A Digital Recursive Filtering Method for Calculating the Base Input
for a Measured Response Acceleration

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March 28, 2012

Introduction

Consider a single-degree-of-freedom (SDOF) system subjected to base excitation, as shown in Figure 1.

There are certain cases where the response of a system is known, but the base input acceleration is unknown. An example would be a seismic sensor which behaved as an SDOF system. The seismometer data would give the acceleration of the mass. A calculation would then be needed to determine the base input which drove the mass to the measured response. This calculation process is a form of deconvolution.

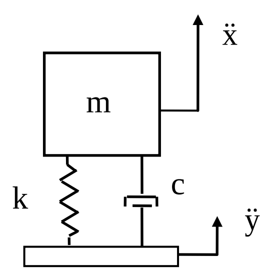

Figure 1.

where

m	=	Mass
c	=	viscous damping coefficient
k	=	Stiffness
x	=	absolute displacement of the mass
y	=	base input displacement

A free-body diagram is shown in Figure 2.

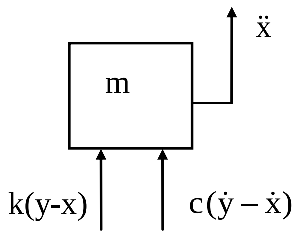

Figure 2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = c(\dot{y}-\dot{x}) + k(y-x) \tag{2}$$

Let

$$u = x - y$$
$$\dot{u} = \dot{x} - \dot{y}$$
$$\ddot{u} = \ddot{x} - \ddot{y}$$
$$\ddot{x} = \ddot{u} + \ddot{y}$$

Substituting the relative displacement terms into equation (2) yields

$$m(\ddot{u}+\ddot{y}) = -c\dot{u} - ku \tag{3}$$

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{y} \tag{4}$$

Dividing through by mass yields

$$\ddot{u} + (c/m)\dot{u} + (k/m)u = -\ddot{y} \tag{5}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{6}$$

$$(k/m) = \omega_n{}^2 \tag{7}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substitute the convention terms into equation (5).

$$\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n{}^2 u = -\ddot{y} \tag{8}$$

Equation (8) does not have a closed-form solution for the general case in which $\ddot{y}$ is an arbitrary function. A convolution integral approach must be used to solve the equation. Note that the impulse response function is embedded in the convolution integral.

Absolute Acceleration

The impulse response function for the acceleration response from Reference 2 is

$$\hat{h}_a(t) = \exp(-\xi\omega_n t)\left[2\xi\omega_n \cos(\omega_d t) + \frac{\omega_n{}^2}{\omega_d}\left(1-2\xi^2\right)\sin(\omega_d t)\right] \tag{9}$$

The corresponding Laplace transform for $H_a(s)$ (response/input) is

$$H_a(s) = \left[\frac{2\xi\omega_n s + \omega_n{}^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right] \tag{10}$$

The corresponding Laplace transform for $H_i(s)$ (input/response) is

$$H_i(s) = \frac{s^2 + 2\xi\omega_n s + \omega_n{}^2}{2\xi\omega_n s + \omega_n{}^2} \tag{11}$$

The Z-transform is found using the bilinear transform.

$$s = \frac{2}{T}\frac{z-1}{z+1} \tag{12}$$

$$H_i(z) = \frac{\left[\frac{2}{T}\frac{(z-1)}{(z+1)}\right]^2 + 2\xi\omega_n\left[\frac{2}{T}\frac{(z-1)}{(z+1)}\right] + \omega_n{}^2}{2\xi\omega_n\left[\frac{2}{T}\frac{(z-1)}{(z+1)}\right] + \omega_n{}^2} \tag{13}$$

$$H_i(z) = \frac{\left[\frac{2}{T}(z-1)\right]^2 + 2\xi\omega_n\left[\frac{2}{T}(z-1)(z+1)\right] + \omega_n{}^2(z+1)^2}{2\xi\omega_n\left[\frac{2}{T}(z-1)(z+1)\right] + \omega_n{}^2(z+1)^2} \tag{14}$$

$$H_i(z) = \frac{[2(z-1)]^2 + 4\xi\omega_n T[(z-1)(z+1)] + T^2\omega_n{}^2(z+1)^2}{4\xi T\omega_n[(z-1)(z+1)] + T^2\omega_n{}^2(z+1)^2} \tag{15}$$

$$H_i(z) = \frac{4\left(z^2-2z+1\right) + 4\xi\omega_n T\left(z^2-1\right) + T^2\omega_n{}^2\left(z^2+2z+1\right)}{4\xi T\omega_n\left(z^2-1\right) + T^2\omega_n{}^2\left(z^2+2z+1\right)} \tag{16}$$

$$H_i(z) = \frac{4z^2 - 8z + 4 + 4\xi\omega_n Tz^2 - 4\xi\omega_n T + T^2\omega_n{}^2 z^2 + 2T^2\omega_n{}^2 z + T^2\omega_n{}^2}{4\xi T\omega_n z^2 - 4\xi T\omega_n + T^2\omega_n{}^2 z^2 + 2T^2\omega_n{}^2 z + T^2\omega_n{}^2} \tag{17}$$

$$H_i(z) = \frac{(4+4\xi\omega_n T + T^2\omega_n{}^2)z^2 + (-8+2T^2\omega_n{}^2)z + \left(4-4\xi\omega_n T + T^2\omega_n{}^2\right)}{(4\xi T\omega_n + T^2\omega_n{}^2)z^2 + (2T^2\omega_n{}^2)z - 4\xi T\omega_n + T^2\omega_n{}^2} \qquad (18)$$

$$H_i(z) = \frac{1}{T\omega_n} \frac{(4+4\xi\omega_n T + T^2\omega_n{}^2)z^2 + 2(-4+T^2\omega_n{}^2)z + \left(4-4\xi\omega_n T + T^2\omega_n{}^2\right)}{(4\xi + T\omega_n)z^2 + (2T\omega_n)z - 4\xi + T\omega_n}$$

(19)

$$H_i(z) =$$

$$\frac{1}{T\omega_n(4\xi + T\omega_n)} \frac{(4+4\xi\omega_n T + T^2\omega_n{}^2)z^2 + 2(-4+T^2\omega_n{}^2)z + \left(4-4\xi\omega_n T + T^2\omega_n{}^2\right)}{z^2 + \left(\frac{2T\omega_n}{4\xi + T\omega_n}\right)z + \left(\frac{-4\xi + T\omega_n}{4\xi + T\omega_n}\right)}$$

(20)

Solve for the filter coefficients using the method in Reference 1.

$$\frac{c_0 z^2 + c_1 z + c_2}{z^2 + a_1 z + a_2} =$$

$$\frac{1}{T\omega_n(4\xi + T\omega_n)} \frac{(4+4\xi\omega_n T + T^2\omega_n{}^2)z^2 + 2(-4+T^2\omega_n{}^2)z + \left(4-4\xi\omega_n T + T^2\omega_n{}^2\right)}{z^2 + \left(\frac{2T\omega_n}{4\xi + T\omega_n}\right)z + \left(\frac{-4\xi + T\omega_n}{4\xi + T\omega_n}\right)}$$

(21)

Solve for a_1.

$$a_1 = \left(\frac{2T\omega_n}{4\xi + T\omega_n} \right) \tag{22}$$

Solve for a_2.

$$a_2 = \left(\frac{-4\xi + T\omega_n}{4\xi + T\omega_n} \right) \tag{23}$$

Solve for c_0.

$$c_0 = \frac{(4 + 4\xi\omega_n T + T^2\omega_n{}^2)}{T\omega_n(4\xi + T\omega_n)} \tag{24}$$

Solve for c_1.

$$c_1 = \frac{2(-4 + T^2\omega_n{}^2)}{T\omega_n(4\xi + T\omega_n)} \tag{25}$$

Solve for c_2.

$$c_2 = \frac{\left(4 - 4\xi\omega_n T + T^2\omega_n{}^2\right)}{T\omega_n(4\xi + T\omega_n)} \tag{26}$$

The digital recursive filtering relationship is

$$\ddot{y}_i = -a_1 \ddot{y}_{i-1} - a_2 \ddot{y}_{i-2}$$

$$+c_0 \ddot{x}_i + c_1 \ddot{x}_{i-1} + c_2 \ddot{x}_{i-2} \tag{27}$$

The digital recursive filtering relationship is

$$\ddot{y}_i = -\left(\frac{2T\omega_n}{4\xi + T\omega_n}\right)\ddot{y}_{i-1} - \left(\frac{-4\xi + T\omega_n}{4\xi + T\omega_n}\right)\ddot{y}_{i-2}$$

$$+\left(\frac{4 + 4\xi\omega_n T + T^2\omega_n{}^2}{T\omega_n(4\xi + T\omega_n)}\right)\ddot{x}_i + \left(\frac{2(-4 + T^2\omega_n{}^2)}{T\omega_n(4\xi + T\omega_n)}\right)\ddot{x}_{i-1} + \left(\frac{4 - 4\xi\omega_n T + T^2\omega_n{}^2}{T\omega_n(4\xi + T\omega_n)}\right)\ddot{x}_{i-2} \tag{28}$$

References

1. David O. Smallwood, An Improved Recursive Formula for Calculating Shock Response Spectra, Shock and Vibration Bulletin, No. 51, May 1981.

2. T. Irvine, The Impulse Response Function for Base Excitation, Vibrationdata, 2012.

3. T. Irvine, The Response of a Single-degree-of-freedom System Subjected to a Wavelet Pulse Base Excitation, Vibrationdata, 2008.

Example

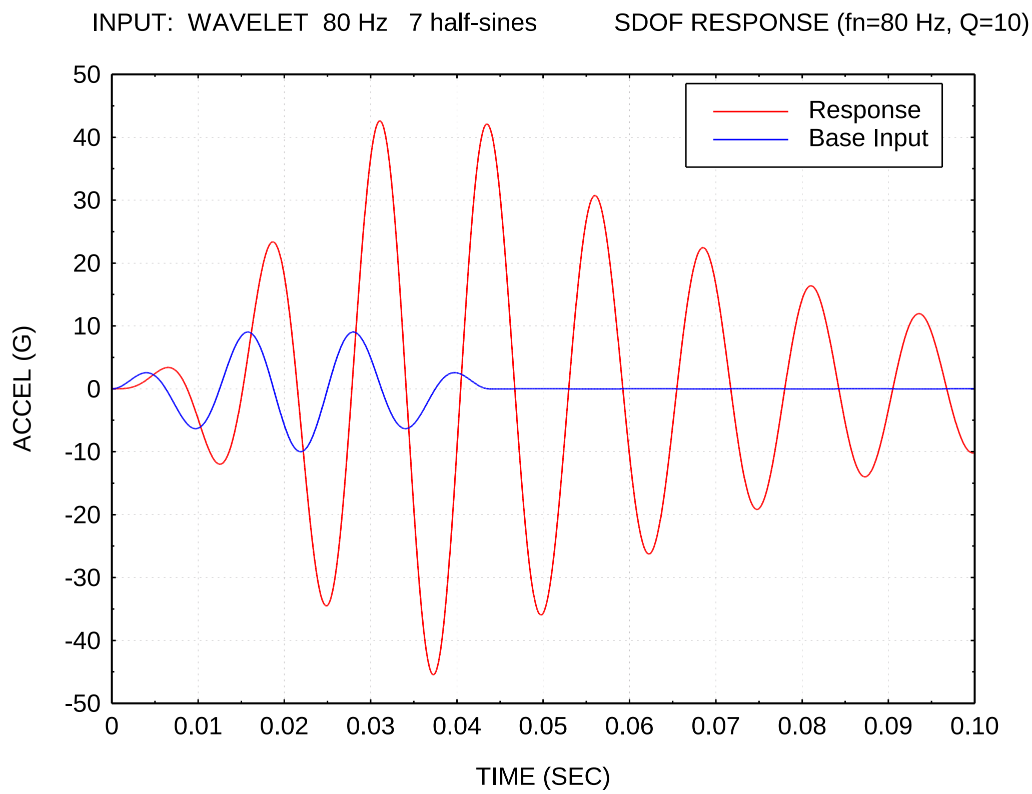

Figure A-1.

An SDOF system is subjected to a wavelet pulse. Both the input and response are shown in Figure A-1.

The response is calculated via Reference 3.

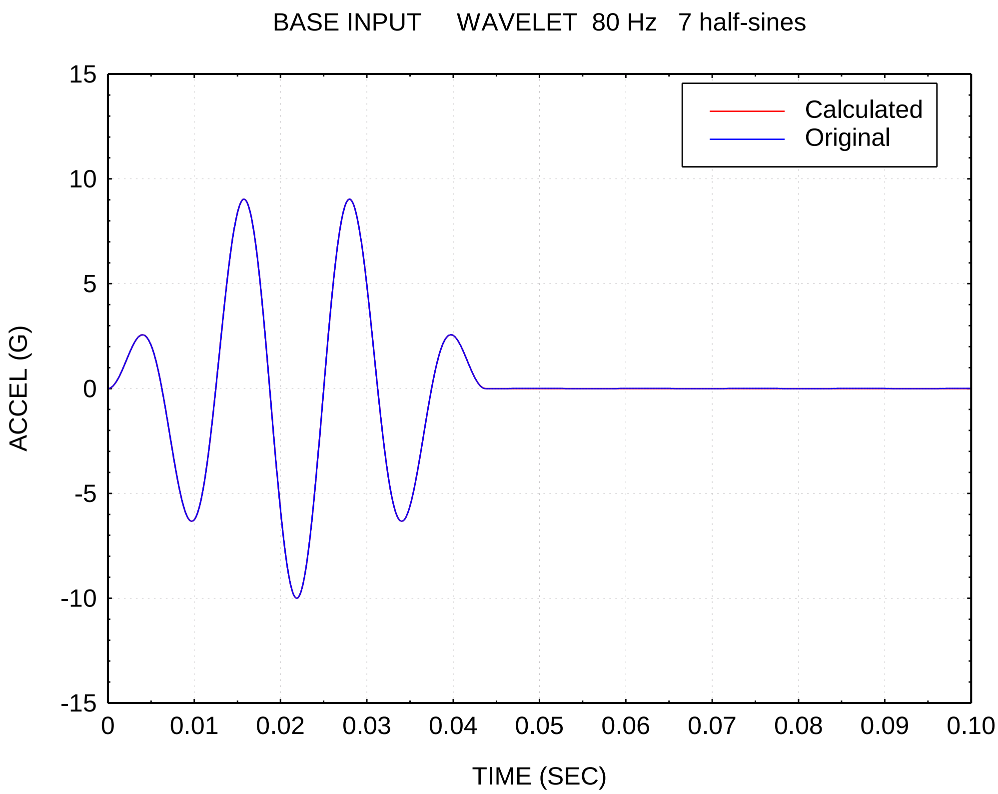

Figure A-2.

The Original and Calculated Base Input curves are nearly identical.

The calculation was performed via equation (28) given the response in Figure A-1.