PLATE BENDING FREQUENCIES VIA THE FINITE ELEMENT METHOD
WITH RECTANGULAR ELEMENTS
Revision A

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Rectangular Plate in Bending

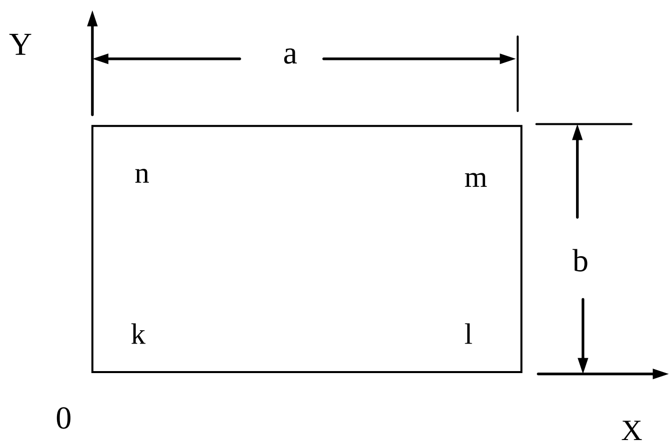

Figure 1.

Stiffness Matrix

Let u represent the out-of-plane displacement. The total strain energy of the plate is

$$U = \frac{D}{2}\int_0^b\int_0^a \left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\mu\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right) + 2(1-\mu)\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2 \right] dxdy \tag{1}$$

Note that the plate stiffness factor D is given by

$$D = \frac{Eh^3}{12(1-\mu^2)} \tag{2}$$

where

E = elastic modulus

h = plate thickness

μ = Poisson's ratio

The plate has one translational and two rotational degrees-of-freedom at each corner, for a total of 12 degrees-of-freedom. The two rotational components are about the x and y-axes respectively.

Let the four corners be labeled *k, l, m,* and *n,* as shown in Figure 1.

The rotation components θ at the corner *1* are

$$\theta_{xl} = \frac{\partial u_l}{\partial x} \tag{3}$$

$$\theta_{yl} = \frac{\partial u_l}{\partial y} \tag{4}$$

The displacement equation has 12 constants to enforce 12 continuity conditions.

$$\begin{aligned} u(x,y,t) = {} & a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 \\ & + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^3 y + a_{12} xy^3 \end{aligned} \tag{5}$$

The displacement equation can be written as

$$u(x,y,t) = \{A\}^T \{Z\} \tag{6}$$

where

$$\{A\}^T = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \tag{7}$$

$$\{Z\}^T = \lfloor 1 \quad x \quad y \quad x^2 \quad xy \quad y^2 \quad x^3 \quad x^2y \quad xy^2 \quad y^3 \quad x^3y \quad xy^3 \rfloor \tag{8}$$

The strain energy is thus

$$U = \frac{D}{2}\{A\}^T \int_0^b\int_0^a \left[S(x,y)\right] dxdy \{A\} \tag{9}$$

where

$$S(x,y) = \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\mu\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right) + 2(1-\mu)\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 \tag{10}$$

Again,

$$\begin{aligned} u(x,y,t) = {} & a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 \\ & + a_8x^2y + a_9xy^2 + a_{10}y^3 + a_{11}x^3y + a_{12}xy^3 \end{aligned} \tag{11}$$

$$\frac{\partial}{\partial x}u(x,y,t) = a_2 + 2a_4x + a_5y + 3a_7x^2 + 2a_8xy + a_9y^2 + 3a_{11}x^2y + a_{12}y^3 \tag{12}$$

$$\frac{\partial}{\partial y}u(x,y,t) = a_3 + a_5x + 2a_6y + a_8x^2 + 2a_9xy + 3a_{10}y^2 + a_{11}x^3 + 3a_{12}xy^2 \tag{13}$$

The displacement and derivatives at each corner are

$$u\{0,0,t\} = a_1 \tag{14}$$

$$\frac{\partial u}{\partial x}\{0,0,t\} = a_2 \tag{15}$$

$$\frac{\partial u}{\partial y}\{0,0,t\} = a_3 \tag{16}$$

$$u\{a,0,t\} = (a_1) + a(a_2) + a^2(a_4) + a^3(a_7) \tag{17}$$

$$\frac{\partial u}{\partial x}\{a,0,t\} = (a_2) + 2a(a_4) + 3a^2(a_7) \tag{18}$$

$$\frac{\partial u}{\partial y}\{a,0,t\} = (a_3) + a(a_5) + a^2(a_8) + a^3(a_{11}) \tag{19}$$

$$\begin{aligned} u\{a,b,t\} = {} & a_1 + a_2 a + a_3 b + a_4 a^2 + a_5 ab + a_6 b^2 + a_7 a^3 \\ & + a_8 a^2 b + a_9 ab^2 + a_{10} b^3 + a_{11} a^3 b + a_{12} ab^3 \end{aligned} \tag{20}$$

$$\frac{\partial u}{\partial x}\{a,b,t\} = (a_2) + 2a(a_4) + b(a_5) + 3a^2(a_7) + 2ab(a_8) + b^2(a_9) + 3a^2 b(a_{11}) + b^3(a_{12}) \tag{21}$$

$$\frac{\partial u}{\partial y}\{a,b,t\} = (a_3) + a(a_5) + 2b(a_6) + a^2(a_8) + 2ab(a_9) + 3b^2(a_{10}) + a^3(a_{11}) + 3ab^2(a_{12}) \tag{22}$$

$$u\{0,b,t\} = (a_1) + (a_3)b + (a_6)b^2 + (a_{10})b^3 \tag{23}$$

$$\frac{\partial u}{\partial x}\{0,b,t\} = (a_2) + (a_5)b + (a_9)b^2 + (a_{12})b^3 \tag{24}$$

$$\frac{\partial u}{\partial y}\{0,b,t\} = (a_3) + 2b(a_6) + 3b^2(a_{10}) \tag{25}$$

The displacement vector for element i is

$$\{u\}_i^T = \left[u_k \quad \theta_{xk} \quad \theta_{yk} \quad u_l \quad \theta_{xl} \quad \theta_{yl} \quad u_m \quad \theta_{xm} \quad \theta_{ym} \quad u_n \quad \theta_{xn} \quad \theta_{yn} \right] \tag{26}$$

The displacement vector can be written as

$$\{u\}_i = [B]\{A\} \tag{27}$$

where

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & a & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2a & 0 & 0 & 3a^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 & a^3 & 0 \\ 1 & a & b & a^2 & ab & b^2 & a^3 & a^2b & ab^2 & b^3 & a^3b & ab^3 \\ 0 & 1 & 0 & 2a & b & 0 & 3a^2 & 2ab & b^2 & 0 & 3a^2b & b^3 \\ 0 & 0 & 1 & 0 & a & 2b & 0 & a2 & 2ab & 3b^2 & a^3 & 3ab^2 \\ 1 & 0 & b & 0 & 0 & b^2 & 0 & 0 & 0 & b^3 & 0 & 0 \\ 0 & 1 & 0 & 0 & b & 0 & 0 & 0 & b^2 & 0 & 0 & b^3 \\ 0 & 0 & 1 & 0 & 0 & 2b & 0 & 0 & 0 & 3b^2 & 0 & 0 \end{bmatrix}$$

(28)

Solve for $\{A\}$.

$$\{A\} = [B]^{-1}\{u\}_i \tag{29}$$

Let

$$[c] = [B]^{-1} \tag{30}$$

$$\{A\} = [c]\{u\}_i \tag{31}$$

The strain energy is thus

$$U = \frac{D}{2}\{u\}^T[c]^T\int_0^b\int_0^a[S(x,y)]\,dxdy[c]\{u\}_i \tag{32}$$

The kinetic energy is

$$K = \frac{\rho h}{2}\int_0^b\int_0^a \dot{u}^2\,dxdy \tag{33}$$

Recall the displacement equation

$$u(x,y,t) = \{A\}^T\{Z\} \tag{34}$$

The velocity is

$$\dot{u} = \{\dot{A}\}^T\{Z\} \tag{35}$$

$$[\dot{u}]^2 = \{\dot{A}\}^T\{Z\}\{Z\}^T\{\dot{A}\} \tag{36}$$

Recall

$$\{A\} = [c]\{u\}_i \tag{37}$$

$$[\dot{u}]^2 = \{\dot{u}\}_i{}^T[c]^T\{Z\}\{Z\}^T[c]\{\dot{u}\}_i \tag{38}$$

Let

$$P(x,y) = \{Z\}\{Z\}^T \tag{39}$$

$$[\dot{u}]^2 = \{\dot{u}\}_i{}^T[c]^T P(x,y)[c]\{\dot{u}\}_i \tag{40}$$

The kinetic energy is thus

$$K = \frac{\rho h}{2}\{\dot{u}\}_i^T [c]^T \int_0^b \int_0^a P(x,y)\,dxdy\,[c]\{\dot{u}\}_i \tag{41}$$

The virtual work due to the nodal forces and moment is

$$\delta W = \{F\}_i^T \{\delta u\}_i = \{\delta u\}_i \{F\}_i \tag{42}$$

where

$$\{F\}_i^T = \begin{bmatrix} F_k & M_{xk} & M_{yk} & F_l & M_{xl} & M_{yl} & F_m & M_{xm} & M_{ym} & F_n & M_{xn} & M_{yn} \end{bmatrix} \tag{43}$$

Hamilton’s principle yields the equation of motion.

$$[m]\{\ddot{u}\}_i + [k]\{u\}_i = \{F\}_{\mathrm{i}} \tag{44}$$

The respective local mass and stiffness matrices are

$$[m] = \rho h [c]^T \int_0^b \int_0^a [\,P(x,y)\,]\,dxdy\,[c] \tag{45}$$

$$[k] = D[c]^T \int_0^b \int_0^a [\,S(x,y)\,]\,dxdy\,[c] \tag{46}$$

The local matrices can then be assembled into global matrices. Then boundary conditions can be applied.

The generalized eigenvalue problem is

$$\{[K]-\omega^{2}[M]\}\{U\}=0 \tag{47}$$

where

K	is the global stiffness matrix
M	is the global mass matrix
ω	is the natural frequency
U	is the corresponding mode shape

Note that there is a natural frequency and mode shape for each degree-of-freedom.

References

1. W. Soedel, Vibrations of Shells and Plates, Third Edition (Dekker Mechanical Engineering), Third Edition, Marcel Dekker, New York, 2004.

2. R. Blevins, Formulas for Natural Frequency and Mode Shape, Krieger, Malabar, Florida, 1979. See Table 11-6.

APPENDIX A

Mass Matrix Core

$$\{Z\}\{Z\}^T = \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2y \\ xy^2 \\ y^3 \\ x^3y \\ xy^3 \end{bmatrix} \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \end{bmatrix} \tag{A-1}$$

$\{Z\}\{Z\}^T =$

$$\begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \\ x & x^2 & xy & x^3 & x^2y & xy^2 & x^4 & x^3y & x^2y^2 & xy^3 & x^4y & x^2y^3 \\ y & xy & y^2 & x^2y & xy^2 & y^3 & x^3y & x^2y^2 & xy^3 & y^4 & x^3y^2 & xy^4 \\ x^2 & x^3 & x^2y^2 & x^4 & x^3y & x^2y^2 & x^5 & x^4y & x^3y^2 & x^2y^3 & x^5y & x^3y^3 \\ xy & x^2y & xy^2 & x^3y & x^2y^2 & xy^3 & x^4y & x^3y^2 & x^2y^3 & xy^4 & x^4y^2 & x^2y^4 \\ y^2 & xy^2 & y^3 & x^2y^2 & xy^3 & y^4 & x^3y^2 & x^2y^3 & xy^4 & y^5 & x^3y^3 & xy^5 \\ x^3 & x^4 & x^3y & x^5 & x^4y & x^3y^2 & x^6 & x^5y & x^4y^2 & x^3y^3 & x^6y & x^4y^3 \\ x^2y & x^3y & x^2y^2 & x^4y & x^3y^2 & x^2y^3 & x^5y & x^4y^2 & x^3y^3 & x^2y^4 & x^5y^2 & x^3y^4 \\ xy^2 & x^2y^2 & xy^3 & x^3y^2 & x^2y^3 & xy^4 & x^4y^2 & x^3y^3 & x^2y^4 & xy^5 & x^4y^3 & x^2y^5 \\ y^3 & xy^3 & y^4 & x^2y^3 & xy^4 & y^5 & x^3y^3 & x^2y^4 & xy^5 & y^6 & x^3y^4 & xy^6 \\ x^3y & x^4y & x^3y^2 & x^5y & x^4y^2 & x^3y^3 & x^6y & x^5y^2 & x^4y^3 & x^3y^4 & x^6y^2 & x^4y^4 \\ xy^3 & x^2y^3 & xy^4 & x^3y^3 & x^2y^4 & xy^5 & x^4y^3 & x^3y^4 & x^2y^5 & xy^6 & x^4y^4 & x^2y^6 \end{bmatrix} \tag{A-2}$$

APPENDIX B

Stiffness Matrix Core

Again,

$$\{Z\}^T = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \end{bmatrix} \tag{B-1}$$

The derivatives are

$$\left\{\frac{\partial Z}{\partial x}\right\}^T = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 3x^2 & 2xy & y^2 & 0 & 3x^2y & y^3 \end{bmatrix} \tag{B-2}$$

$$\left\{\frac{\partial^2 Z}{\partial x^2}\right\}^T = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 & 0 & 6xy & 0 \end{bmatrix} \tag{B-3}$$

$$\left\{\frac{\partial Z}{\partial y}\right\}^T = \begin{bmatrix} 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 & 2xy & 3y^2 & x^3 & 3xy^2 \end{bmatrix} \tag{B-4}$$

$$\left\{\frac{\partial^2 Z}{\partial y^2}\right\}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x & 6y & 0 & 6xy \end{bmatrix} \tag{B-5}$$

$$\left\{\frac{\partial^2 Z}{\partial x \partial y}\right\}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2x & 2y & 0 & 3x^2 & 3y^2 \end{bmatrix} \tag{B-6}$$

$$\left\{\frac{\partial^2 Z}{\partial x^2}\right\}\left\{\frac{\partial^2 Z}{\partial x^2}\right\}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 6x \\ 2y \\ 0 \\ 0 \\ 6xy \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 & 0 & 6xy & 0 \end{bmatrix}$$

(B-7)

$$\left\{\frac{\partial^2 Z}{\partial x^2}\right\}\left\{\frac{\partial^2 Z}{\partial x^2}\right\}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 12x & 4y & 0 & 0 & 12xy & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12x & 0 & 0 & 36x^2 & 12xy & 0 & 0 & 36x^2y & 0 \\ 0 & 0 & 0 & 4y & 0 & 0 & 12xy & 4y^2 & 0 & 0 & 12xy^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12xy & 0 & 0 & 36x^2y & 12xy^2 & 0 & 0 & 36x^2y^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(B-8)

$$\left\{\frac{\partial^2 Z}{\partial y^2}\right\}\left\{\frac{\partial^2 Z}{\partial y^2}\right\}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 2x \\ 6y \\ 0 \\ 6xy \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x & 6y & 0 & 6xy \end{bmatrix}$$

(B-9)

$$\left\{\frac{\partial^2 Z}{\partial y^2}\right\}\left\{\frac{\partial^2 Z}{\partial y^2}\right\}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 4x & 12y & 0 & 12xy \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4x & 0 & 0 & 4x^2 & 12xy & 0 & 12x^2y \\ 0 & 0 & 0 & 0 & 0 & 12y & 0 & 0 & 12xy & 36y^2 & 0 & 36xy^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12xy & 0 & 0 & 12x^2y & 36xy^2 & 0 & 36x^2y^2 \end{bmatrix}$$

(B-10)

$$\left\{\frac{\partial^2 Z}{\partial x^2}\right\}\left\{\frac{\partial^2 Z}{\partial x^2}\right\}^T + \left\{\frac{\partial^2 Z}{\partial y^2}\right\}\left\{\frac{\partial^2 Z}{\partial y^2}\right\}^T =$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 12x & 4y & 0 & 0 & 12xy & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 4x & 12y & 0 & 12xy \\
0 & 0 & 0 & 12x & 0 & 0 & 36x^2 & 12xy & 0 & 0 & 36x^2y & 0 \\
0 & 0 & 0 & 4y & 0 & 0 & 12xy & 4y^2 & 0 & 0 & 12xy^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 4x & 0 & 0 & 4x^2 & 12xy & 0 & 12x^2y \\
0 & 0 & 0 & 0 & 0 & 12y & 0 & 0 & 12xy & 36y^2 & 0 & 36xy^2 \\
0 & 0 & 0 & 12xy & 0 & 0 & 36x^2y & 12xy^2 & 0 & 0 & 36x^2y^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 12xy & 0 & 0 & 12x^2y & 36xy^2 & 0 & 36x^2y^2
\end{bmatrix}$$

(B-11)

$$\left\{\frac{\partial^2 Z}{\partial x^2}\right\}\left\{\frac{\partial^2 Z}{\partial y^2}\right\}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 6x \\ 2y \\ 0 \\ 0 \\ 6xy \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x & 6y & 0 & 6xy \end{bmatrix}$$

(B-12)

$$\left\{\frac{\partial^2 Z}{\partial x^2}\right\}\left\{\frac{\partial^2 Z}{\partial y^2}\right\}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 4x & 12y & 0 & 12xy \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12x & 0 & 0 & 12x^2 & 36xy & 0 & 36x^2y \\ 0 & 0 & 0 & 0 & 0 & 4y & 0 & 0 & 4xy & 12y^2 & 0 & 12xy^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12xy & 0 & 0 & 12x^2y & 36xy^2 & 0 & 36x^2y^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(B-13)

$$\left\{\frac{\partial^2 Z}{\partial y^2}\right\}\left\{\frac{\partial^2 Z}{\partial x^2}\right\}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 2x \\ 6y \\ 0 \\ 6xy \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 & 0 & 6xy & 0 \end{bmatrix}$$

(B-14)

$$\left\{\frac{\partial^2 Z}{\partial y^2}\right\}\left\{\frac{\partial^2 Z}{\partial x^2}\right\}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 12x & 4y & 0 & 0 & 12xy & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4x & 0 & 0 & 12x^2 & 4xy & 0 & 0 & 12x^2y & 0 \\ 0 & 0 & 0 & 12y & 0 & 0 & 36xy & 12y^2 & 0 & 0 & 36xy^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12xy & 0 & 0 & 36x^2y & 12xy^2 & 0 & 0 & 36x^2y^2 & 0 \end{bmatrix}$$

(B-15)

$$\left\{\frac{\partial^2 Z}{\partial x^2}\right\}\left\{\frac{\partial^2 Z}{\partial y^2}\right\}^T + \left\{\frac{\partial^2 Z}{\partial y^2}\right\}\left\{\frac{\partial^2 Z}{\partial x^2}\right\}^T =$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 4x & 12y & 0 & 12xy \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 12x & 4y & 0 & 0 & 12xy & 0 \\
0 & 0 & 0 & 0 & 0 & 12x & 0 & 0 & 12x^2 & 36xy & 0 & 36x^2y \\
0 & 0 & 0 & 0 & 0 & 4y & 0 & 0 & 4xy & 12y^2 & 0 & 12xy^2 \\
0 & 0 & 0 & 4x & 0 & 0 & 12x^2 & 4xy & 0 & 0 & 12x^2y & 0 \\
0 & 0 & 0 & 12y & 0 & 0 & 36xy & 12y^2 & 0 & 0 & 36xy^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 12xy & 0 & 0 & 12x^2y & 36xy^2 & 0 & 36x^2y^2 \\
0 & 0 & 0 & 12xy & 0 & 0 & 36x^2y & 12xy^2 & 0 & 0 & 36x^2y^2 & 0
\end{bmatrix}$$

(B-16)

$$\left\{\frac{\partial^2 Z}{\partial x \partial y}\right\}\left\{\frac{\partial^2 Z}{\partial x \partial y}\right\}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 2x \\ 2y \\ 0 \\ 3x^2 \\ 3y^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2x & 2y & 0 & 3x^2 & 3y^2 \end{bmatrix}$$

(B-17)

$$\left\{\frac{\partial^2 Z}{\partial x \partial y}\right\}\left\{\frac{\partial^2 Z}{\partial x \partial y}\right\}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2x & 2y & 0 & 3x^2 & 3y^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2x & 0 & 0 & 4x^2 & 4xy & 0 & 6x^3 & 6xy^2 \\ 0 & 0 & 0 & 0 & 2y & 0 & 0 & 4xy & 4y^2 & 0 & 6x^2y & 6y^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3x^2 & 0 & 0 & 6x^3 & 6x^2y & 0 & 9x^4 & 9x^2y^2 \\ 0 & 0 & 0 & 0 & 3y^2 & 0 & 0 & 6xy^2 & 6y^3 & 0 & 9x^2y^2 & 9y^4 \end{bmatrix}$$

(B-18)

Let $\beta = 2(1-\mu)$

$$\left\{\frac{\partial^2 Z}{\partial x^2}\right\}\left\{\frac{\partial^2 Z}{\partial x^2}\right\}^T + \left\{\frac{\partial^2 Z}{\partial y^2}\right\}\left\{\frac{\partial^2 Z}{\partial y^2}\right\}^T + \mu\left\{\left\{\frac{\partial^2 Z}{\partial x^2}\right\}\left\{\frac{\partial^2 Z}{\partial y^2}\right\}^T + \left\{\frac{\partial^2 Z}{\partial y^2}\right\}\left\{\frac{\partial^2 Z}{\partial x^2}\right\}^T\right\} + \beta\left\{\frac{\partial^2 Z}{\partial x \partial y}\right\}\left\{\frac{\partial^2 Z}{\partial x \partial y}\right\}^T =$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 4\mu & 12x & 4y & 4x\mu & 12y\mu & 12xy & 12xy\mu \\
0 & 0 & 0 & 0 & 1\beta & 0 & 0 & 2x\beta & 2y\beta & 0 & 3x^2\beta & 3y^2\beta \\
0 & 0 & 0 & 4\mu & 0 & 4 & 12x\mu & 4y\mu & 4x & 12y & 12xy\mu & 12xy \\
0 & 0 & 0 & 12x & 0 & 12x\mu & 36x^2 & 12xy & 12x^2\mu & 36xy\mu & 36x^2y & 36x^2y\mu \\
0 & 0 & 0 & 4y & 2x\beta & 4y\mu & 12xy & 4y^2+4x^2\beta & 4xy\mu+4xy\beta & 12y^2\mu & 12xy^2+6x^3\beta & 12xy^2\mu+6xy^2\beta \\
0 & 0 & 0 & 4x\mu & 2y\beta & 4x & 12x^2\mu & 4xy\mu+4xy\beta & 4x^2+4y^2\beta & 12xy & 12x^2y\mu+6x^2y\beta & 12x^2y+6y^3\beta \\
0 & 0 & 0 & 12y\mu & 0 & 12y & 36xy\mu & 12y^2\mu & 12xy & 36y^2 & 36xy^2\mu & 36xy^2 \\
0 & 0 & 0 & 12xy & 3x^2\beta & 12xy\mu & 36x^2y & 12xy^2+6x^3\beta & 12x^2y\mu+6x^2y\beta & 36xy^2\mu & 36x^2y^2+9x^4\beta & 36x^2y^2\mu+9x^2y^2\beta \\
0 & 0 & 0 & 12xy\mu & 3y^2\beta & 12xy & 36x^2y\mu & 12xy^2\mu+6xy^2\beta & 12x^2y+6y^3\beta & 36xy^2 & 36x^2y^2\mu+9x^2y^2\beta & 36x^2y^2+9y^4\beta
\end{bmatrix}$$

(B-19)

APPENDIX C

Example 1

Consider an aluminum plate, with dimensions 4 x 6 x 0.125 inches. The plate is simply-supported on all sides.

The natural frequencies are

$$\omega = \sqrt{\frac{D}{\rho h}}\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right) \quad , \qquad m = 1,2,3,\ldots, n = 1,2,3,\ldots$$

(C-1)

$$D = \frac{Eh^3}{12(1-\mu^2)}$$

$$= \frac{(1.0e+07 \text{ lbf / in\^{}2})(0.125 \text{ in})\text{\^{}}3}{12(1-0.3\text{\^{}}2)}$$

$$= 1789 \text{ lbf in}$$

(C-2)

$$\rho h = \left((0.1/386) \text{ lbf sec\^{}2/in\^{}4}\right)(0.125 \text{ in})$$

$$= 3.238\text{e - }05 \text{ lbf sec\^{}2/in\^{}3}$$

(C-3)

$$\omega_{mn} = \sqrt{\frac{1789 \text{ lbf in}}{3.238\text{e-}05 \text{ lbf sec\^{}2/in\^{}3}}} \left(\left(\frac{m\pi}{6\,\text{in}} \right)^2 + \left(\frac{n\pi}{4\,\text{in}} \right)^2 \right) , \quad m = 1,2,3,\ldots, n = 1,2,3,\ldots$$

(C-4)

Table C-1. Natural Frequencies, Hand Calculation		
fn(Hz)	m	n
1054	1	1
2027	2	1
3243	1	2
3649	3	1
4216	2	2

Now perform a finite element analysis with 13 nodes along the X-axis and 9 nodes along the Y-axis. The analysis is performed using the mass and stiffness matrices derived in the main text, as implemented in Matlab script: plate_fea.m

The fundamental mode shape is shown in Figure C-1. The natural frequency comparison is shown in Table C-2.

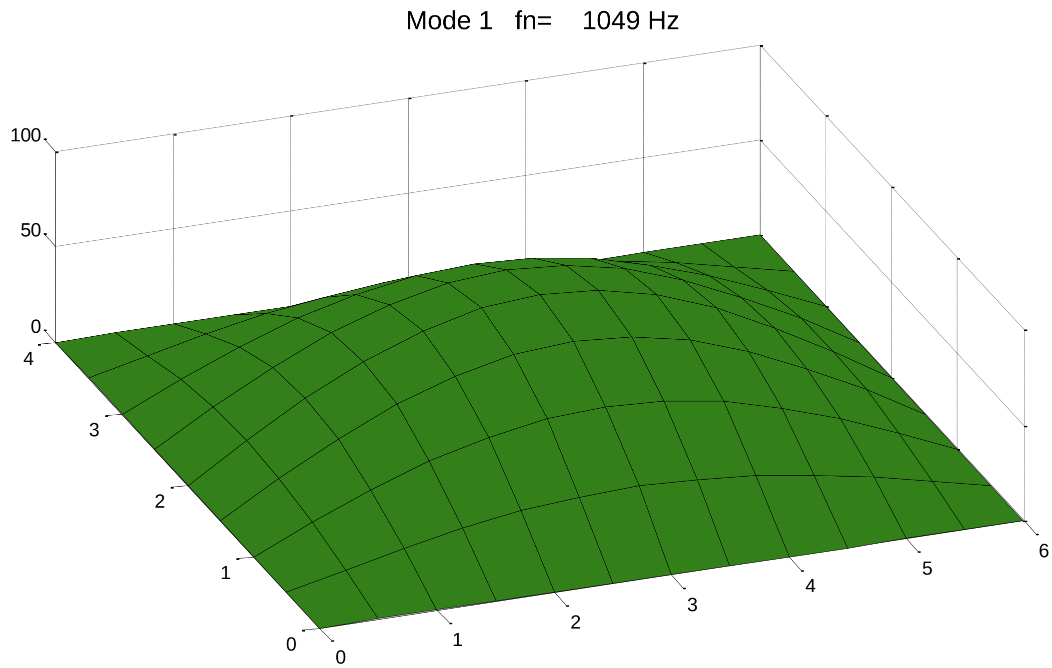

Figure C-1.

Table C-2. Natural Frequencies		
Mode	Hand Calculation fn(Hz)	FEA fn(Hz)
1	1054	1049
2	2027	2006
3	3243	3223
4	3649	3603
5	4216	4136

APPENDIX D

Example 2

Repeat example 1 but change the boundary conditions such that the corners are pinned. The edges are otherwise free.

The fundamental frequency is 291 Hz using the hand calculation formula from Reference 2.

Again, the analysis is performed using the mass and stiffness matrices derived in the main text, as implemented in Matlab script: plate_fea.m

The natural frequency comparison is shown in Table D-1. The mode shapes are shown in the following figures.

Table D-1. Natural Frequencies		
Mode	Hand Calculation fn(Hz)	FEA fn(Hz)
1	291	294
2	-	708
3	-	852
4	-	1110
5	-	1739

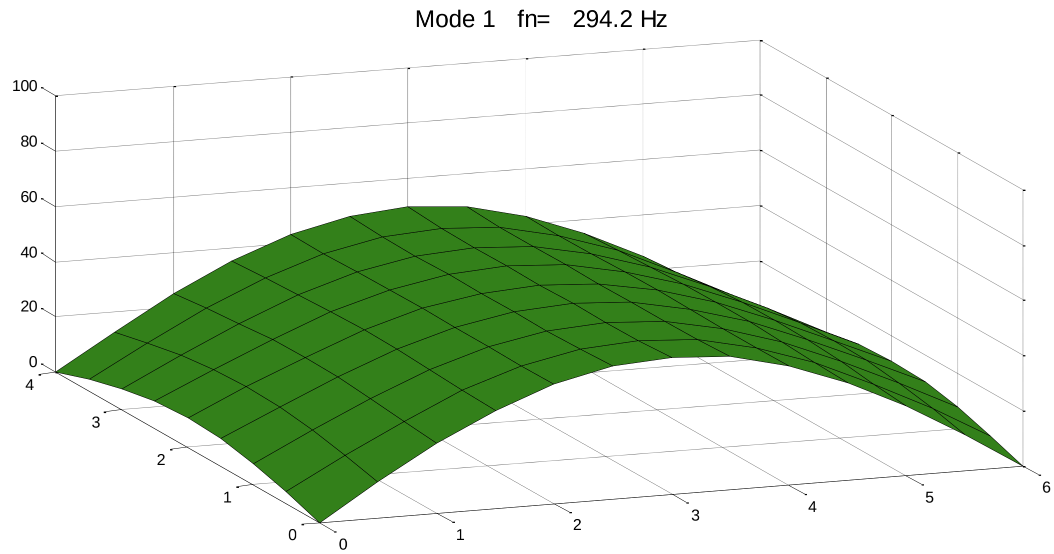

Figure D-1.

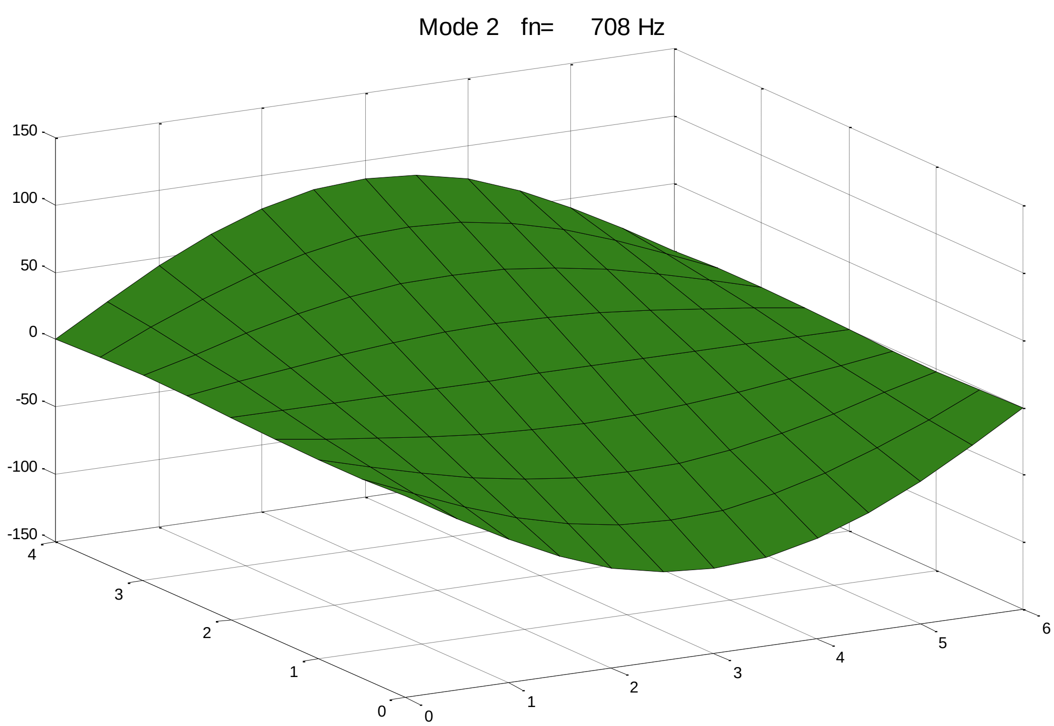

Figure D-2.

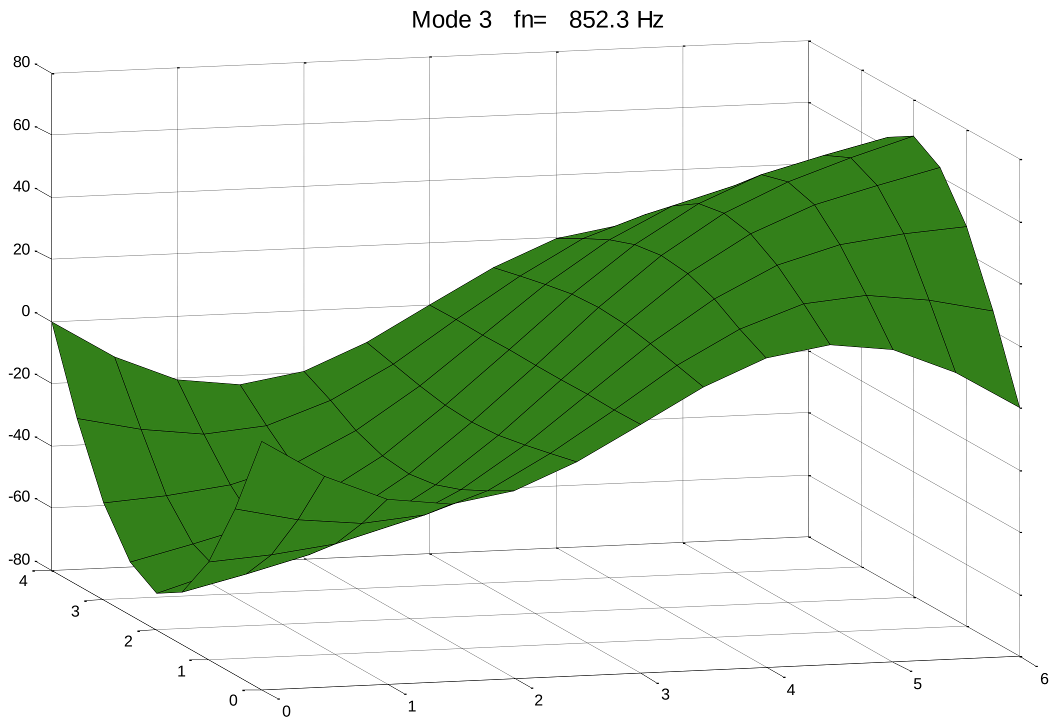

Figure D-3.

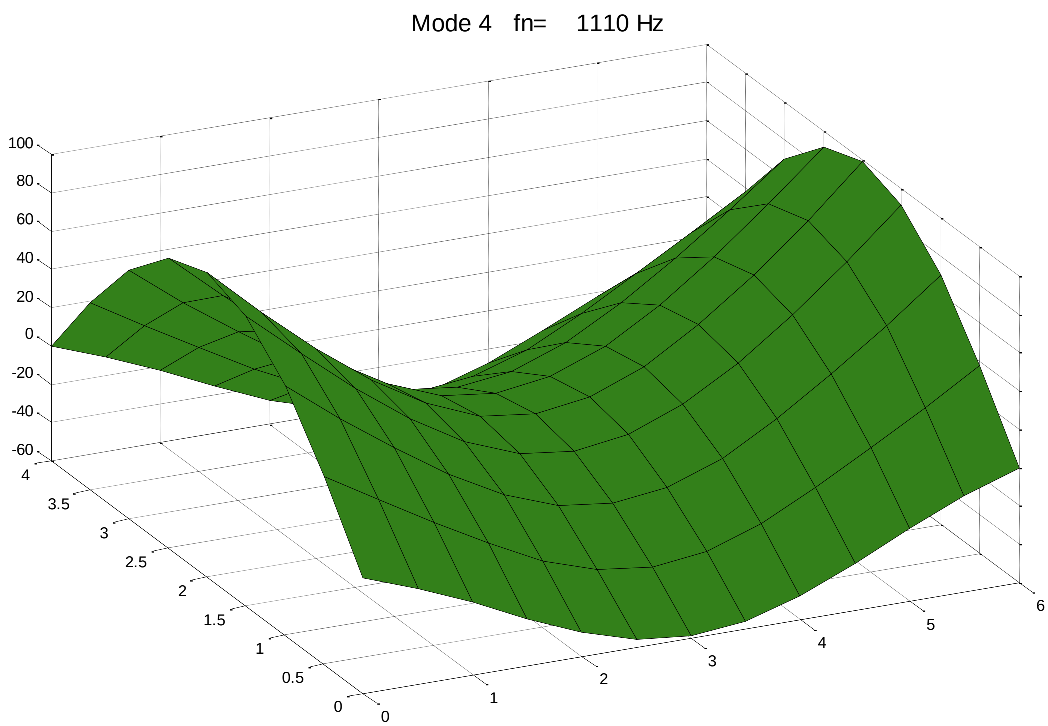

Figure D-4.

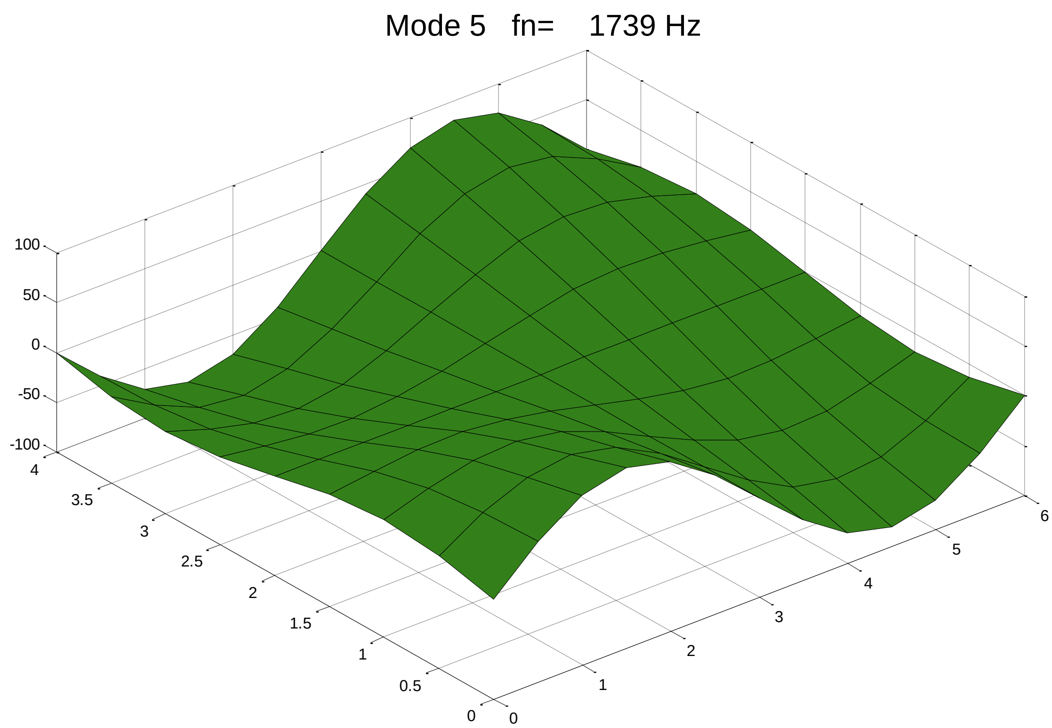

Figure D-5.