A Digital Recursive Filtering Method for Calculating the Applied Force for a Measured Response Acceleration

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Introduction

Consider a single-degree-of-freedom (SDOF) system subjected to an applied force, as shown in Figure 1.

There are certain cases where the response of a system is known, but applied force is unknown. Note that response acceleration is usually easier to measure than applied force. The corresponding force can be calculated via a deconvolution process in the form of a digital recursive filtering relationship.

![Figure 1.](image)

The variables are

- m is the mass
- c is the viscous damping coefficient
- k is the stiffness
- x is the absolute displacement of the mass
- f(t) is the applied force
Note that the double-dot denotes acceleration.

The free-body diagram is

![Free-body diagram](image)

Figure 2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x}$$  \hspace{1cm} (1)

$$m\ddot{x} = -c\dot{x} - kx + f(t)$$  \hspace{1cm} (2)

$$m\ddot{x} + c\dot{x} + kx = f(t)$$  \hspace{1cm} (3)

Divide through by $m$,

$$\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = \left(\frac{1}{m}\right)f(t)$$  \hspace{1cm} (4)

By convention,

$$(c/m) = 2\xi\omega_n$$  \hspace{1cm} (5)

$$(k/m) = \omega_n^2$$  \hspace{1cm} (6)

where

$\omega_n$ is the natural frequency in (radians/sec)

$\xi$ is the damping ratio.
By substitution,

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \frac{1}{m} f(t) \]  

(7)

Equation (7) does not have a closed-form solution for the general case in which \( f(t) \) is an arbitrary function. A convolution integral approach must be used to solve the equation.

**Absolute Acceleration**

The acceleration response for a unit impulse from Reference 1 is

\[ \ddot{y}(t) = \frac{1}{m} \left\{ \delta(t) + \exp(-\zeta \omega_n t) \left[ -2\zeta \omega_n \cos(\omega_d t) + \frac{\omega_n^2}{\omega_d} \left( (2\zeta)^2 - 1 \right) \sin(\omega_d t) \right] \right\} \]

(8)

The corresponding Laplace transform for \( H_a(s) \) (acceleration/force) is

\[ H_a(s) = \frac{1}{m} \left[ \frac{s^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \right] \]  

(9)

The corresponding Laplace transform for \( H_i(s) \) (force/acceleration) is

\[ H_i(s) = m \left[ \frac{s^2 + 2\zeta \omega_n s + \omega_n^2}{s^2} \right] \]

(10)

The Z-transform is found using the bilinear transform.

\[ s = \frac{2}{T} \frac{z - 1}{z + 1} \]  

(11)
\( H_i(z) = m \left[ \frac{2z - 1}{T z + 1} \right]^2 + 2 \xi \omega_n \left[ \frac{2z - 1}{T z + 1} \right] z + [\omega_n^2 z + 1]^2 \right] \)  

(12)

\( H_i(z) = m \left[ \frac{2z - 1}{T z + 1} \right]^2 \frac{2 \xi \omega_n [2z - 1] z + 1 + \omega_n^2 [z + 1]^2}{\left[ \frac{2z - 1}{T z + 1} \right]^2} \right] \)  

(13)

\( H_i(z) = m \left[ \frac{4\left[\frac{1}{T} z - 1\right]^2 + 4 \xi \omega_n [\frac{1}{T} z - 1] z + 1 + \omega_n^2 [z + 1]^2}{4\left[\frac{1}{T} z - 1\right]^2} \right] \)  

(14)

\( H_i(z) = m \left[ \frac{4\left[z - 1\right]^2 + 4 \xi \omega_n z - 1 + \omega_n^2 z^2 + 1}{4\left[z - 1\right]^2} \right] \)  

(15)

\( H_i(z) = m \left[ \frac{4\left[z^2 - 2z + 1\right] + 4 \xi \omega_n z^2 - 1 + \omega_n^2 z^2 + 1}{4\left[z^2 - 2z + 1\right]} \right] \)  

(16)

\( H_i(z) = m \left[ \frac{4z^2 - 8z + 4 + 4 \xi \omega_n z^2 - 4 \xi \omega_n z^2 + \omega_n^2 z^2 + 2 \omega_n^2 z^2 + \omega_n^2 z^2}{4z^2 - 8z + 4} \right] \)  

(17)
\[ H_1(z) = m \left[ \frac{4 + 4\zeta\omega_n T + \omega_n^2 T^2}{4z^2 - 8z + 4} \right] \] (18)

\[ H_1(z) = m \left[ \frac{4 + 4\zeta\omega_n T + \omega_n^2 T^2}{4z^2 - 8z + 4} \right] \] (19)

Solve for the filter coefficients using the method in Reference 1.

\[ \frac{c_0 z^2 + c_1 z + c_2}{z^2 + a_1 z + a_2} = m \left[ \frac{4 + 4\zeta\omega_n T + \omega_n^2 T^2}{4z^2 - 8z + 4} \right] \] (20)

\[ \frac{c_0 z^2 + c_1 z + c_2}{z^2 + a_1 z + a_2} = m \left[ \frac{4 + 4\zeta\omega_n T + \omega_n^2 T^2}{z^2 - 2z + 1} \right] \] (21)

Solve for \( a_1 \).

\[ a_1 = -2 \] (22)

Solve for \( a_2 \).

\[ a_2 = 1 \] (23)

Solve for \( c_0 \).

\[ c_0 = \frac{m}{4} \left( 4 + 4\zeta\omega_n T + \omega_n^2 T^2 \right) \] (24)
Solve for $c_1$.

$$c_1 = \frac{m}{2} \left( -4 + \omega_n^2 T^2 \right) \quad (25)$$

Solve for $c_2$.

$$c_2 = \frac{m}{4} \left( 4 - 4\xi\omega_n T + \omega_n^2 T^2 \right) \quad (26)$$

The digital recursive filtering relationship is

$$f_i = -a_1 f_{i-1} - a_2 f_{i-2}$$

$$+ c_0 \ddot{x}_i + c_1 \ddot{x}_{i-1} + c_2 \ddot{x}_{i-2} \quad (27)$$

The digital recursive filtering relationship is

$$f_i = +2f_{i-1} - f_{i-2}$$

$$+ \frac{m}{4} \left( 4 + 4\xi\omega_n T + \omega_n^2 T^2 \right) \dddot{x}_i$$

$$+ \frac{m}{2} \left( -4 + \omega_n^2 T^2 \right) \dddot{x}_{i-1}$$

$$+ \frac{m}{4} \left( 4 - 4\xi\omega_n T + \omega_n^2 \right) \dddot{x}_{i-2} \quad (28)$$

References


APPENDIX A

Example

Figure A-1.

An SDOF system is subjected to an applied force in the form of a wavelet pulse. Both the input and response are shown in Figure A-1.

The response is calculated via Reference 3.
Figure A-2.

The Original and Calculated Applied Force curves are nearly identical.

The calculation was performed via equation (28) given the response in Figure A-1.