Introduction

Consider a single-degree-of-freedom (SDOF) system subjected to base excitation, as shown in Figure 1.

There are certain cases where the response of a system is known, but the base input acceleration is unknown. An example would be a seismic sensor which behaved as an SDOF system. The seismometer data would give the acceleration of the mass. A calculation would then be needed to determine the base input which drove the mass to the measured response. This calculation process is a form of deconvolution.

where

\[
\begin{align*}
\text{m} & = \text{Mass} \\
\text{c} & = \text{viscous damping coefficient} \\
\text{k} & = \text{Stiffness} \\
\text{x} & = \text{absolute displacement of the mass} \\
\text{y} & = \text{base input displacement}
\end{align*}
\]
A free-body diagram is shown in Figure 2.

![Free Body Diagram](image)

**Figure 2.**

Summation of forces in the vertical direction

\[
\sum F = \sum m \ddot{x}
\]

(1)

\[
m \ddot{x} = c (\dot{y} - \dot{x}) + k (y - x)
\]

(2)

Let

\[
\begin{align*}
    u &= x - y \\
    \dot{u} &= \dot{x} - \dot{y} \\
    \ddot{u} &= \ddot{x} - \ddot{y} \\
    \dddot{x} &= \dddot{u} + \dddot{y}
\end{align*}
\]

Substituting the relative displacement terms into equation (2) yields

\[
m(\dddot{u} + \dddot{y}) = -cu - ku
\]

(3)

\[
m \dddot{u} + c \dddot{u} + ku = -m \dddot{y}
\]

(4)

Dividing through by mass yields

\[
\dddot{u} + (c/m)\dddot{u} + (k/m)u = -\dddot{y}
\]

(5)
By convention,

\[
(c / m) = 2\xi \omega_n
\]  

(6)

\[
(k / m) = \omega_n^2
\]  

(7)

where \( \omega_n \) is the natural frequency in (radians/sec), and \( \xi \) is the damping ratio.

Substitute the convention terms into equation (5).

\[
\ddot{u} + 2\xi \omega_n \dot{u} + \omega_n^2 u = -\ddot{y}
\]  

(8)

Equation (8) does not have a closed-form solution for the general case in which \( \ddot{y} \) is an arbitrary function. A convolution integral approach must be used to solve the equation. Note that the impulse response function is embedded in the convolution integral.

**Absolute Acceleration**

The impulse response function for the acceleration response from Reference 2 is

\[
\hat{h}_a(t) = \exp(-\xi \omega_n t) \left[ 2\xi \omega_n \cos(\omega_d t) + \frac{\omega_n^2}{\omega_d} \left(1 - 2\xi^2\right) \sin(\omega_d t) \right]
\]  

(9)

The corresponding Laplace transform for \( H_a(s) \) (response/input) is

\[
H_a(s) = \frac{2\xi \omega_n s + \omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2}
\]  

(10)

The corresponding Laplace transform for \( H_i(s) \) (input/response) is

\[
H_i(s) = \frac{s^2 + 2\xi \omega_n s + \omega_n^2}{2\xi \omega_n s + \omega_n^2}
\]  

(11)
The Z-transform is found using the bilinear transform.

\[ s = \frac{2}{T} \frac{z - 1}{z + 1} \]  

(12)

\[ H_i(z) = \frac{\left[ \frac{2}{T}(z-1) \right]^2 + 2\xi \omega_n \left[ \frac{2}{T}(z-1)(z+1) \right] + \omega_n^2 (z+1)^2}{2\xi \omega_n \left[ \frac{2}{T}(z-1)(z+1) \right] + \omega_n^2 (z+1)^2} \]  

(13)

\[ H_i(z) = \frac{\left[ \frac{2}{T}(z-1) \right]^2 + 2\xi \omega_n \left[ \frac{2}{T}(z-1)(z+1) \right] + \omega_n^2 (z+1)^2}{2\xi \omega_n \left[ \frac{2}{T}(z-1)(z+1) \right] + \omega_n^2 (z+1)^2} \]  

(14)

\[ H_i(z) = \frac{2(z-1)^2 + 4\xi \omega_n T[(z-1)(z+1)] + T^2 \omega_n^2 (z+1)^2}{4\xi T \omega_n [(z-1)(z+1)] + T^2 \omega_n^2 (z+1)^2} \]  

(15)

\[ H_i(z) = \frac{4(z^2 - 2z + 1) + 4\xi \omega_n T(z^2 - 1) + T^2 \omega_n^2 (z^2 + 2z + 1)}{4\xi T \omega_n (z^2 - 1) + T^2 \omega_n^2 (z^2 + 2z + 1)} \]  

(16)

\[ H_i(z) = \frac{4z^2 - 8z + 4 + 4\xi \omega_n Tz^2 - 4\xi \omega_n T + T^2 \omega_n^2 z^2 + 2T^2 \omega_n^2 z + T^2 \omega_n^2}{4\xi T \omega_n z^2 - 4\xi T \omega_n + T^2 \omega_n^2 z^2 + 2T^2 \omega_n^2 z + T^2 \omega_n^2} \]  

(17)
\[ H_i(z) = \frac{(4 + 4\xi \omega_n T + T^2 \omega_n^2) z^2 + (-8 + 2T^2 \omega_n^2) z + \left(4 - 4\xi \omega_n T + T^2 \omega_n^2 \right)}{(4\xi T \omega_n + T^2 \omega_n^2) z^2 + (2T^2 \omega_n^2) z - 4\xi T \omega_n + T^2 \omega_n^2} \]  

(18)

\[ H_i(z) = \frac{1}{T \omega_n} \frac{(4 + 4\xi \omega_n T + T^2 \omega_n^2) z^2 + 2(-4 + T^2 \omega_n^2) z + \left(4 - 4\xi \omega_n T + T^2 \omega_n^2 \right)}{(4\xi + T \omega_n) z^2 + (2T \omega_n) z - 4\xi + T \omega_n} \]  

(19)

\[ H_i(z) = \frac{1}{T \omega_n (4\xi + T \omega_n)} \frac{(4 + 4\xi \omega_n T + T^2 \omega_n^2) z^2 + 2(-4 + T^2 \omega_n^2) z + \left(4 - 4\xi \omega_n T + T^2 \omega_n^2 \right)}{z^2 + \left(\frac{2T \omega_n}{4\xi + T \omega_n}\right) z + \left(\frac{-4\xi + T \omega_n}{4\xi + T \omega_n}\right)} \]  

(20)

Solve for the filter coefficients using the method in Reference 1.

\[ \frac{c_0 z^2 + c_1 z + c_2}{z^2 + a_1 z + a_2} = \]

\[ \frac{1}{T \omega_n (4\xi + T \omega_n)} \frac{(4 + 4\xi \omega_n T + T^2 \omega_n^2) z^2 + 2(-4 + T^2 \omega_n^2) z + \left(4 - 4\xi \omega_n T + T^2 \omega_n^2 \right)}{z^2 + \left(\frac{2T \omega_n}{4\xi + T \omega_n}\right) z + \left(\frac{-4\xi + T \omega_n}{4\xi + T \omega_n}\right)} \]  

(21)
Solve for $a_1$.

$$a_1 = \left( \frac{2T\omega_n}{4\xi + T\omega_n} \right)$$  \hspace{1cm} (22)

Solve for $a_2$.

$$a_2 = \left( \frac{-4\xi + T\omega_n}{4\xi + T\omega_n} \right)$$  \hspace{1cm} (23)

Solve for $c_0$.

$$c_0 = \frac{(4 + 4\xi\omega_n T + T^2\omega_n^2)}{T\omega_n (4\xi + T\omega_n)}$$  \hspace{1cm} (24)

Solve for $c_1$.

$$c_1 = \frac{2(-4 + T^2\omega_n^2)}{T\omega_n (4\xi + T\omega_n)}$$  \hspace{1cm} (25)

Solve for $c_2$.

$$c_2 = \frac{(4 - 4\xi\omega_n T + T^2\omega_n^2)}{T\omega_n (4\xi + T\omega_n)}$$  \hspace{1cm} (26)
The digital recursive filtering relationship is

\[ \ddot{y}_i = -a_1 \ddot{y}_{i-1} - a_2 \ddot{y}_{i-2} \]
\[ + c_0 \ddot{x}_i + c_1 \ddot{x}_{i-1} + c_2 \ddot{x}_{i-2} \]

(27)

The digital recursive filtering relationship is

\[ \ddot{y}_i = - \left( \frac{2T \omega_n}{4 \xi + T \omega_n} \right) \ddot{y}_{i-1} - \left( \frac{-4 \xi + T \omega_n}{4 \xi + T \omega_n} \right) \ddot{y}_{i-2} \]
\[ + \left( \frac{4 + 4 \xi \omega_n T + T^2 \omega_n^2}{T \omega_n (4 \xi + T \omega_n)} \right) \ddot{x}_i + \left( \frac{2(-4 + T^2 \omega_n^2)}{T \omega_n (4 \xi + T \omega_n)} \right) \ddot{x}_{i-1} + \left( \frac{4 - 4 \xi \omega_n T + T^2 \omega_n^2}{T \omega_n (4 \xi + T \omega_n)} \right) \ddot{x}_{i-2} \]

(28)

References


Example

An SDOF system is subjected to a wavelet pulse. Both the input and response are shown in Figure A-1.

The response is calculated via Reference 3.
The Original and Calculated Base Input curves are nearly identical.

The calculation was performed via equation (28) given the response in Figure A-1.