KINETIC & STRAIN ENERGY FORMULAS Revision E

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Uniform Bar under a Constant Axial Load

The total strain energy U for a bar in tension or compression is

$$U = \frac{1}{2} \frac{\sigma^2}{E} LA$$
(1.1)

where

- σ is axial stress
- E is Young's modulus of elasticity
- L is length
- A is cross-sectional area

Equation (1.1) is taken from Reference 1.

The axial stress σ is related to the applied load P by

$$\sigma = \frac{P}{A}$$
(1.2)

Thus,

$$U = \frac{1}{2} \frac{P^2 L}{E A}$$
(1.3)

Derive an alternate formula. The displacement e is

$$e = \frac{PL}{EA}$$
(1.4)

Thus,

$$\mathbf{U} = \frac{1}{2} \mathbf{P} \mathbf{e} \tag{1.5}$$

Bar under a General Axial Load

$$U = \frac{1}{2} \int_{0}^{L} \frac{P^2}{EA} dx$$
 (2.1)

Equation (2.1) is taken from Reference 1.

Another form is

$$U = \frac{1}{2} \int_0^L EA\left(\frac{du}{dx}\right)^2 dx$$
 (2.2)

Note that lower case u is the axial displacement. Equation (2.2) is taken from Reference 3.

Beam in Bending

The strain energy for a beam in bending undergoing a displacement y is

$$U = \frac{1}{2} \int_0^L EI\left(\frac{d^2 y}{dx^2}\right)^2 dx$$
(2.3)

An alternate form is

$$U = \frac{1}{2} \int_{0}^{L} \frac{1}{EI} \left[M(x) \right]^{2} dx$$
(3.2)

$$T = \frac{1}{2} \rho \Omega^2 \int_0^L [y]^2 dx$$
 (3.3)

where

- ρ = mass per length
- Ω = angular natural frequency

Circular Rod in Torsion

The strain energy for a circular rod in torsion is

$$U = \frac{1}{2} \int_{0}^{L} \frac{T^{2}}{GJ} dx$$
 (4.1)

where

- T is the torque
- J is the polar moment of inertia
- G is the shear modulus of elasticity

Beam or Rod in Shear

The shear strain energy is

$$U = \frac{1}{2} \int_{0}^{L} \frac{V^2}{GA} dx$$
 (5.1)

where V is the shear force.

Rectangular Plate in Bending



Let Z represent the out-of-plane displacement. The total strain energy of the plate is

$$U = \frac{D}{2} \int_{0}^{b} \int_{0}^{a} \left[\left(\frac{\partial^{2} Z}{\partial X^{2}} \right)^{2} + \left(\frac{\partial^{2} Z}{\partial Y^{2}} \right)^{2} + 2\mu \left(\frac{\partial^{2} Z}{\partial X^{2}} \right) \left(\frac{\partial^{2} Z}{\partial Y^{2}} \right) + 2\left(1 - \mu \right) \left(\frac{\partial^{2} Z}{\partial X \partial Y} \right)^{2} \right] dXdY$$

$$(6.1)$$

Note that the plate stiffness factor D is given by

$$D = \frac{Eh^3}{12(1-\mu^2)}$$
(6.2)

where

E = elastic modulus

h = plate thickness

 μ = Poisson's ratio

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2} \int_0^b \int_0^a Z^2 \, dX \, dY$$
 (6.3)

where

 Ω = angular natural frequency

An alternate form for thick plates which accounts for rotary inertia is

$$T = \frac{\rho}{2} \int_0^b \int_0^a \left(\frac{h^3}{12} \dot{\alpha}^2 + \frac{h^3}{12} \dot{\beta}^2 + h\dot{z}^2 \right) dx \, dy$$
(6.4)

where

$$\alpha$$
 = angular rotation about the x-axis

 β = angular rotation about the y-axis

Equation (6.4) is taken from Reference 6, equation (6.64).

Rectangular Plate in Bending & Shear, Stress-Strain Formulation

The following is taken from Reference 4.

$$U = \frac{1}{2} \int_{A} \int_{-h/2}^{h/2} \left\{ \begin{bmatrix} \epsilon_{xx} & \epsilon_{yy} & \gamma_{xy} \end{bmatrix} \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} \right\} dz dA$$
$$+ \frac{k}{2} \int_{A} \int_{-h/2}^{h/2} \left\{ \begin{bmatrix} \gamma_{yz} & \gamma_{zx} \end{bmatrix} \begin{bmatrix} \tau_{yz} \\ \tau_{zx} \end{bmatrix} \right\} dz dA$$

where

- h is the plate thickness
- A is the surface area
- k is the shear factor
- z is the axis perpendicular to the plate

(7.1)

Circular Plate in Bending



The strain energy for a circular plate in bending undergoing an out-of-plane displacement Z is

$$U = \frac{D}{2} \int_{0}^{2\pi} \int_{0}^{R} \left[\left(\frac{\partial^{2}Z}{\partial r^{2}} + \frac{1}{r} \frac{\partial Z}{\partial r} + \frac{1}{r^{2}} \frac{\partial^{2}Z}{\partial \theta^{2}} \right)^{2} - 2(1-\mu) \frac{\partial^{2}Z}{\partial^{2}r} \left(\frac{1}{r} \frac{\partial Z}{\partial r} + \frac{1}{r^{2}} \frac{\partial^{2}Z}{\partial \theta^{2}} \right) \right] + 2(1-\mu) \left\{ \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial Z}{\partial \theta} \right) \right\}^{2} r dr d\theta$$

(8.1)

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2} \int_0^{2\pi} \int_0^R Z^2 r \, dr \, d\theta$$
 (8.2)

where

h = thickness

 ρ = mass per volume

 Ω = angular natural frequency

Rectangular Plate, In-plane Displacement



Let u and v be the displacement in the x and y-axes, respectively.

The strain energy is

$$U = \frac{Eh}{2(1-\mu^2)} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + 2\mu \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial v}{\partial y} \right) + \left(\frac{\partial v}{\partial y} \right)^2 + \frac{1}{2} (1-\mu) \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] dx \, dy$$
(9.1)

The kinetic energy is

$$T = \frac{\rho h}{2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 \right] dx \, dy$$
(9.2)

where

- E = elastic modulus
- h = plate thickness
- μ = Poisson's ratio
- ρ = mass per volume

The in-plane equations are taken from Reference 7.

Axial Spring

The strain energy is

$$U = \frac{1}{2}kx^2$$
 (10.1)

where k is the spring stiffness and x is the displacement.

Ring, In-Plane, Extensional Displacement

u	is the radial displacement
Е	is the elastic modulus
А	is the cross-sectional area
R	is the radius

$$U = \frac{\pi AE}{r} u^2$$
(11.1)

Equation (11.1) is taken from Reference 2.

The kinetic energy T is

$$T = \frac{\rho A}{2} \dot{u}^2 \ 2\pi r$$
 (11.2)

References

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