RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A TERMINAL SAWTOOTH APPLIED FORCE

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February 25, 2006

Introduction

Consider the single-degree-of-freedom system in Figure 1.

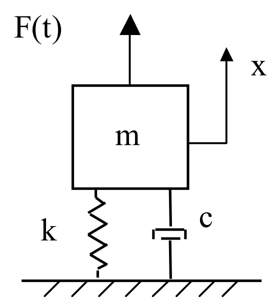

Figure 1.

where

m is the mass
c is the viscous damping coefficient
k is the stiffness
x is the absolute displacement of the mass
F(t) is the applied force

A free-body diagram is shown in Figure 2.

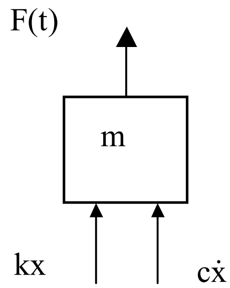

Figure 2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = -c\dot{x} - kx + F(t) \tag{2}$$

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{3}$$

$$\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = \left(\frac{1}{m}\right)F(t) \tag{4}$$

By convention

$$(c/m) = 2\xi\omega_n$$

$$(k/m) = \omega_n^{\;2}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substitute the convention terms into equation (5).

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^{\;2}x = \left(\frac{1}{m}\right)F(t) \tag{5}$$

Terminal Sawtooth Pulse

Consider the pulse given by equation (6).

$$F(t) = \begin{cases} \hat{F}\left(\dfrac{t}{T}\right), & 0 \le t \le T \\ 0, & t > T \end{cases} \tag{6}$$

The equation of motion becomes

$$\ddot{x} + 2\xi\omega_n \dot{x} + \omega_n{}^2 x = \left(\frac{\hat{F}}{m}\right)\left(\frac{t}{T}\right), \qquad 0 \le t \le T \tag{7}$$

Now take the Laplace transform.

$$L\left\{ \ddot{x} + 2\xi\omega_n \dot{x} + \omega_n{}^2 x \right\} = L\left\{ \left(\frac{\hat{F}}{m}\right)\left(\frac{t}{T}\right) \right\} \tag{8}$$

$$\begin{aligned} &s^2 X(s) - s\,x(0) - \dot{x}(0) \\ &+2\xi\omega_n s\, X(s) - 2\xi\omega_n x(0) \\ &+\omega_n{}^2 X(s) = \left(\frac{\hat{F}}{mT}\right)\frac{1}{s^2} \end{aligned} \tag{9}$$

$$\left\{ s^2 + 2\xi\omega_n s + \omega_n{}^2 \right\} X(s) + \{-1\}\dot{x}(0) + \{-s - 2\xi\omega_n\} x(0) = \left(\frac{\hat{F}}{mT}\right)\frac{1}{s^2} \tag{10}$$

$$\left\{ s^2 + 2\xi\omega_n s + \omega_n{}^2 \right\} X(s) = \dot{x}(0) + \{s + 2\xi\omega_n\} x(0) + \left(\frac{\hat{F}}{mT}\right)\frac{1}{s^2} \tag{11}$$

$$X(s)=\left\{\frac{\dot{x}(0)+\{s+2\xi\omega_n\}x(0)}{s^2+2\xi\omega_n s+\omega_n^2}\right\}+\left\{\left(\frac{\hat{F}}{mT}\right)\frac{1}{s^2}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}\right\} \tag{12}$$

Let

$$X(s)=X_n(s)+X_f(s) \tag{13}$$

where

$$X_n(s)=\left\{\frac{\dot{x}(0)+\{s+2\xi\omega_n\}x(0)}{s^2+2\xi\omega_n s+\omega_n^2}\right\} \tag{14}$$

$$X_f(s)=\left\{\left(\frac{\hat{F}}{mT}\right)\frac{1}{s^2}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}\right\} \tag{15}$$

Consider the denominator term,

$$s^2+2\xi\omega_n s+\omega_n^2=\left(s+\xi\omega_n\right)^2+\omega_n^2-\left(\xi\omega_n\right)^2 \tag{16}$$

$$s^2+2\xi\omega_n s+\omega_n^2=\left(s+\xi\omega_n\right)^2+\omega_n^2\left(1-\xi^2\right) \tag{17}$$

Now define the damped natural frequency,

$$\omega_d=\omega_n\sqrt{1-\xi^2} \tag{18}$$

Substitute equation (18) into (17),

$$s^2+2\xi\omega_n s+\omega_n^2=(s+\xi\omega_n)^2+\omega_d^2 \tag{19}$$

Substitute equation (19) into (15).

$$X_n(s) = \left\{ \frac{\dot{x}(0) + \{s + 2\xi\omega_n\}x(0)}{(s+\xi\omega_n)^2 + \omega_d{}^2} \right\} \tag{20}$$

Rearrange the terms into a convenient format prior to the inverse Laplace transform.

$$X_n(s) = \left\{ \frac{(s+\xi\omega_n)x(0)}{(s+\xi\omega_n)^2 + \omega_d{}^2} \right\} + \left\{ \frac{\dot{x}(0) + (\xi\omega_n)x(0)}{(s+\xi\omega_n)^2 + \omega_d{}^2} \right\} \tag{21}$$

$$X_n(s) = \left\{ \frac{(s+\xi\omega_n)x(0)}{(s+\xi\omega_n)^2 + \omega_d{}^2} \right\} + \left\{ \frac{\left\{ \frac{\dot{x}(0) + (\xi\omega_n)x(0)}{\omega_d} \right\}\omega_d}{(s+\xi\omega_n)^2 + \omega_d{}^2} \right\} \tag{22}$$

Take the inverse Laplace transform using Reference 1.

$$x_n(t) = x(0)\exp(-\xi\omega_n t)\cos(\omega_d t) + \left\{ \frac{\dot{x}(0) + (\xi\omega_n)x(0)}{\omega_d} \right\}\exp(-\xi\omega_n t)\sin(\omega_d t) \tag{23}$$

$$x_n(t) = \exp(-\xi\omega_n t)\left\{ x(0)\cos(\omega_d t) + \left\{ \frac{\dot{x}(0) + (\xi\omega_n)x(0)}{\omega_d} \right\}\sin(\omega_d t) \right\} \tag{24}$$

Recall equation (15).

$$X_f(s) = \left\{ \left(\frac{\hat{F}}{mT} \right) \frac{1}{s^2} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2} \right\} \tag{25}$$

Expand into partial fractions using Appendix A.

$$\left\{ \frac{1}{s^2} \right\} \left\{ \frac{1}{s^2 + \alpha s + \beta} \right\} = \left\{ \frac{1}{\beta^2} \right\} \left\{ \frac{-\alpha s + \beta}{s^2} \right\} + \left\{ \frac{1}{\beta^2} \right\} \left\{ \frac{\alpha s + \alpha^2 - \beta}{s^2 + \alpha s + \beta} \right\} \tag{26}$$

$$\alpha = 2\xi\omega_n \tag{27}$$

$$\beta = \omega_n{}^2 \tag{28}$$

$$X_f(s) = \left\{\frac{\hat{F}}{m\,T\,\omega_n{}^4}\right\}\left\{\frac{-2\,\xi\,\omega_n s + \omega_n{}^2}{s^2} + \frac{2\,\xi\,\omega_n s + (2\,\xi\,\omega_n)^2 - \omega_n{}^2}{s^2 + 2\,\xi\,\omega_n s + \omega_n{}^2}\right\} \tag{29}$$

$$X_f(s) = \left\{\frac{\hat{F}}{m\,T\,\omega_n{}^3}\right\}\left\{\frac{-2\,\xi\;s + \omega_n}{s^2} + \frac{2\,\xi\;s + (2\,\xi)^2\,\omega_n - \omega_n}{s^2 + 2\,\xi\,\omega_n s + \omega_n{}^2}\right\} \tag{30}$$

$$X_f(s) = \left\{\frac{\hat{F}}{m\,T\,\omega_n{}^3}\right\}\left\{\frac{-2\,\xi}{s} + \frac{\omega_n}{s^2} + \frac{2\,\xi\;s + (2\,\xi)^2\,\omega_n - \omega_n}{s^2 + 2\,\xi\,\omega_n s + \omega_n{}^2}\right\} \tag{31}$$

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = (s + \xi\omega_n)^2 + \omega_d{}^2 \tag{32}$$

$$X_f(s) = \left\{\frac{\hat{F}}{m\,T\,\omega_n{}^3}\right\}\left\{\frac{-2\,\xi}{s} + \frac{\omega_n}{s^2} + \frac{2\,\xi\;s + (2\,\xi)^2\,\omega_n - \omega_n}{(s + \xi\,\omega_n)^2 + \omega_d{}^2}\right\} \tag{33}$$

$$X_f(s) = \left\{\frac{\hat{F}}{m\,T\,\omega_n{}^3}\right\}\left\{\frac{-2\,\xi}{s} + \frac{\omega_n}{s^2} + \frac{2\,\xi\;s + \omega_n\left[(2\,\xi)^2 - 1\right]}{(s + \xi\,\omega_n)^2 + \omega_d{}^2}\right\} \tag{34}$$

$$X_f(s) = \left\{\frac{\hat{F}}{m T \omega_n^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + [2\xi]\left[\frac{s + \omega_n\left[(2\xi) - \frac{1}{2\xi}\right]}{(s+\xi\omega_n)^2 + \omega_d^2}\right]\right\} \quad (35)$$

$$X_f(s) = \left\{\frac{\hat{F}}{m T \omega_n^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + [2\xi]\left[\frac{s + \frac{\omega_n}{2\xi}\left[(2\xi)^2 - 1\right]}{(s+\xi\omega_n)^2 + \omega_d^2}\right]\right\} \quad (36)$$

$$X_f(s) = \left\{\frac{\hat{F}}{m T \omega_n^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + [2\xi]\left[\frac{s + \frac{\omega_n}{2\xi}\left[(2\xi)^2 - 1\right]}{(s+\xi\omega_n)^2 + \omega_d^2}\right]\right\} \quad (37)$$

$$X_f(s) = \left\{\frac{\hat{F}}{m T \omega_n^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + 2\xi\left[\frac{s + 2\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}\right] + \left[\frac{-\ \omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}\right]\right\}$$

(38)

$$X_f(s) = \left\{\frac{\hat{F}}{m T \omega_n^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + 2\xi\left[\frac{s + 2\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}\right] + \left[\frac{-\ \omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}\right]\right\}$$

(39)

$$X_f(s) = \left\{\frac{\hat{F}}{m T \omega_n^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + 2\xi\left[\frac{s + \xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}\right] + \left[\frac{2\xi^2\omega_n -\ \omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}\right]\right\}$$

(40)

$$X_f(s) = \left\{\frac{\hat{F}}{m T \omega_n^{3}}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + 2\xi\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right] + \omega_n\left[\frac{2\xi^2 - 1}{(s+\xi\omega_n)^2+\omega_d^2}\right]\right\}$$

(41)

$$X_f(s) = \left\{\frac{\hat{F}}{m T \omega_n^{3}}\right\}\left\{\frac{-2\xi}{\omega_n s} + \frac{1}{s^2} + \left[\frac{2\xi}{\omega_n}\right]\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right] + \left[\frac{2\xi^2 - 1}{(s+\xi\omega_n)^2+\omega_d^2}\right]\right\}$$

(42)

$$X_f(s) = \left\{\frac{\hat{F}}{m T \omega_n^{2}}\right\}\left\{\frac{-2\xi}{\omega_n s} + \frac{1}{s^2} + \left[\frac{2\xi}{\omega_n}\right]\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right] + \left[\frac{2\xi^2 - 1}{(s+\xi\omega_n)^2+\omega_d^2}\right]\right\}$$

(43)

Take the inverse Laplace transform using Reference 1. The displacement is

$$x_f(t) = \left\{\frac{\hat{F}}{m T \omega_n^{2}}\right\}\left\{\frac{-2\xi}{\omega_n} + t + \exp(-\xi\omega_n t)\left[\frac{2\xi}{\omega_n}\cos(\omega_d t) + \frac{1}{\omega_d}\left(2\xi^2 - 1\right)\sin(\omega_d t)\right]\right\}$$

(44)

$$x_n(t) = \exp(-\xi\omega_n t)\left\{x(0)\cos(\omega_d t) + \left\{\frac{\dot{x}(0) + (\xi\omega_n)x(0)}{\omega_d}\right\}\sin(\omega_d t)\right\} \quad (45)$$

The total displacement is

$$x(t) = \exp(-\xi\omega_n t)\left\{x(0)\cos(\omega_d t) + \left\{\frac{\dot{x}(0) + (\xi\omega_n)x(0)}{\omega_d}\right\}\sin(\omega_d t)\right\}$$

$$+\left\{\frac{\hat{F}}{m T \omega_n{}^2}\right\}\left\{\frac{-2\xi}{\omega_n} + t + \exp(-\xi\omega_n t)\left[\frac{2\xi}{\omega_n}\cos(\omega_d t) + \frac{1}{\omega_d}\left(2\xi^2 - 1\right)\sin(\omega_d t)\right]\right\},$$

$$0 \le t \le T$$

(46)

The solution for t > T is the free vibration solution.

The total velocity is

$$\dot{x}(t) = -\xi\omega_n \exp(-\xi\omega_n t)\left\{x(0)\cos(\omega_d t) + \left\{\frac{\dot{x}(0) + (\xi\omega_n)x(0)}{\omega_d}\right\}\sin(\omega_d t)\right\}$$

$$+\exp(-\xi\omega_n t)\left\{-x(0)\omega_d \sin(\omega_d t) + \left[\dot{x}(0) + (\xi\omega_n)x(0)\right]\cos(\omega_d t)\right\}$$

$$+\left\{\frac{\hat{F}}{m T \omega_n{}^2}\right\}\left\{1 - \xi\omega_n \exp(-\xi\omega_n t)\left[\frac{2\xi}{\omega_n}\cos(\omega_d t) + \frac{1}{\omega_d}\left(2\xi^2 - 1\right)\sin(\omega_d t)\right]\right\}$$

$$+\left\{\frac{\hat{F}}{m T \omega_n{}^2}\right\}\left\{\exp(-\xi\omega_n t)\right\}\left\{\frac{-2\xi\omega_d}{\omega_n}\sin(\omega_d t) + \left(2\xi^2 - 1\right)\cos(\omega_d t)\right\},$$

$$0 \le t \le T$$

(47)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\frac{\omega_n}{\omega_d}\right][-\xi\,\dot{x}(0) - \omega_n x(0)]\sin(\omega_d t)\right\}$$

$$+\left\{\frac{\hat{F}}{m\,T\,\omega_n{}^2}\right\}\left\{\ 1 + \exp(-\xi\,\omega_n t)\left[-\cos(\omega_d t) + \frac{\xi\,\omega_n}{\omega_d}\left[\left(-2\xi^2\ +\ 1\right) - \frac{2\,\omega_d{}^2}{\omega_n{}^2}\right]\sin(\omega_d t)\right]\right\}$$

$$0 \le t \le T$$

,

(48)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\frac{\omega_n}{\omega_d}\right][-\xi\,\dot{x}(0) - \omega_n x(0)]\sin(\omega_d t)\right\}$$

$$+\left\{\frac{\hat{F}}{m\,T\,\omega_n{}^2}\right\}\left\{\ 1 + \exp(-\xi\,\omega_n t)\left[-\cos(\omega_d t) + \frac{\xi\,\omega_n}{\omega_d}\left[\left(-2\xi^2\ +\ 1\right) - 2\left(1 - \xi^2\right)\right]\sin(\omega_d t)\right]\right\}$$

$$0 \le t \le T$$

,

(49)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\frac{\omega_n}{\omega_d}\right]\left[-\xi\,\dot{x}(0) - \omega_n x(0)\right]\sin(\omega_d t)\right\}$$

$$+\left\{\frac{\hat{F}}{m\,T\,\omega_n{}^2}\right\}\left\{\,1 - \exp(-\xi\,\omega_n t)\left[\,\cos(\omega_d t) + \frac{\xi\,\omega_n}{\omega_d}\sin(\omega_d t)\right]\right\},$$

$$0 \le t \le T \tag{50}$$

Example 1

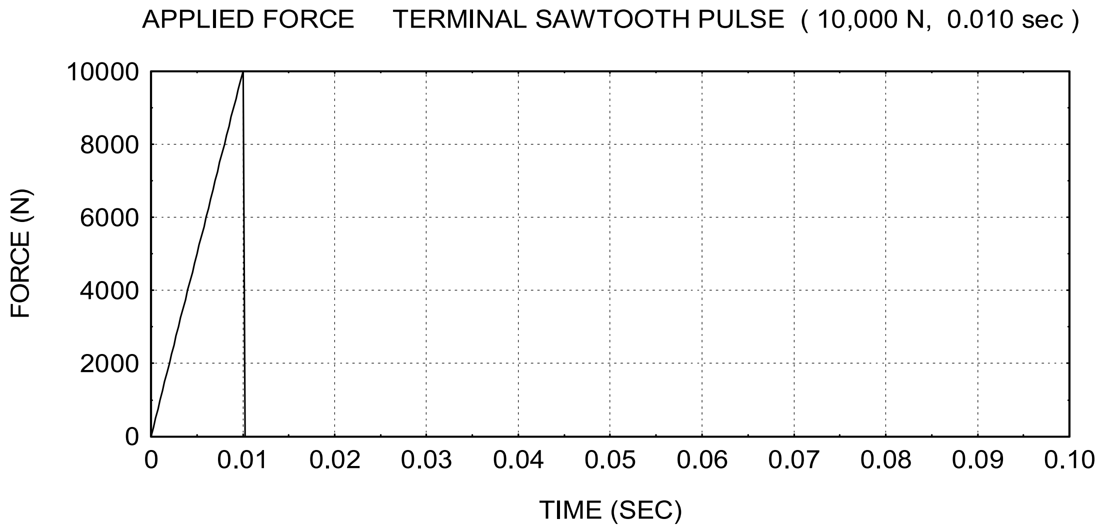

Figure 3.

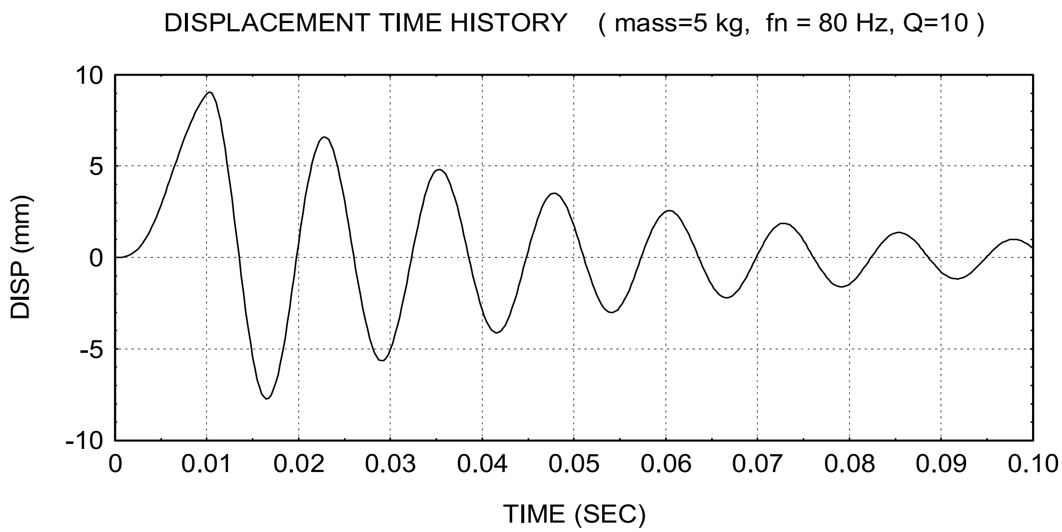

Figure 4.

A single-degree-of-freedom system is subjected to the applied force in Figure 3.

The response is given in Figure 4. The characteristics of the system are given in the plot title.

Example 2

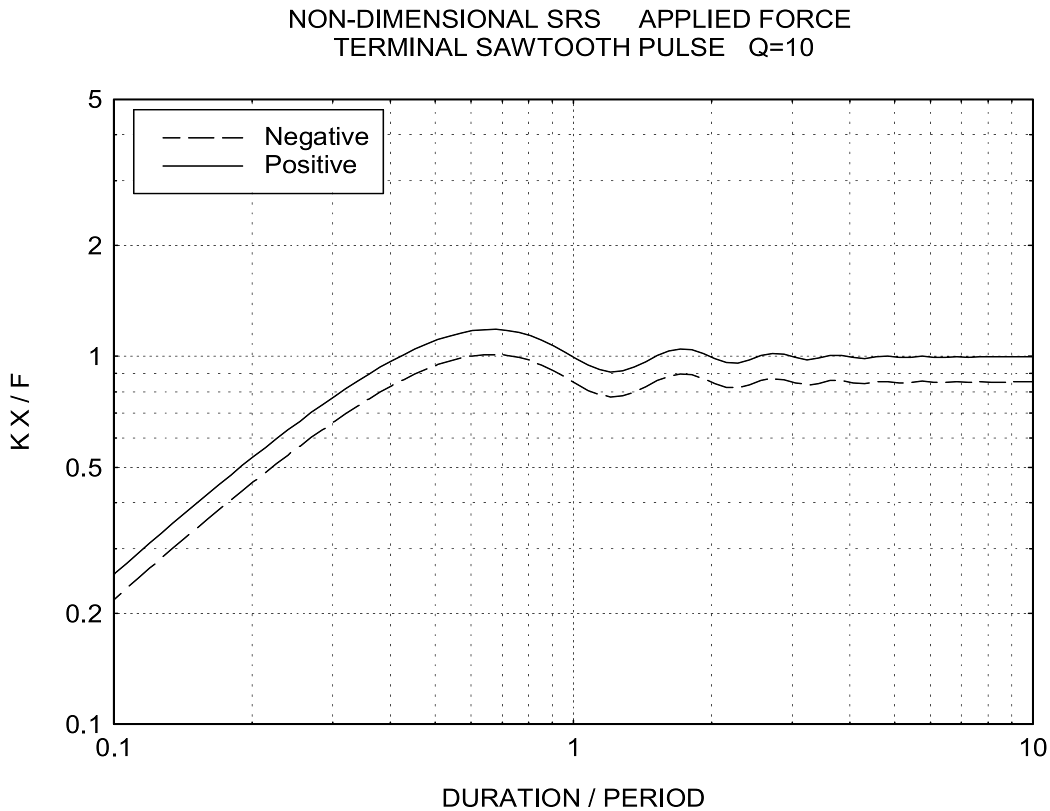

Figure 5.

A non-dimensional displacement SRS is given in Figure 5.

The period is the inverse of the natural frequency.

References

1. T. Irvine, Table of Laplace Transforms, Vibrationdata, 1999.
2. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Vibrationdata, 1999.

APPENDIX A

Partial Fraction Expansion

$$\left\{\frac{1}{s^2}\right\}\left\{\frac{1}{s^2+\alpha s+\beta}\right\} = \frac{as+b}{s^2} + \frac{cs+d}{s^2+\alpha s+\beta} \tag{A-1}$$

$$1 = [as+b]\left[s^2+\alpha s+\beta\right] + [cs+d]\left[s^2\right] \tag{A-2}$$

$$1 = \left[as^3+a\alpha s^2+a\beta s\right] + \left[bs^2+b\alpha s+b\beta\right] + \left[cs^3+ds^2\right] \tag{A-3}$$

$$1 = [a+c]s^3+[a\alpha+b+d]s^2+[a\beta+b\alpha]s+b\beta \tag{A-4}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \alpha & 1 & 0 & 1 \\ \beta & \alpha & 0 & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{A-5}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -\alpha & 1 \\ 0 & \alpha & -\beta & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{A-6}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & \alpha & -\beta & 0 \\ 0 & 1 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{A-7}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -\beta/\alpha & 0 \\ 0 & 1 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\beta \\ 0 \\ 0 \end{bmatrix} \tag{A-8}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\beta/\alpha & 0 \\ 0 & 0 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\beta \\ -1/\beta \\ -1/\beta \end{bmatrix} \tag{A-9}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/\alpha \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\beta \\ \alpha/\beta^2 \\ 1/(\alpha\beta) \end{bmatrix} \tag{A-10}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1/\alpha \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -\alpha/\beta^2 \\ 1/\beta \\ \alpha/\beta^2 \\ -\left[\alpha/\beta^2\right] + \left[1/(\alpha\beta)\right] \end{bmatrix} \tag{A-11}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -\alpha/\beta^2 \\ 1/\beta \\ \alpha/\beta^2 \\ \left[\alpha^2/\beta^2\right] - [1/\beta] \end{bmatrix} \tag{A-12}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -\alpha \\ \beta \\ \alpha \\ \alpha^2 - \beta \end{bmatrix} \left[\frac{1}{\beta^2}\right] \tag{A-13}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -\alpha & 1 \\ 0 & \alpha & -\beta & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{A-14}$$

$$\left\{\frac{1}{s^2}\right\}\left\{\frac{1}{s^2 + \alpha s + \beta}\right\} = \left\{\frac{1}{\beta^2}\right\}\left\{\frac{-\alpha s + \beta}{s^2}\right\} + \left\{\frac{1}{\beta^2}\right\}\left\{\frac{\alpha s + \alpha^2 - \beta}{s^2 + \alpha s + \beta}\right\} \tag{A-15}$$