{P}(x) = \sqrt{\rho L}\, Y_1(x) \tag{A-3}$$

$$\frac{Y(x,\omega)}{F(\omega)} = \sqrt{\rho L}\sum_{n=1}^{m} \frac{Y_n(x)\int_0^L Y_n(x) Y_1(x)\,dx}{\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega} \tag{A-4}$$

Note that the orthogonality condition yields

$$\int_0^L Y_n(x) Y_1(x)\,dx = \begin{cases} L, & \text{for } n = 1 \\ 0, & \text{for } n = 2, 3, 4, \ldots \end{cases} \tag{A-5}$$

The displacement transfer function is

$$\frac{Y(x,\omega)}{F(\omega)} = \frac{1}{\rho}\left\{\frac{[\cosh(\beta_n x) - \cos(\beta_n x)] - \sigma_n\,[\sinh(\beta_n x) - \sin(\beta_n x)]}{\left(\omega_1{}^2 - \omega^2\right) + j\,2\,\xi_1\omega_1\omega}\right\} \tag{A-6}$$

The bending moment transfer function is

$$\frac{M(x,\omega)}{F(\omega)} = \frac{EI\,Y''(x,\omega)}{F(\omega)} = \frac{EI\,\beta_n{}^2}{\rho}\left\{\frac{[\cosh(\beta_n x) + \cos(\beta_n x)] - \sigma_n\,[\sinh(\beta_n x) + \sin(\beta_n x)]}{\left(\omega_1{}^2 - \omega^2\right) + j\,2\,\xi_1\omega_1\omega}\right\} \tag{A-7}$$

The bending stress transfer function is

$$\frac{\sigma(x,\omega)}{F(\omega)} = \left(\frac{c}{I}\right)\frac{M(x,\omega)}{F(\omega)} \tag{A-8}$$

APPENDIX B

Example

Consider a beam with the following properties:

Cross-Section	Rectangular
Boundary Conditions	Fixed-Fixed
Material	Aluminum

Thickness	T	=	0.125 inch
Width	W	=	1.0 inch
Length	L	=	27.5 inch
Cross-Section Area	A	=	0.125 in^2
Area Moment of Inertia	I	=	0.000163 in^4
Elastic Modulus	E	=	10E+06 lbf/in^2
Stiffness	EI	=	1628 lbf in^2
Mass per Volume	ρ_V	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)
Mass per Length	ρ	=	0.0125 lbm / in (0.00003237 lbf sec^2/in^4)
Mass	ρL	=	0.3438 lbm (0.0008906 lbf sec^2/in)
Viscous Damping Ratio	ξ	=	0.05

The normal modes and frequency response function analysis are performed via a Matlab script.

The normal modes results are:

Table B-1. Natural Frequency Results, Fixed-Fixed, Beam		
Mode	fn (Hz)	Participation Factor
1	33.38	0.02479
2	92.02	0
3	180.4	0.01086
4	298.2	0
5	445.4	0.006908

Note that the mode shape and participation factors are considered as dimensionless, but they must be consistent with respect to one another.

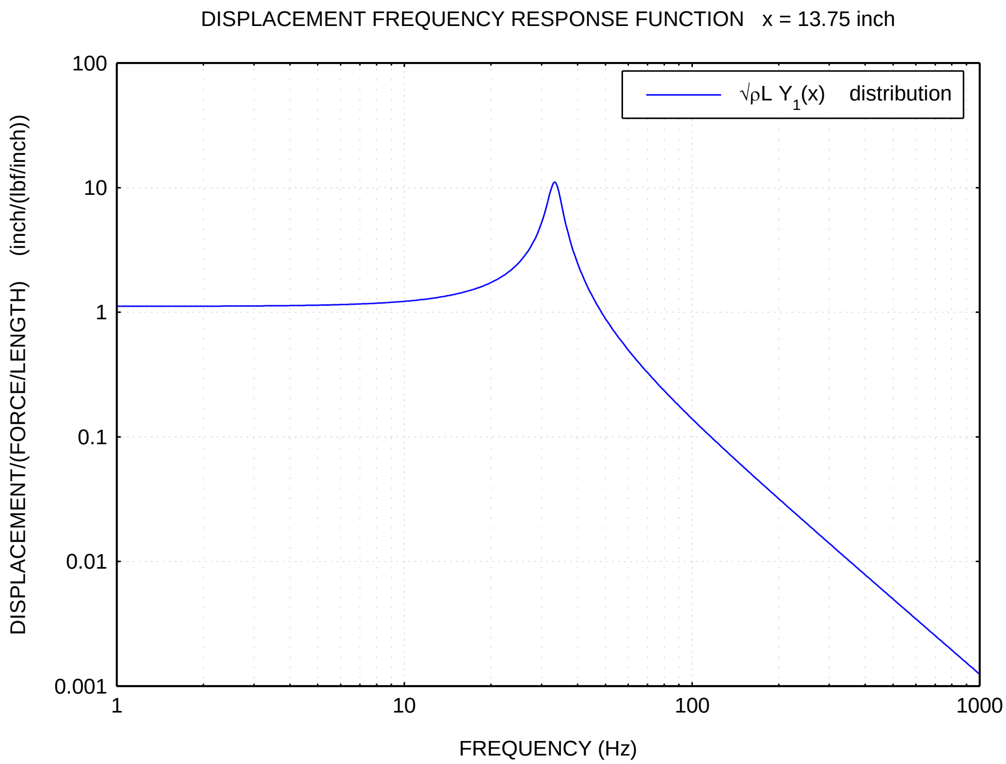

Figure B-1.

Now consider that the pressure field has the spatial distribution:

$$\hat{P}(x) = \sqrt{\rho L}\, Y_1(x) \qquad \text{(B-1)}$$

The resulting transfer function magnitude is shown in Figure B-1.

The maximum displacement response is: 11.13 [in/(lbf/in)] at 33.42 Hz

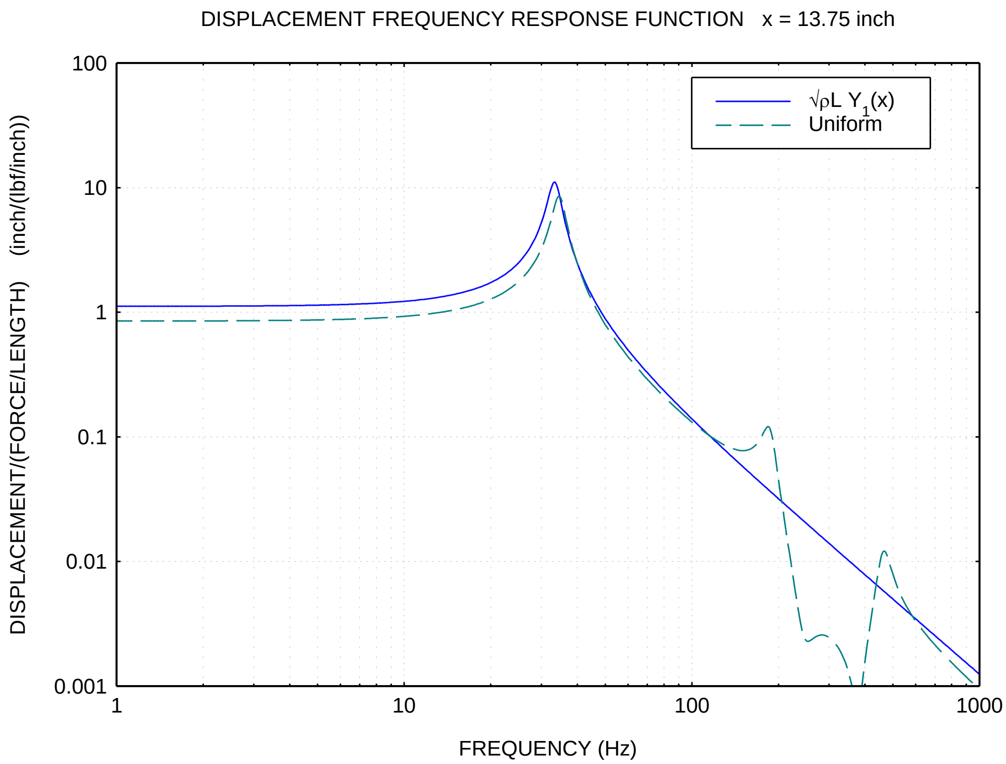

Figure B-2.

The displacement frequency response functions for two pressure distributions are shown in Figure B-2.

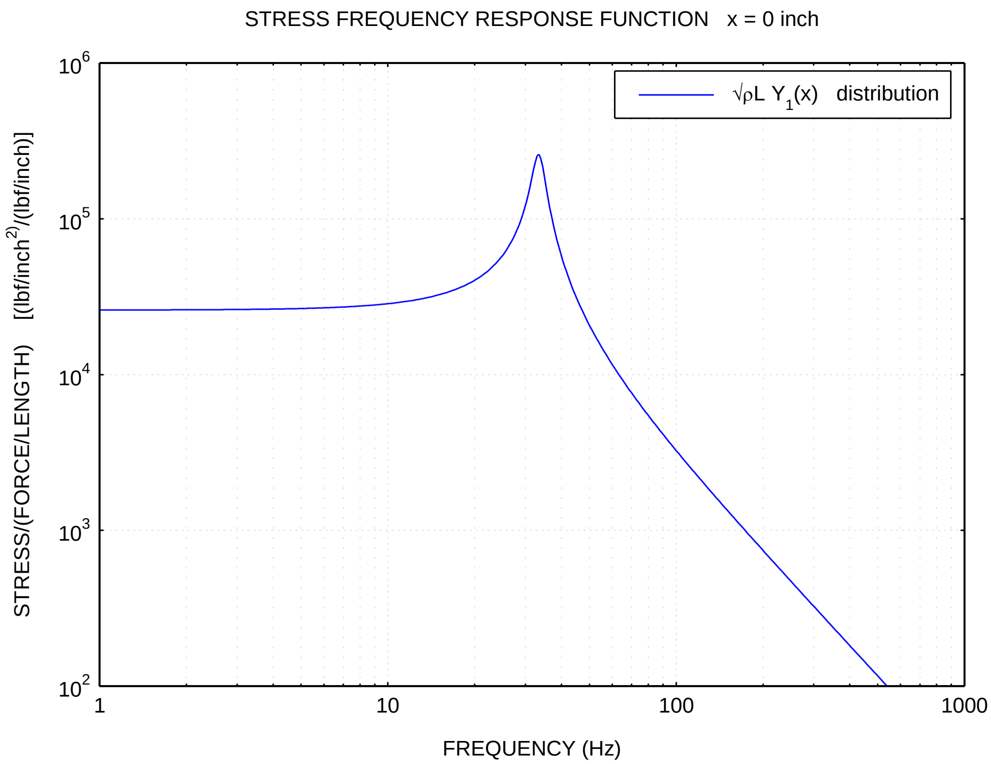

Figure B-3.

The maximum stress response is: 2.59e+05 [(lbf/in^2)/(lbf/in)] at 33.42 Hz