STEADY-STATE VIBRATION RESPONSE OF A SIMPLY-SUPPORTED BEAM SUBJECTED TO AN APPLIED FORCE WITH SPATIAL VARIATION

Revision F

By Tom Irvine
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Email: tom@vibrationdata.com

The simply-supported beam in Figure 1 is subjected to an applied force with spatial variation.

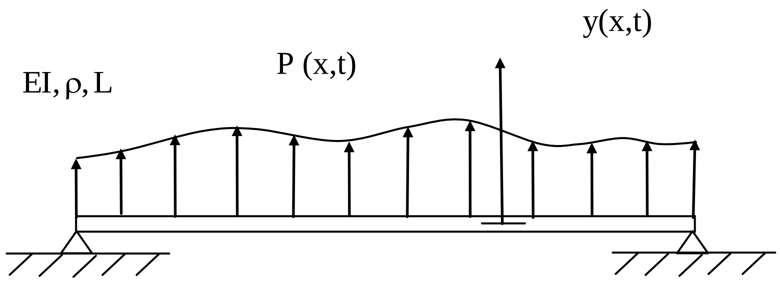

Figure 1.

The following equations are taken from References 1 and 2.

The governing differential equation is

$$EI \frac{\partial^4 y}{\partial x^4} + \rho \frac{\partial^2 y}{\partial t^2} = P(x, t) \tag{1}$$

where

E	is the modulus of elasticity
I	is the area moment of inertia
L	is the length
ρ	is the mass density (mass/length)
P	is the applied force per length

Assume that the pressure field can be represented as

$$P(x,t) = \hat{P}(x) f(t) \tag{2}$$

where $\hat{P}(x)$ is dimensionless

Assume that the displacement field can be represented as

$$y(x,t) = Y(x) T(t) \tag{3}$$

$$EI \frac{\partial^4}{\partial x^4} Y(x) T(t) + \rho \frac{\partial^2}{\partial t^2} Y(x) T(t) = \hat{P}(x) f(t) \tag{4}$$

$$EI \frac{d^4}{dx^4} Y(x) T(t) + \rho Y(x) \ddot{T}(t) = \hat{P}(x) f(t) \tag{5}$$

The homogeneous equation is

$$EI \frac{d^4}{dx^4} Y(x) T(t) + \rho Y(x) \ddot{T}(t) = 0 \tag{6}$$

$$\rho Y(x) \ddot{T}(t) = -EI \frac{d^4}{dx^4} Y(x) T(t) \tag{7}$$

$$\frac{\ddot{T}(t)}{T(t)} = -\frac{EI}{\rho Y(x)} \frac{d^4}{dx^4} Y(x) = -c^2 \tag{8}$$

Thus

$$\frac{d^2}{dt^2} T(t) + c^2 T(t) = 0 \tag{9}$$

$$\left\{\frac{-EI}{\rho}\right\}\frac{\left\{\frac{d^4}{dx^4}Y(x)\right\}}{Y(x)} = -c^2 \tag{10}$$

$$\frac{d^4}{dx^4}Y(x) - c^2\left\{\frac{\rho}{EI}\right\}Y(x) = 0 \tag{11}$$

A solution for equation (11) is

$$Y(x) = a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x) \tag{12}$$

Intermediate steps are given in Reference 1.

The natural frequency term ω_n is

$$\omega_n = \beta_n{}^2 \sqrt{\frac{EI}{\rho}} \tag{13}$$

$$\beta_n = \left[\frac{n\pi}{L}\right] \tag{14}$$

The natural frequency is

$$\omega_n = \left[\frac{n\pi}{L}\right]^2 \sqrt{\frac{EI}{\rho}}\ ,\ n = 1,2,3,... \tag{15}$$

$$f_n = \left[\frac{1}{2\pi}\right]\left[\frac{n\pi}{L}\right]^2 \sqrt{\frac{EI}{\rho}}, \quad n = 1, 2, 3, \ldots \tag{16}$$

The mass-normalized mode shapes are

$$Y_n(x) = \sqrt{\frac{2}{\rho L}} \sin(n\pi x / L) \tag{17}$$

$$Y_n'(x) = \left(\frac{n\pi}{L}\right)\sqrt{\frac{2}{\rho L}} \cos(n\pi x / L) \tag{18}$$

$$Y_n''(x) = -\left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{2}{\rho L}} \sin(n\pi x / L) \tag{19}$$

Recall

$$EI \frac{\partial^4}{\partial x^4} y(x,t) + \rho \frac{\partial^2}{\partial t^2} y(x,t) = \hat{P}(x) f(t) \tag{20}$$

Let

$$y(x,t) = \sum_{n=1}^{\infty} Y_n(x) T_n(t) \tag{21}$$

$$EI \frac{\partial^4}{\partial x^4}\left[\sum_{n=1}^{\infty} Y_n(x) T_n(t)\right] + \rho \frac{\partial^2}{\partial t^2}\left[\sum_{n=1}^{\infty} Y_n(x) T_n(t)\right] = \hat{P}(x) f(t) \tag{22}$$

$$EI\left[\sum_{n=1}^{\infty} T_n(t)\frac{\partial^4}{\partial x^4}Y_n(x)\right]+\rho\left[\sum_{n=1}^{\infty} Y_n(x)\frac{\partial^2}{\partial t^2}T_n(t)\right]=\hat{P}(x)f(t) \tag{23}$$

$$EI\left[\sum_{n=1}^{\infty} T_n(t)\frac{d^4}{dx^4}Y_n(x)\right]+\rho\left[\sum_{n=1}^{\infty} Y_n(x)\frac{d^2}{dt^2}T_n(t)\right]=\hat{P}(x)f(t) \tag{24}$$

Note that

$$\frac{d^4}{dx^4}Y_n(x)=\beta_n{}^4 Y_n(x) \tag{25}$$

By substitution,

$$EI\left[\sum_{n=1}^{\infty} \beta_n{}^4 T_n(t)Y_n(x)\right]+\rho\left[\sum_{n=1}^{\infty} Y_n(x)\frac{d^2}{dt^2}T_n(t)\right]=\hat{P}(x)f(t) \tag{26}$$

$$EI\left[\sum_{n=1}^{\infty} \beta_n{}^4 T_n(t)Y_n(x)\right]+\rho\left[\sum_{n=1}^{\infty} Y_n(x)\frac{d^2}{dt^2}T_n(t)\right]=-\hat{P}(x)f(t) \tag{27}$$

Multiply each term by $Y_p(x)$.

$$EI\left[\sum_{n=1}^{\infty} \beta_n{}^4 T_n(t)Y_n(x)Y_p(x)\right]+\rho\left[\sum_{n=1}^{\infty} Y_n(x)Y_p(x)\frac{d^2}{dt^2}T_n(t)\right]=Y_p(x)\hat{P}(x)f(t) \tag{28}$$

Integrate with respect to length.

$$\int_0^L \left\{ EI \left[\sum_{n=1}^{\infty} \beta_n{}^4 \, T_n(t) Y_n(x) Y_p(x) \right] + \rho \left[\sum_{n=1}^{\infty} Y_n(x) Y_p(x) \frac{d^2}{dt^2} T_n(t) \right] \right\} dx$$

$$= \int_0^L Y_p(x) \hat{P}(x) f(t) \, dx$$

(29)

$$EI \int_0^L \left[\sum_{n=1}^{\infty} \beta_n{}^4 \, T_n(t) Y_n(x) Y_p(x) \right] dx + \rho \int_0^L \left[\sum_{n=1}^{\infty} Y_n(x) Y_p(x) \frac{d^2}{dt^2} T_n(t) \right] dx$$

$$= f(t) \int_0^L Y_p(x) \hat{P}(x) \, dx$$

(30)

$$EI \sum_{n=1}^{\infty} \left\{ \beta_n{}^4 T_n(t) \int_0^L Y_n(x) Y_p(x) \, dx \right\} + \rho \sum_{n=1}^{\infty} \left\{ \frac{d^2}{dt^2} T_n(t) \int_0^L Y_n(x) Y_p(x) dx \right\}$$

$$= f(t) \int_0^L Y_p(x) \hat{P}(x) \, dx$$

(31)

$$\frac{EI}{\rho} \sum_{n=1}^{\infty} \left\{ \beta_n{}^4 T_n(t) \int_0^L \rho Y_n(x) Y_p(x) \, dx \right\} + \sum_{n=1}^{\infty} \left\{ \frac{d^2}{dt^2} T_n(t) \int_0^L \rho Y_n(x) Y_p(x) dx \right\}$$

$$= f(t) \int_0^L Y_p(x) \hat{P}(x) \, dx$$

(32)

The eigenvectors are orthogonal such that

$$\int_0^L \rho Y_n(x) Y_p(x) \, dx = 0 \qquad \text{for } n \neq p \tag{33}$$

$$\int_0^L \rho Y_n(x) Y_p(x) \, dx = 1 \qquad \text{for } n = p \tag{34}$$

$$\frac{d^2}{dt^2} T_n(t) + \frac{EI \beta_n{}^4}{\rho} T_n(t) = f(t) \int_0^L Y_n(x) \hat{P}(x)\, dx \tag{35}$$

$$\omega_n = \beta_n{}^2 \sqrt{\frac{EI}{\rho}} \tag{36}$$

$$\omega_n{}^2 = \frac{EI}{\rho} \beta_n{}^4 \tag{37}$$

$$\frac{d^2}{dt^2} T_n(t) + \omega_n{}^2\, T_n(t) = f(t) \int_0^L Y_n(x) \hat{P}(x)\, dx \tag{38}$$

Add a damping term.

$$\ddot{T}_n(t) + 2\xi_n \omega_n \dot{T}_n(t) + \omega_n{}^2\, T_n(t) = f(t) \int_0^L Y_n(x) \hat{P}(x)\, dx \tag{39}$$

<u>Steady-State Solution</u>

Take a Fourier transform of both sides of (39).

$$\int_{-\infty}^{\infty} \left\{ \ddot{T}_n(t) + 2\xi_n \omega_n \dot{T}(t) + \omega_n{}^2\, T_n(t) \right\} \exp(-j\omega t)\, dt = \int_{-\infty}^{\infty} f(t) \int_0^L Y_n(x) \hat{P}(x)\, dx dt \tag{40}$$

$$\int_{-\infty}^{\infty} \ddot{T}_n(t) \exp(-j\omega t)\, dt + 2\xi_n \omega_n \int_{-\infty}^{\infty} \dot{T}(t) \exp(-j\omega t)\, dt + \omega_n{}^2 \int_{-\infty}^{\infty} T_n(t) \exp(-j\omega t)\, dt$$

$$= \left\{ \int_0^L Y_n(x) \hat{P}(x)\, dx \right\} \left\{ \int_{-\infty}^{\infty} f(t) \exp(-j\omega t)\, dt \right\}$$

(41)

Note that

$$\int_{-\infty}^{\infty} \dot{T}_n(t)\exp(-j\omega t)dt = j\omega\int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt \tag{42}$$

$$\int_{-\infty}^{\infty} \ddot{T}_n(t)\exp(-j\omega t)dt = -\omega^2\int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt \tag{43}$$

$$-\omega^2\int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt + j2\xi_n\omega_n\omega\int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt + \omega_n{}^2\int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt$$
$$= \left\{\int_0^L Y_n(x)\hat{P}(x)\,dx\right\}\left\{\int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt\right\} \tag{44}$$

$$\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega\right]\int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt = \left\{\int_0^L Y_n(x)\hat{P}(x)\,dx\right\}\left\{\int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt\right\} \tag{45}$$

$$\hat{T}_n(\omega) = \int_{-\infty}^{\infty} T_n(t)\exp(-j\omega t)dt \tag{46}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt \tag{47}$$

$$\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega\right]\hat{T}_n(\omega) = \left\{\int_0^L Y_n(x)\hat{P}(x)\,dx\right\}F(\omega) \tag{48}$$

$$\hat{T}_n(\omega) = F(\omega) \frac{\int_0^L Y_n(x)\hat{P}(x)\,dx}{\left(\omega_n{}^2 - \omega^2\right) + j\,2\,\xi_n \omega_n \omega} \tag{49}$$

$$Y(x,\omega) = \sum_{n=1}^{\infty} Y_n(x)\hat{T}_n(\omega) \tag{50}$$

The Fourier transform of the displacement is

$$Y(x,\omega) = F(\omega) \sum_{n=1}^{\infty} \frac{Y_n(x)\int_0^L Y_n(x)\hat{P}(x)\,dx}{\left(\omega_n{}^2 - \omega^2\right) + j\,2\,\xi_n \omega_n \omega} \tag{51}$$

The frequency response function relating the displacement to the distributed force is

$$\frac{Y(x,\omega)}{F(\omega)} = \sum_{n=1}^{\infty} \frac{Y_n(x)\int_0^L Y_n(x)\hat{P}(x)\,dx}{\left(\omega_n{}^2 - \omega^2\right) + j\,2\,\xi_n \omega_n \omega} \tag{52}$$

The bending moment transfer function is

$$\frac{M(x,\omega)}{F(\omega)} = \frac{EI\ Y''(x,\omega)}{F(\omega)} = \frac{EI}{\rho} \sum_{n=1}^{\infty} \left\{ \frac{\Gamma_n Y_n{}''(x)}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n \omega \omega_n} \right\} \tag{53}$$

The bending stress transfer function is

$$\frac{\sigma(x,\omega)}{F(\omega)} = \left(\frac{c}{I}\right) \frac{M(x,\omega)}{F(\omega)} \tag{54}$$

where c is the distance from the neutral axis

References

1. T. Irvine, Bending Frequencies of Beams, Rod, and Pipes, Revision P, Vibrationdata, 2011.

2. T. Irvine, Steady-State Vibration Response of a Cantilever Beam Subjected to Base Excitation, Rev A, Vibrationdata, 2009.

APPENDIX A

Special Case

$$\frac{Y(x,\omega)}{F(\omega)} = \sum_{n=1}^{\infty} \frac{Y_n(x)\int_0^L Y_n(x)\hat{P}(x)\,dx}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n\omega_n\omega} \tag{A-1}$$

$$Y_n(x) = \sqrt{\frac{2}{\rho L}}\sin(n\pi x/L) \tag{A-2}$$

$$\frac{Y(x,\omega)}{F(\omega)} = \sum_{n=1}^{\infty} \frac{\sqrt{\frac{2}{\rho L}}\sin(n\pi x/L)\int_0^L \sqrt{\frac{2}{\rho L}}\sin(n\pi x/L)\hat{P}(x)\,dx}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n\omega_n\omega} \tag{A-3}$$

$$\frac{Y(x,\omega)}{F(\omega)} = \frac{2}{\rho L}\sum_{n=1}^{\infty} \frac{\sin(n\pi x/L)\int_0^L \sin(n\pi x/L)\hat{P}(x)\,dx}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n\omega_n\omega} \tag{A-4}$$

Now assume that

$$\hat{P}(x) = \sin(\pi x/L) \tag{A-5}$$

$$\frac{Y(x,\omega)}{F(\omega)} = \frac{2}{\rho L}\sum_{n=1}^{\infty} \frac{\sin(n\pi x/L)\int_0^L \sin(n\pi x/L)\sin(\pi x/L)\,dx}{\left(\omega_n^2 - \omega^2\right) + j2\xi_n\omega_n\omega} \tag{A-6}$$

For n = 1,

$$\int_0^L [\sin(\pi x/L)]^2 \, dx = \frac{1}{2}\int_0^L [1-\cos(2\pi x/L)] \, dx \tag{A-7}$$

$$\int_0^L [\sin(\pi x/L)]^2 \, dx = \frac{1}{2}\left\{ x - \left(\frac{2\pi}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\right\}\Bigg|_0^L \tag{A-8}$$

$$\int_0^L [\sin(\pi x/L)]^2 \, dx = L/2 \tag{A-9}$$

For n = 2,3,4….

$$\int_0^L \sin(n\pi x/L)\sin(\pi x/L)dx = \frac{1}{2}\int_0^L [-\cos((n+1)\pi x/L) + \cos((n-1)\pi x/L)]dx \tag{A-10}$$

$$\int_0^L \sin(n\pi x/L)\sin(\pi x/L)dx = \frac{1}{2}\left\{-\frac{L}{(n+1)\pi}\sin((n+1)\pi x/L) + \frac{L}{(n-1)\pi}\sin((n-1)\pi x/L)\right\}\Bigg|_0^L \tag{A-11}$$

$$\int_0^L \sin(n\pi x/L)\sin(\pi x/L)dx = 0 \tag{A-12}$$

In summary,

$$\int_0^L \sin(n\pi x/L)\sin(\pi x/L)dx = \begin{cases} L/2, & \text{for } n = 1 \\ 0, & \text{for } n = 2,3,4,... \end{cases} \tag{A-13}$$

The transfer function is

$$\frac{Y(x,\omega)}{F(\omega)} = \frac{2}{\rho L} \frac{\left(\frac{L}{2}\right)\sin(\pi x / L)}{\left(\omega_1{}^2 - \omega^2\right) + j2\xi_1\omega_1\omega} \tag{A-14}$$

$$\frac{Y(x,\omega)}{F(\omega)} = \frac{1}{\rho} \frac{\sin(\pi x / L)}{\left(\omega_1{}^2 - \omega^2\right) + j2\xi_1\omega_1\omega} \tag{A-15}$$

$$\omega_1 = \left[\frac{\pi}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{A-16}$$

$$\frac{M(x,\omega)}{F(\omega)} = \frac{EIY''(x,\omega)}{F(\omega)} = -\left(\frac{\pi}{L}\right)^2 \left(\frac{EI}{\rho}\right) \frac{\sin(\pi x / L)}{\left(\omega_1{}^2 - \omega^2\right) + j2\xi_1\omega_1\omega} \tag{A-17}$$

APPENDIX B

Example

Consider a beam with the following properties:

Cross-Section	Rectangular
Boundary Conditions	Simply Supported at Each End
Material	Aluminum

Thickness	T	=	0.125 inch
Width	W	=	1.0 inch
Length	L	=	27.5 inch
Cross-Section Area	A	=	0.125 in^2
Area Moment of Inertia	I	=	0.000163 in^4
Elastic Modulus	E	=	10E+06 lbf/in^2
Stiffness	EI	=	1.042E+05 lbf/in^2
Mass per Volume	ρ_V	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)
Mass per Length	ρ	=	0.0125 lbm / in (0.00003237 lbf sec^2/in^4)
Mass	ρL	=	0.3438 lbm (0.0008906 lbf sec^2/in)
Viscous Damping Ratio	ξ	=	0.05

The normal modes and frequency response function analysis are performed via a Matlab script.

The normal modes results are:

Table B-1. Natural Frequency Results, Beam Simply-Supported at Each End		
Mode	fn (Hz)	Participation Factor
1	14.7	0.02687
2	58.9	0
3	132.5	0.008956
4	235.6	0
5	368.1	0.005373
6	530.1	0

Note that the mode shape and participation factors are considered as dimensionless, but they must be consistent with respect to one another.

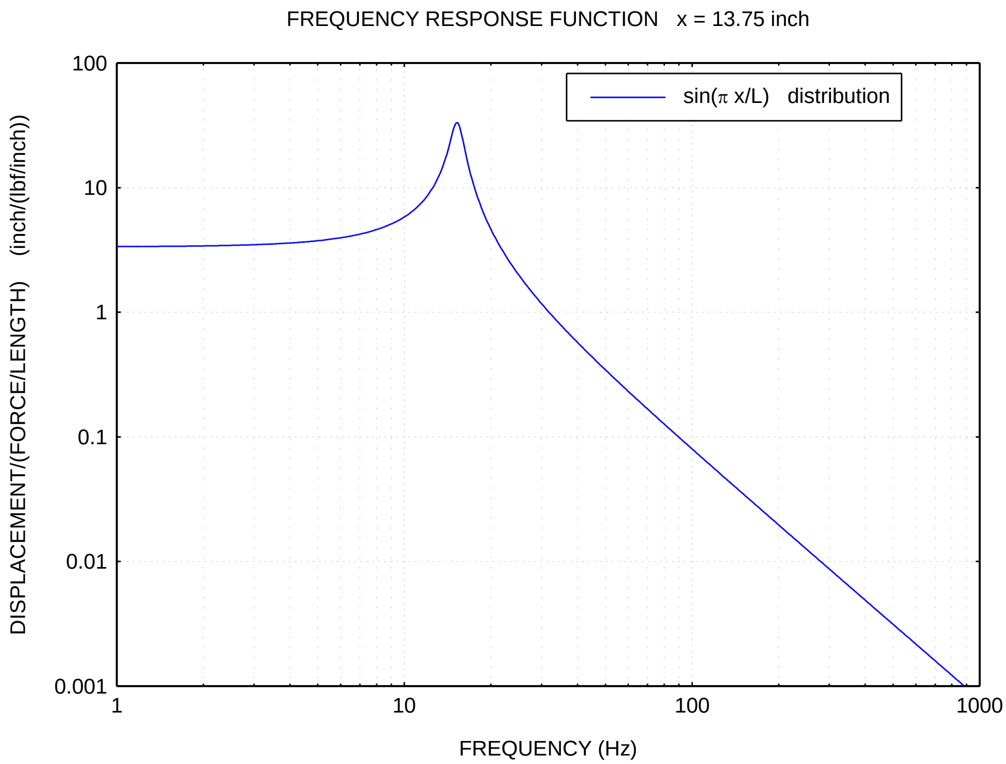

Figure B-1.

Now consider that the pressure field has the spatial distribution:

$$\hat{P}(x) = \sin\left(\pi\, x / L\right) \tag{B-1}$$

The resulting transfer function magnitude is shown in Figure B-1.

The maximum displacement response is: 33.35 [in/(lbf/in)] at 15.32 Hz

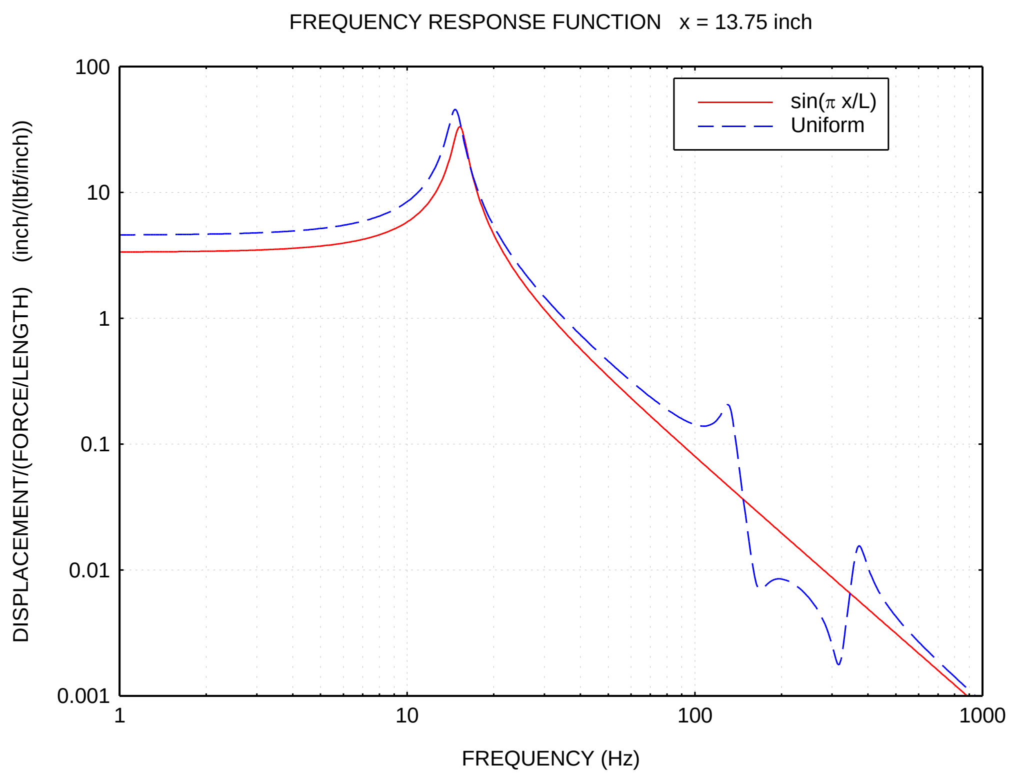

Figure B-2.

The displacement frequency response functions for two pressure distributions are shown in Figure B-2.

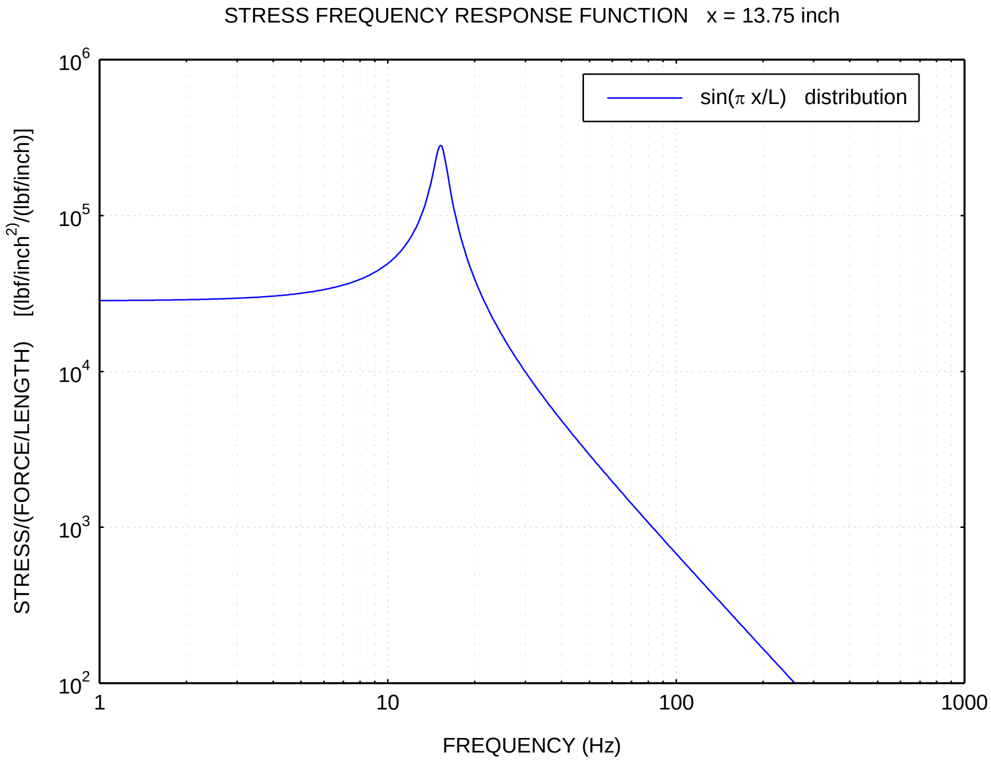

Figure B-3.

The maximum stress response is: 2.822e+05 [(lbf/in^2) /(lbf/in)] at 15.32 Hz