The simply-supported beam in Figure 1 is subject to base excitation.

![Figure 1](image)

The following equations are taken from References 1 and 2.

The governing differential equation is

\[
EI \frac{\partial^4 y}{\partial x^4} + \rho \frac{\partial^2 y}{\partial t^2} = -\rho \frac{\partial^2 w}{\partial t^2}
\]  

where

- \( E \) is the modulus of elasticity
- \( I \) is the area moment of inertia
- \( L \) is the length
- \( \rho \) is the mass density (mass/length)
The mass-normalized mode shapes are

\[ Y_n(x) = \frac{2}{\sqrt{\rho L}} \sin(\beta_n x) \]  

(2)

\[ \frac{d^2}{dx^2} Y_n(x) = -\beta_n^2 \frac{2}{\sqrt{\rho L}} \sin(\beta_n x) \]  

(3)

The eigenvalues are

\[ \beta_n = \frac{n \pi}{L}, \quad n = 1, 2, 3, \ldots \]  

(4)

The natural frequencies are

\[ \omega_n = \beta_n^2 \sqrt{EI/\rho} \]  

(5)

The relative displacement response \( Y(x, \omega) \) to base acceleration is

\[ Y(x, \omega) = \tilde{W}(\omega) \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n Y_n(x)}{\left( \omega_n^2 - \omega^2 \right) + j 2\xi_n \omega \omega_n} \right\} \]  

(6)

The participation factors for constant mass density are

\[ \Gamma_n = \rho \int_0^L Y_n(x) \, dx \]  

(7)
\[ \Gamma_n = \rho \int_0^L \sqrt{\frac{2}{\rho L}} \sin(n\pi x / L) \, dx \] \hspace{1cm} (8)

\[ \Gamma_n = -\sqrt{2\rho L} \left[ \frac{1}{n\pi} \right] \left[ \cos(n\pi) - 1 \right] \right] \hspace{1cm} , \ n=1, 2, 3, \ldots \] \hspace{1cm} (9)

**Effective Modal Mass**

The effective modal mass is

\[ m_{\text{eff},n} = \left[ \frac{\int_0^L m(x) Y_n(x) \, dx}{\int_0^L m(x) [Y_n(x)]^2 \, dx} \right]^2 \] \hspace{1cm} (10)

The eigenvectors are already normalized such that

\[ \int_0^L m(x) [Y_n(x)]^2 \, dx = 1 \] \hspace{1cm} (11)

Thus,

\[ m_{\text{eff},n} = \left[ \Gamma_n \right]^2 = \left[ \int_0^L m(x) Y_n(x) \, dx \right]^2 \] \hspace{1cm} (12)

\[ m_{\text{eff},n} = \left[ -\sqrt{2\rho L} \left[ \frac{1}{n\pi} \right] \left[ \cos(n\pi) - 1 \right] \right]^2 \] \hspace{1cm} (13)

\[ m_{\text{eff},n} = 2\rho L \frac{1}{(n\pi)^2} \left[ \cos(n\pi) - 1 \right]^2 \hspace{1cm} , \ n=1, 2, 3, \ldots \] \hspace{1cm} (14)
Example

Consider a beam with the following properties:

<table>
<thead>
<tr>
<th>Cross-Section</th>
<th>Circular</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary Conditions</td>
<td>Simply-Supported at Each End</td>
</tr>
<tr>
<td>Material</td>
<td>Aluminum</td>
</tr>
</tbody>
</table>

| Diameter | D     | 0.5 inch |
| Cross-Section Area | A     | 0.1963 in^2 |
| Length    | L     | 24 inch  |
| Area Moment of Inertia | I     | 0.003068 in^4 |
| Elastic Modulus | E     | 1.0e+07 lbf/in^2 |
| Stiffness  | EI    | 30680 lbf in^2 |
| Mass per Volume | \( \rho_v \) | 0.1 lbm / in^3 (0.000259 lbf sec^2/in^4) |
| Mass per Length | \( \rho \) | 0.01963 lbm/in (5.08e-05 lbf sec^2/in^2) |
| Mass       | \( \rho L \) | 0.471 lbm (1.22E-03 lbf sec^2/in) |
| Viscous Damping Ratio | \( \xi \) | 0.05 |

The normal modes and frequency response function analysis are performed via Matlab script: ss_beam_stress.m. The normal modes results are:

<table>
<thead>
<tr>
<th>Mode</th>
<th>fn (Hz)</th>
<th>Participation Factor</th>
<th>Effective Modal Mass (lbf sec^2/in)</th>
<th>Effective Modal Mass (lbm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>66.97</td>
<td>0.0315</td>
<td>0.9896e-03</td>
<td>0.3820</td>
</tr>
<tr>
<td>2</td>
<td>267.9</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>603.8</td>
<td>0.0105</td>
<td>0.1100e-03</td>
<td>0.0424</td>
</tr>
<tr>
<td>4</td>
<td>1072</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that the mode shape and participation factors are considered as dimensionless, but they must be consistent with respect to one another.
ACCELERATION at \( x = 0.5 \) L  TRANSFER FUNCTION

Figure 2.
Figure 3.

RELATIVE DISPLACEMENT AT $x = 0.5 \, L$ TRANSFER FUNCTION

[ REL DISP / BASE ACCEL ] (inch / G)

RELATIVE DISPLACEMENT AT $x = 0.5 \, L$ TRANSFER FUNCTION

FREQUENCY (Hz)

$\langle R E L \, D I S P \rangle / \langle B A S E \, A C C E L \rangle$ (inch / G)

RELATIVE DISPLACEMENT AT $x = 0.5 \, L$ TRANSFER FUNCTION
The simply-supported beam is subjected to the base input PSD in Table 2. The resulting response PSD curves are shown in Figures 5 & 6 for the midpoint.
References
