STEADY-STATE VIBRATION RESPONSE OF A
SIMPLY-SUPPORTED BEAM SUBJECTED TO BASE EXCITATION
Revision A

By Tom Irvine
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Email: tomirvine@aol.com

The simply-supported beam in Figure 1 is subject to base excitation.

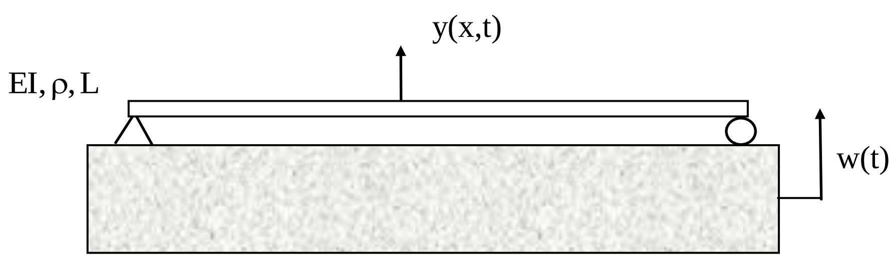

Figure 1.

The following equations are taken from References 1 and 2.

The governing differential equation is

$$EI\frac{\partial^4 y}{\partial x^4} + \rho\frac{\partial^2 y}{\partial t^2} = -\rho\frac{\partial^2 w}{\partial t^2} \qquad (1)$$

where

- E is the modulus of elasticity
- I is the area moment of inertia
- L is the length
- ρ is the mass density (mass/length)

The mass-normalized mode shapes are

$$Y_n(x) = \sqrt{\frac{2}{\rho L}} \sin(\beta_n x) \tag{2}$$

$$\frac{d^2}{dx^2} Y_n(x) = -\beta_n{}^2 \sqrt{\frac{2}{\rho L}} \sin(\beta_n x) \tag{3}$$

The eigenvalues are

$$\beta_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots \tag{4}$$

The natural frequencies are

$$\omega_n = \beta_n{}^2 \sqrt{EI/\rho} \tag{5}$$

The relative displacement response Y(x, ω) to base acceleration is

$$Y(x,\omega) = \ddot{W}(\omega) \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n Y_n(x)}{\left(\omega_n^2 - \omega^2\right) + j 2 \xi_n \omega \omega_n} \right\} \tag{6}$$

The participation factors for constant mass density are

$$\Gamma_n = \rho \int_0^L Y_n(x)\, dx \tag{7}$$

$$\Gamma_n = \rho \int_0^L \sqrt{\frac{2}{\rho L}} \sin(n\pi x / L) dx \tag{8}$$

$$\Gamma_n = -\sqrt{2\rho L} \left[\frac{1}{n\pi}\right] [\cos(n\pi) - 1] \quad , \text{n=1, 2, 3, ...} \tag{9}$$

Effective Modal Mass

The effective modal mass is

$$m_{eff,n} = \frac{\left[\int_0^L m(x) Y_n(x) dx\right]^2}{\int_0^L m(x) [Y_n(x)]^2 dx} \tag{10}$$

The eigenvectors are already normalized such that

$$\int_0^L m(x) [Y_n(x)]^2 dx = 1 \tag{11}$$

Thus,

$$m_{eff,n} = [\Gamma_n]^2 = \left[\int_0^L m(x) Y_n(x) dx\right]^2 \tag{12}$$

$$m_{eff,n} = \left[-\sqrt{2\rho L} \left[\frac{1}{n\pi}\right] [\cos(n\pi) - 1]\right]^2 \tag{13}$$

$$m_{eff,n} = 2\rho L \frac{1}{(n\pi)^2} [\cos(n\pi) - 1]^2 \quad , \text{n=1, 2, 3, ...} \tag{14}$$

Example

Consider a beam with the following properties:

Cross-Section	Circular
Boundary Conditions	Simply-Supported at Each End
Material	Aluminum

Diameter	D	=	0.5 inch
Cross-Section Area	A	=	0.1963 in^2
Length	L	=	24 inch
Area Moment of Inertia	I	=	0.003068 in^4
Elastic Modulus	E	=	1.0e+07 lbf/in^2
Stiffness	EI	=	30680 lbf in^2
Mass per Volume	ρ_V	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)
Mass per Length	ρ	=	0.01963 lbm/in (5.08e-05 lbf sec^2/in^2)
Mass	ρL	=	0.471 lbm (1.22E-03 lbf sec^2/in)
Viscous Damping Ratio	ξ	=	0.05

The normal modes and frequency response function analysis are performed via Matlab script: ss_beam_stress.m. The normal modes results are:

Table 1.
Natural Frequency Results, Beam Simply-Supported at Each End

Mode	fn (Hz)	Participation Factor	Effective Modal Mass (lbf sec^2/in)	Effective Modal Mass (lbm)
1	66.97	0.0315	0.9896e-03	0.3820
2	267.9	0	0	0
3	603.8	0.0105	0.1100e-03	0.0424
4	1072	0	0	0

Note that the mode shape and participation factors are considered as dimensionless, but they must be consistent with respect to one another.

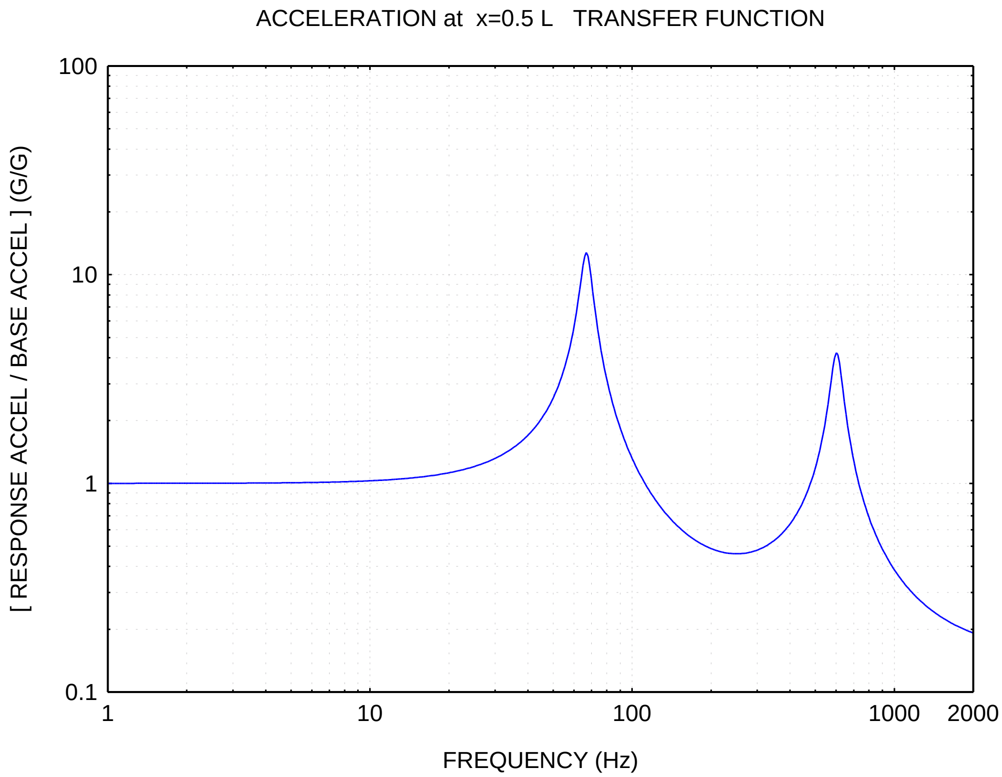
ACCELERATION at x=0.5 L TRANSFER FUNCTION
100
10
1
0.1
[RESPONSE ACCEL / BASE ACCEL] (G/G)
1
10
100
1000
2000
FREQUENCY (Hz)

Figure 2.

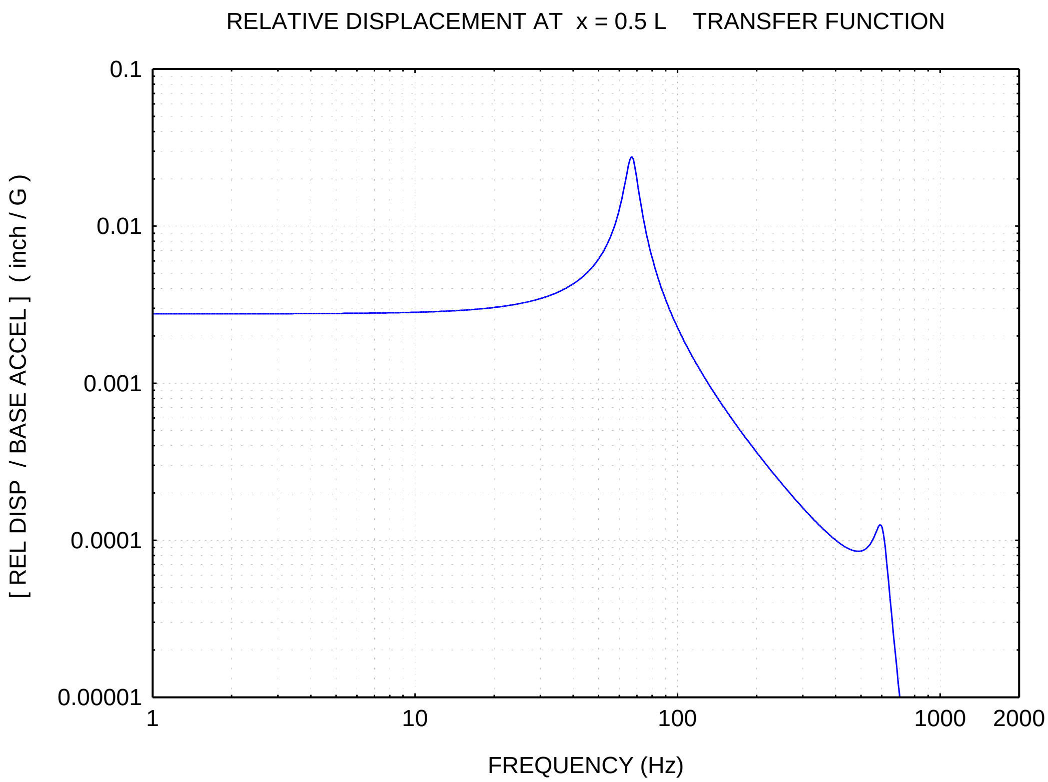
RELATIVE DISPLACEMENT AT x = 0.5 L TRANSFER FUNCTION
[REL DISP / BASE ACCEL] (inch / G)
0.1
0.01
0.001
0.0001
0.00001
1
10
100
1000
2000
FREQUENCY (Hz)

Figure 3.

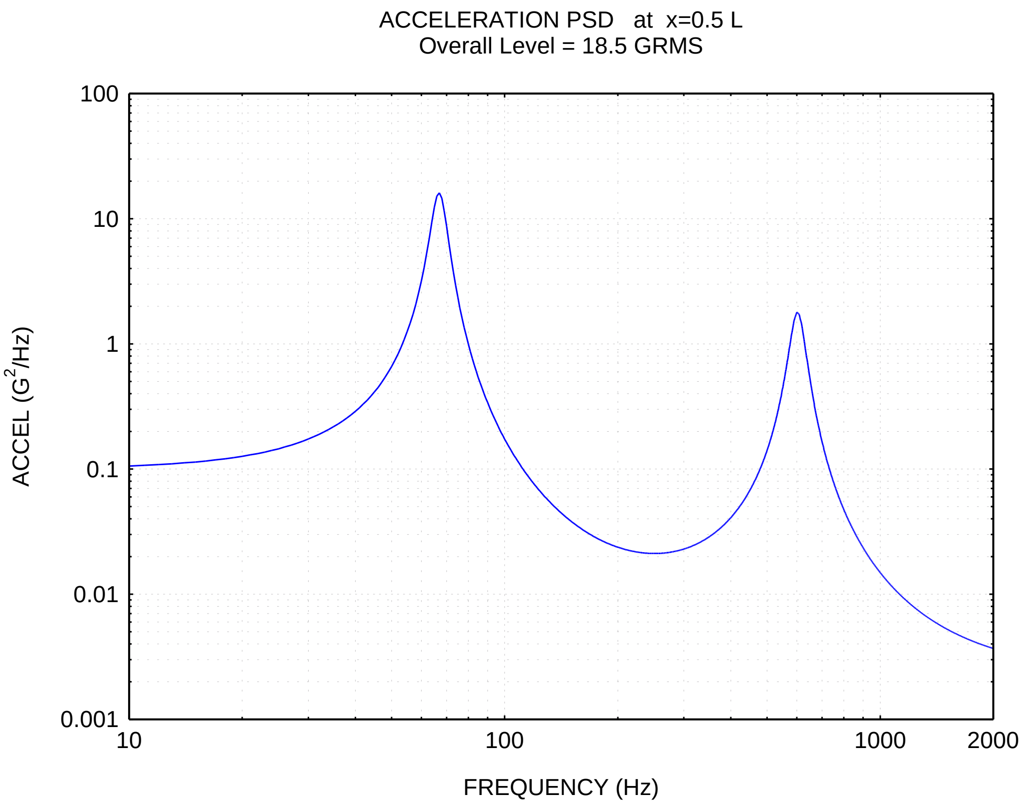

Figure 4.

Table 2. PSD, Base Input, 14.1 GRMS	
Freq (Hz)	Accel (G^2/Hz)
10	0.1
2000	0.1

The simply-supported beam is subjected to the base input PSD in Table 2. The resulting response PSD curves are shown in Figures 5 & 6 for the midpoint.

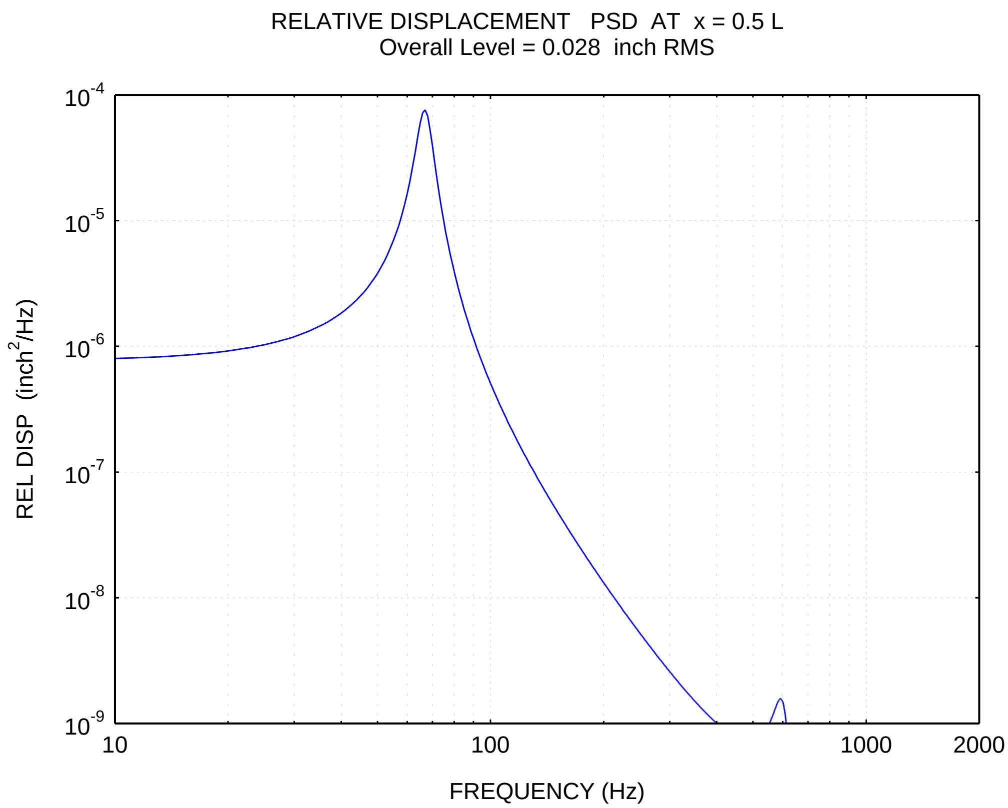

Figure 5.

References

1. T. Irvine, Bending Frequencies of Beams, Rod, and Pipes, Revision M, Vibrationdata, 2010.

2. T. Irvine, Steady-State Vibration Response of a Cantilever Beam Subjected to Base Excitation, Rev A, Vibrationdata, 2009.