A Digital Recursive Filtering Method for Calculating the Applied Force for a Measured Response Acceleration

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Introduction

Consider a single-degree-of-freedom (SDOF) system subjected to an applied force, as shown in Figure 1.

There are certain cases where the response of a system is known, but applied force is unknown. Note that response acceleration is usually easier to measure than applied force. The corresponding force can be calculated via a deconvolution process in the form of a digital recursive filtering relationship.

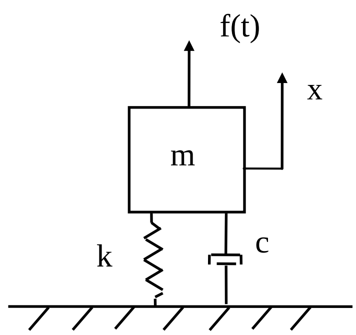

Figure 1.

The variables are

m	is the mass
c	is the viscous damping coefficient
k	is the stiffness
x	is the absolute displacement of the mass
f(t)	is the applied force

Note that the double-dot denotes acceleration.

The free-body diagram is

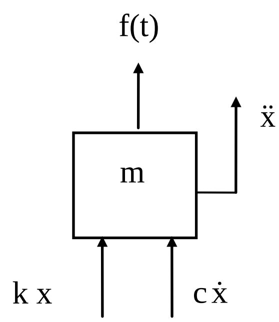

Figure 2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = -c\dot{x} - kx + f(t) \tag{2}$$

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{3}$$

Divide through by m,

$$\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = \left(\frac{1}{m}\right)f(t) \tag{4}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{5}$$

$$(k/m) = \omega_n{}^2 \tag{6}$$

where

ω_n is the natural frequency in (radians/sec)

ξ is the damping ratio.

By substitution,

$$\ddot{x} + 2\,\xi\,\omega_n\,\dot{x} + \omega_n^{\,2}\,x = \frac{1}{m}\,f(t) \tag{7}$$

Equation (7) does not have a closed-form solution for the general case in which $f(t)$ is an arbitrary function. A convolution integral approach must be used to solve the equation.

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<u>Absolute Acceleration</u>

The acceleration response for a unit impulse from Reference 1 is

$$\ddot{y}(t) = \frac{1}{m}\left\{ \delta(t) + \exp\left(-\xi\omega_n t\right)\left[-2\xi\omega_n \cos\left(\omega_d t\right) + \frac{\omega_n^{\,2}}{\omega_d}\left[(2\xi)^2 - 1\right]\sin\left(\omega_d t\right)\right]\right\} \tag{8}$$

The corresponding Laplace transform for $H_a(s)$ (acceleration/force) is

$$H_a(s) = \frac{1}{m}\left[\frac{s^2}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}\right] \tag{9}$$

The corresponding Laplace transform for $H_i(s)$ (force/acceleration) is

$$H_i(s) = m\left[\frac{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}{s^2}\right] \tag{10}$$

The Z-transform is found using the bilinear transform.

$$s = \frac{2}{T}\,\frac{z-1}{z+1} \tag{11}$$

$$H_i(z) = m\left[\frac{\left[\frac{2}{T}\frac{z-1}{z+1}\right]^2 + 2\xi\omega_n\left[\frac{2}{T}\frac{z-1}{z+1}\right] + \omega_n{}^2}{\left[\frac{2}{T}\frac{z-1}{z+1}\right]^2}\right] \tag{12}$$

$$H_i(z) = m\left[\frac{\left[\frac{2}{T}z-1\right]^2 + 2\xi\omega_n\left[\frac{2}{T}z-1\right][z+1] + \omega_n{}^2[z+1]^2}{\left[\frac{2}{T}z-1\right]^2}\right] \tag{13}$$

$$H_i(z) = m\left[\frac{4\left[\frac{1}{T}z-1\right]^2 + 4\xi\omega_n\left[\frac{1}{T}z-1\right][z+1] + \omega_n{}^2[z+1]^2}{4\left[\frac{1}{T}z-1\right]^2}\right] \tag{14}$$

$$H_i(z) = m\left[\frac{4[z-1]^2 + 4\xi\omega_n T[z-1][z+1] + \omega_n{}^2T^2[z+1]^2}{4[z-1]^2}\right] \tag{15}$$

$$H_i(z) = m\left[\frac{4\left(z^2-2z+1\right) + 4\xi\omega_n T\left(z^2-1\right) + \omega_n{}^2T^2\left(z^2+2z+1\right)}{4\left(z^2-2z+1\right)}\right] \tag{16}$$

$$H_i(z) = m\left[\frac{4z^2-8z+4+4\xi\omega_n Tz^2-4\xi\omega_n T+\omega_n{}^2T^2z^2+2\omega_n{}^2T^2z+\omega_n{}^2T^2}{4z^2-8z+4}\right] \tag{17}$$

$$H_i(z) = m\left[\frac{\left(4+4\xi\omega_n T+\omega_n{}^2T^2\right)z^2+\left(-8+2\omega_n{}^2T^2\right)z+4-4\xi\omega_n T+\omega_n{}^2T^2}{4z^2-8z+4}\right] \tag{18}$$

$$H_i(z) = m\left[\frac{\left(4+4\xi\omega_n T+\omega_n{}^2T^2\right)z^2+2\left(-4+\omega_n{}^2T^2\right)z+4-4\xi\omega_n T+\omega_n{}^2T^2}{4z^2-8z+4}\right] \tag{19}$$

Solve for the filter coefficients using the method in Reference 1.

$$\frac{c_0 z^2+c_1 z+c_2}{z^2+a_1 z+a_2} = m\left[\frac{\left(4+4\xi\omega_n T+\omega_n{}^2T^2\right)z^2+2\left(-4+\omega_n{}^2T^2\right)z+4-4\xi\omega_n T+\omega_n{}^2T^2}{4z^2-8z+4}\right] \tag{20}$$

$$\frac{c_0 z^2+c_1 z+c_2}{z^2+a_1 z+a_2} = \frac{m}{4}\left[\frac{\left(4+4\xi\omega_n T+\omega_n{}^2T^2\right)z^2+2\left(-4+\omega_n{}^2T^2\right)z+4-4\xi\omega_n T+\omega_n{}^2T^2}{z^2-2z+1}\right] \tag{21}$$

Solve for a_1.

$$a_1 = -2 \tag{22}$$

Solve for a_2.

$$a_2 = 1 \tag{23}$$

Solve for c_0.

$$c_0 = \frac{m}{4}\left(4+4\xi\omega_n T+\omega_n{}^2T^2\right) \tag{24}$$

Solve for c_1.

$$c_1 = \frac{m}{2}\left(-4 + \omega_n{}^2 T^2\right) \tag{25}$$

Solve for c_2.

$$c_2 = \frac{m}{4}\left(4 - 4\xi\omega_n T + \omega_n{}^2 T^2\right) \tag{26}$$

The digital recursive filtering relationship is

$$f_i = -a_1 f_{i-1} - a_2 f_{i-2}$$

$$+c_0 \ddot{x}_i + c_1 \ddot{x}_{i-1} + c_2 \ddot{x}_{i-2} \tag{27}$$

The digital recursive filtering relationship is

$$f_i = +2 f_{i-1} - f_{i-2}$$

$$+\frac{m}{4}\left(4 + 4\xi\omega_n T + \omega_n{}^2 T^2\right)\ddot{x}_i + \frac{m}{2}\left(-4 + \omega_n{}^2 T^2\right)\ddot{x}_{i-1} + \frac{m}{4}\left(4 - 4\xi\omega_n T + \omega_n{}^2\right)\ddot{x}_{i-2} \tag{28}$$

References

1. T. Irvine, The Time-Domain Response of a Single-degree-of-freedom System Subjected to an Impulse Force, Revision B, Vibrationdata, 2012.

2. T. Irvine, An Introduction to the Shock Response Spectrum, Revision R, Vibrationdata, 2010.

3. T. Irvine, Modal Transient Analysis of a System Subjected to an Applied Force via a Ramp Invariant Digital Recursive Filtering Relationship, Revision J, Vibrationdata, 2012

APPENDIX A

Example

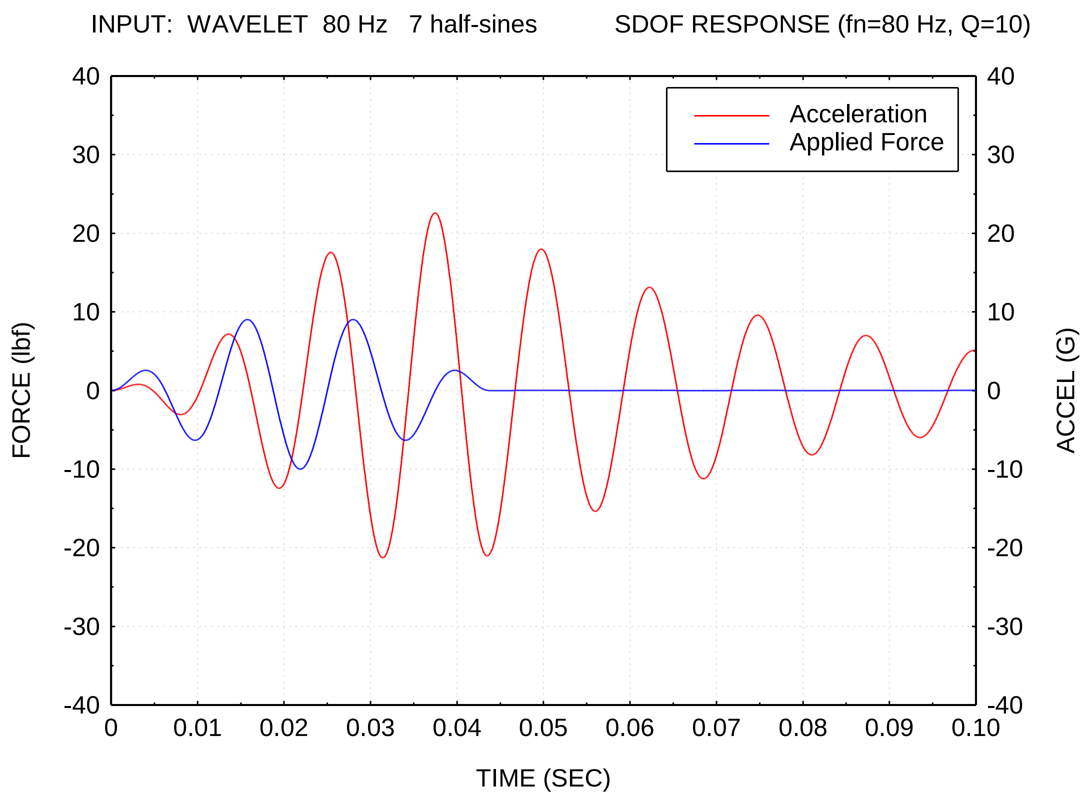

Figure A-1.

An SDOF system is subjected to an applied force in the form of a wavelet pulse. Both the input and response are shown in Figure A-1.

The response is calculated via Reference 3.

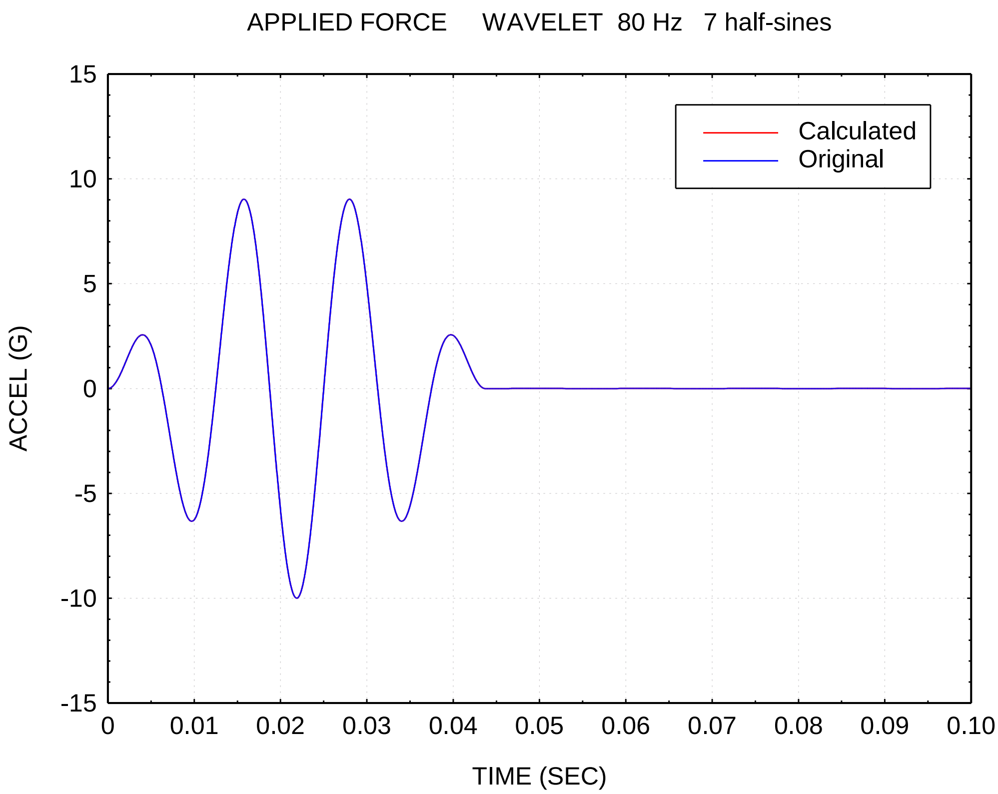

Figure A-2.

The Original and Calculated Applied Force curves are nearly identical.

The calculation was performed via equation (28) given the response in Figure A-1.