FREQUENCY RESPONSE FUNCTION ANALYSIS OF A MULTI-DEGREE-OF-FREEDOM SYSTEM WITH ENFORCED MOTION

By Tom Irvine
Email: tomirvine@aol.com

August 6, 2011

Variables

M	Mass matrix
K	Stiffness matrix
F	Applied forces
F_d	Forces at driven nodes
F_f	Forces at free nodes
I	Identity matrix
Π	Transformation matrix
Q	Eigenvector matrix
u	Displacement vector
u_d	Displacements at driven nodes
u_f	Displacements at free nodes
ω	Excitation frequency
ω_i	Natural frequency for mode i
ξ_i	Damping ratio for mode i

The equation of motion for a multi-degree-of-freedom system is

$$[M][\ddot{u}]+[K][u]=F \tag{1}$$

$$[u]=\begin{bmatrix} u_d \\ u_f \end{bmatrix} \tag{2}$$

Partition the matrices and vectors as follows

$$\begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix}\begin{bmatrix} \ddot{u}_d \\ \ddot{u}_f \end{bmatrix}+\begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix}\begin{bmatrix} u_d \\ u_f \end{bmatrix}=\begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{3}$$

The equations of motions for enforced displacement and acceleration are given in Appendices A and B, respectively.

Create a transformation matrix such that

$$\begin{bmatrix} u_d \\ u_f \end{bmatrix}=\Pi\begin{bmatrix} u_d \\ u_w \end{bmatrix} \tag{4}$$

$$\Pi=\begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix}\Pi\begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix}+\begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix}\Pi\begin{bmatrix} u_d \\ u_w \end{bmatrix}=\begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{6}$$

Premultiply by Π^T,

$$\Pi^T\begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix}\Pi\begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix}+\Pi^T\begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix}\Pi\begin{bmatrix} u_d \\ u_w \end{bmatrix}=\Pi^T\begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{7}$$

APPENDIX A

Enforced Acceleration

Again, the partitioned equation of motion is

$$\Pi^T \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \Pi \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \Pi^T \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \Pi^T \begin{bmatrix} F_d \\ F_f \end{bmatrix} \quad \text{(A-1)}$$

Transform the equation of motion to uncouple the stiffness matrix so that the resulting stiffness matrix is

$$\begin{bmatrix} \hat{K}_{dd} & 0 \\ 0 & \hat{K}_{ww} \end{bmatrix} \quad \text{(A-2)}$$

$$\Pi^T \ K \ \Pi = \begin{bmatrix} I & T_1{}^T \\ 0 & T_2{}^T \end{bmatrix} \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \quad \text{(A-3)}$$

$$\Pi^T \ K \ \Pi = \begin{bmatrix} I & T_1{}^T \\ 0 & T_2{}^T \end{bmatrix} \begin{bmatrix} K_{dd} + K_{df} T_1 & K_{df} T_2 \\ K_{fd} + K_{ff} T_1 & K_{ff} T_2 \end{bmatrix} \quad \text{(A-4)}$$

$$\Pi^T \ K \ \Pi = \begin{bmatrix} K_{dd} + K_{df} T_1 + T_1{}^T \left(K_{fd} + K_{ff} T \right) & K_{df} T_2 + T_1{}^T K_{ff} T_2 \\ T_2{}^T \left(K_{fd} + K_{ff} T_1 \right) & T_2{}^T \left(K_{ff} T_2 \right) \end{bmatrix} \quad \text{(A-5)}$$

$$\Pi^T K \Pi = \begin{bmatrix} K_{dd} + K_{df} T_1 + T_1^T (K_{fd} + K_{ff} T_1) & (K_{df} + T_1^T K_{ff}) T_2 \\ T_2^T (K_{fd} + K_{ff} T_1) & T_2^T (K_{ff} T_2) \end{bmatrix} \tag{A-6}$$

$$\Pi^T K \Pi = \begin{bmatrix} K_{dd} + T_1^T K_{fd} + (K_{df} + T_1^T K_{ff}) T_1 & (K_{df} + T_1^T K_{ff}) T_2 \\ T_2^T (K_{fd} + K_{ff} T_1) & T_2^T (K_{ff} T_2) \end{bmatrix} \tag{A-7}$$

Let

$$T_2 = I \tag{A-8}$$

$$\Pi^T K \Pi = \begin{bmatrix} K_{dd} + T_1^T K_{fd} + (K_{df} + T_1^T K_{ff}) T_1 & (K_{df} + T_1^T K_{ff}) \\ (K_{fd} + K_{ff} T_1) & K_{ff} \end{bmatrix} = \begin{bmatrix} \hat{K}_{dd} & 0 \\ 0 & \hat{K}_{ww} \end{bmatrix} \tag{A-9}$$

$$K_{df} + T_1^T K_{ff} = 0 \tag{A-10}$$

$$T_1^T = -K_{df} K_{ff}^{-1} \tag{A-11}$$

$$T_1 = -K_{ff}^{-1} K_{fd} \tag{A-12}$$

$$\Pi = \begin{bmatrix} I_{dd} & 0 \\ T_1 & I_{ff} \end{bmatrix} \tag{A-13}$$

$$\hat{K}_{dd} = K_{dd} + T_1{}^T K_{fd} + \left(K_{df} + T_1{}^T K_{ff}\right) T_1 \tag{A-14}$$

$$\hat{K}_{ww} = K_{ff} \tag{A-15}$$

$$\Pi^T M \Pi = \begin{bmatrix} I_{dd} & T_1{}^T \\ 0 & I_{ff} \end{bmatrix} \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \begin{bmatrix} I_{dd} & 0 \\ T_1 & I_{ff} \end{bmatrix} \tag{A-16}$$

By similarity, the transformed mass matrix is

$$\begin{bmatrix} \hat{m}_{dd} & \hat{m}_{dw} \\ \hat{m}_{wd} & \hat{m}_{ww} \end{bmatrix} = \begin{bmatrix} M_{dd} + T_1{}^T M_{fd} + \left(M_{df} + T_1{}^T M_{ff}\right) T_1 & \left(M_{df} + T_1{}^T M_{ff}\right) \\ \left(M_{fd} + M_{ff} T_1\right) & M_{ff} \end{bmatrix} \tag{A-17}$$

$$\begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} I_{dd} & T_1 \\ 0 & I_{ff} \end{bmatrix} \begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{A-18}$$

$$\begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} I_{dd} F_d + T_1 F_f \\ I_{ff} F_f \end{bmatrix} \tag{A-19}$$

$$\begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} F_d + T_1 F_f \\ F_f \end{bmatrix} \tag{A-20}$$

$$\begin{bmatrix} \hat{m}_{dd} & \hat{m}_{dw} \\ \hat{m}_{wd} & \hat{m}_{ww} \end{bmatrix} \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \begin{bmatrix} \hat{K}_{dd} & 0 \\ 0 & \hat{K}_{ww} \end{bmatrix} \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} \tag{A-21}$$

$$\hat{m}_{wd}\,\ddot{u}_d + \hat{m}_{ww}\,\ddot{u}_w + \hat{K}_{ww}\,u_w = \hat{F}_w \quad \text{(A-22)}$$

The equation of motion is thus

$$\hat{m}_{ww}\,\ddot{u}_w + \hat{K}_{ww}\,u_w = \hat{F}_w - \hat{m}_{wd}\,\ddot{u}_d \quad \text{(A-23)}$$

Now solve the generalized eigenvalue problem.

Let Q be the eigenvector matrix.

Let

$$u_w = Q\,\eta_w \quad \text{(A-24)}$$

$$\hat{m}_{ww}\,Q\,\ddot{\eta}_w + \hat{K}_{ww}\,Q\,\eta_w = \hat{F}_w - \hat{m}_{wd}\,\ddot{u}_d \quad \text{(A-25)}$$

Premultiply by Q^T to decouple the equations of motion.

$$Q^T\,\hat{m}_{ww}\,Q\,\ddot{\eta}_w + Q^T\,\hat{K}_{ww}\,Q\,\eta_w = Q^T\left\{\hat{F}_w - \hat{m}_{wd}\,\ddot{u}_d\right\} \quad \text{(A-26)}$$

Assume that the applied forces are all zero. Thus the only excitation is the enforced acceleration.

$$Q^T\,\hat{m}_{ww}\,Q\,\ddot{\eta}_w + Q^T\,\hat{K}_{ww}\,Q\,\eta_w = -Q^T\left\{\hat{m}_{wd}\,\ddot{u}_d\right\} \quad \text{(A-27)}$$

Assume that the applied forces are all zero. Thus the only excitation is the enforced acceleration.

$$Q^T \hat{m}_{ww} Q \ddot{\eta}_w + Q^T \hat{K}_{ww} Q \eta_w = -Q^T \{\hat{m}_{wd} \ddot{u}_d\} \qquad \text{(A-28)}$$

$$\tilde{M} = Q^T \hat{m}_{ww} Q \ddot{\eta}_w \qquad \text{(A-29)}$$

$$\tilde{K} = Q^T \hat{K}_{ww} Q \eta_w \qquad \text{(A-30)}$$

The decoupled equations of motion with an added modal damping matrix $\tilde{C}$ are

$$\tilde{M} \ddot{\eta}_w + \tilde{C} \dot{\eta}_w + \tilde{K} \eta_w = -Q^T \{\hat{m}_{wd} \ddot{u}_d\} \qquad \text{(A-31)}$$

Note that

$\tilde{M}$ is the identity matrix is a diagonal matrix

$\tilde{C}$ is a diagonal matrix containing the terms $2\xi_i\omega_i$

$\tilde{K}$ is a diagonal matrix containing the terms $\omega_i{}^2$

Thus

$$\ddot{\eta}_{w,i} + 2\xi_i\omega_i \dot{\eta}_{w,i} + \omega_i{}^2 \eta_{w,i} = -Q^T \{\hat{m}_{wd,i} \ddot{u}_d\} \qquad \text{(A-32)}$$

Perform a steady-state analysis. Represent the enforced motion via a Fourier transform.

$$\ddot{u}_d = \hat{U}_d \exp(j\omega t) \qquad \text{(A-33)}$$

Assume

$$\eta_w = N_w \exp(j\omega t) \qquad \text{(A-34)}$$

By substitution,

$$-\omega^2 N_{w,i} \exp(j\omega t) + j 2\xi_i \omega_i \omega N_{w,i} \exp(j\omega t) + \omega_i^2 N_{w,i} \exp(j\omega t) = -Q^T \{\hat{m}_{wd,i}\} \hat{U}_d \exp(j\omega t)$$

(A-35)

$$-\omega^2 N_{w,i} + j 2\xi_i \omega_i \omega N_{w,i} + \omega_i^2 N_{w,i} = -Q^T \{\hat{m}_{wd,i}\} \hat{U}_d \quad \text{(A-36)}$$

$$\{-\omega^2 + j 2\xi_i \omega_i \omega + \omega_i^2\} N_{w,i} = -Q^T \{\hat{m}_{wd,i}\} \hat{U}_d \quad \text{(A-37)}$$

$$\{\omega_i^2 - \omega^2 + j 2\xi_i \omega_i \omega\} N_{w,i} = -Q^T \{\hat{m}_{wd,i}\} \hat{U}_d \quad \text{(A-38)}$$

$$N_{w,i} = \frac{-Q^T \{\hat{m}_{wd}\}}{\omega_i^2 - \omega^2 + j 2\xi_i \omega_i \omega} \hat{U}_d \quad \text{(A-39)}$$

The final acceleration frequency response functions are found via

$$U_w = Q N_w \quad \text{(A-40)}$$

$$\begin{bmatrix} U_d \\ U_f \end{bmatrix} = \Pi \begin{bmatrix} U_d \\ U_w \end{bmatrix} \quad \text{(A-41)}$$

$$\Pi = \begin{bmatrix} I_{dd} & 0 \\ -K_{ff}^{-1} K_{fd} & I_{ff} \end{bmatrix} \quad \text{(A-42)}$$

APPENDIX B

Enforced Acceleration Example

<u>Equation of Motion</u>

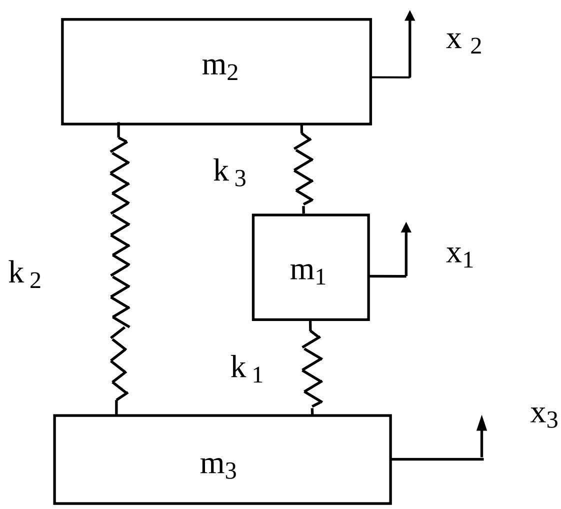


Figure B-1.

Consider the system in Figure B-1. Assign the values in Table B-1.

Table B-1. Parameters	
Variable	Value
m_1	2.0 kg
m_2	1.0 kg
m_3	1.0 kg *
k_1	100,000 N/m
k_2	200,000 N/m
k_3	300,000 N/m

The value of m_3 is arbitrary because its motion will be enforced.

Furthermore, assume that each mode has a damping value of 5%.

The following equations of motion are derived in Appendix C for the system in Figure B-1.

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 & -k_1 \\ -k_3 & k_2 + k_3 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{B-1}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 400{,}000 & -300{,}000 & -100{,}000 \\ -300{,}000 & 500{,}000 & -200{,}000 \\ -100{,}000 & -200{,}000 & 300{,}000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{B-2}$$

<u>FRF Analysis</u>

The analysis is performed via Matlab script: enforced_acceleration_frf.m.

```
>> enforced_acceleration_frf

   enforced_acceleration_frf.m  ver 1.0  August 6, 2011
   by Tom Irvine

 Enter the units system
 1=English  2=metric
 2
 Assume symmetric mass and stiffness matrices.
 mass unit = kg
 stiffness unit = N/m

 Select file input method
   1=file preloaded into Matlab
   2=Excel file
1

 Mass Matrix
 Enter the matrix name:  mmm

 Stiffness Matrix
 Enter the matrix name:  kkk
 Input Matrices
```

```
mass =

     2     0     0
     0     1     0
     0     0     1


stiff =

      400000     -300000     -100000
     -300000      500000     -200000
     -100000     -200000      300000

 Select modal damping input method
  1=uniform damping for all modes
  2=damping vector
 1

 Enter damping ratio
 0.05

 number of dofs =3

 Enter the enforced acceleration dof
 3

MT =

    4.0000    2.0000    1.0000
    2.0000    2.0000         0
    1.0000         0    1.0000


KT =

  1.0e+005 *

    0.0000         0         0
         0    4.0000   -3.0000
         0   -3.0000    5.0000

 Natural Frequencies
 No.      f(Hz)
1.    4.293e-007
2.        90.982
3.        130.59
```

```
  Modes Shapes (column format)

ModeShapes =

    0.5000   -0.7691   -0.3981
    0.0000    1.2131    0.1683
    0.0000    0.6501    1.2559


Mwd =

    2.0000
    1.0000


Mww =

     2     0
     0     1


Kww =

      400000     -300000
     -300000      500000

 Natural Frequencies
 No.      f(Hz)
1.        47.797
2.        124.28

  Modes Shapes (column format)

ModeShapes =

    0.6280   -0.3251
    0.4597    0.8881

 Participation Factors

part =

    1.7156
    0.2380
```

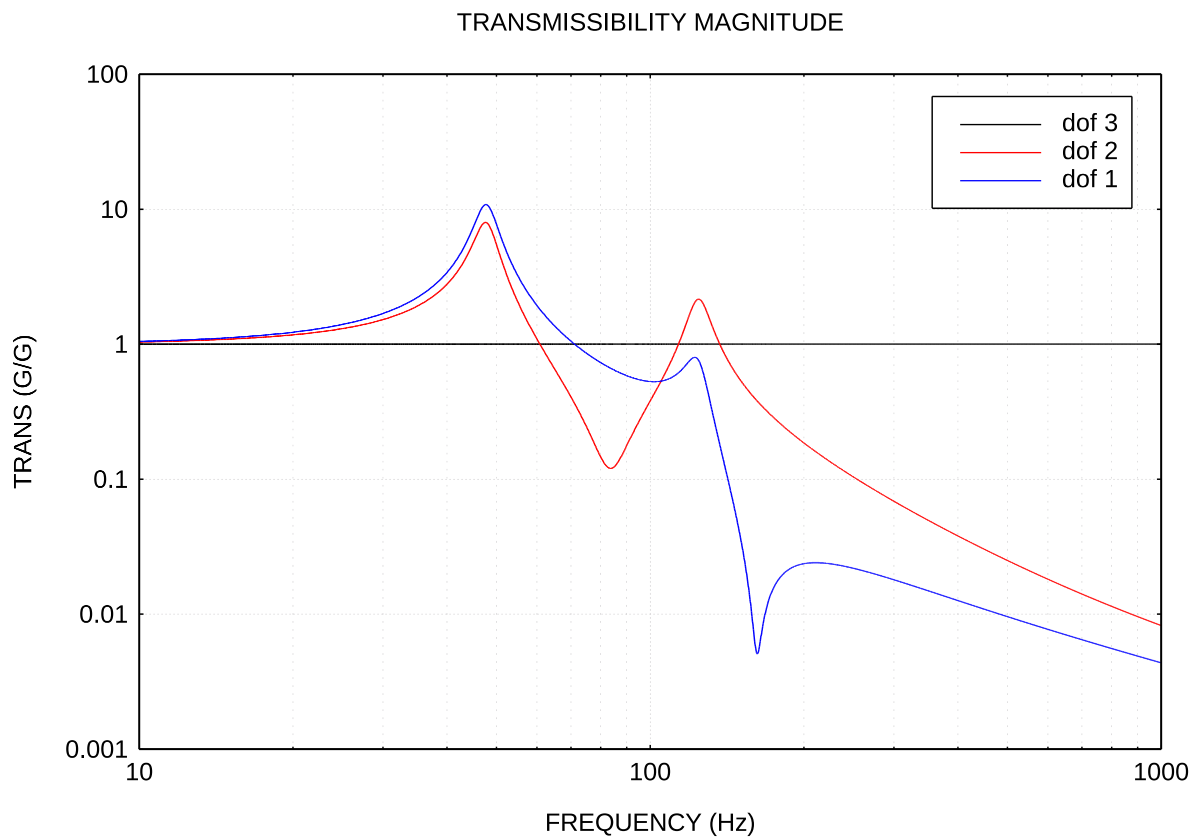
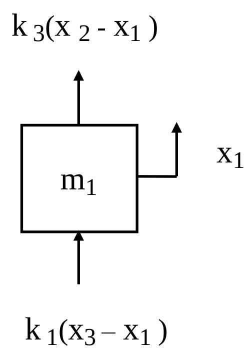

TRANSMISSIBILITY MAGNITUDE
100
10
1
0.1
0.01
0.001
10
100
1000
TRANS (G/G)
FREQUENCY (Hz)
dof 3
dof 2
dof 1

Figure B-2.

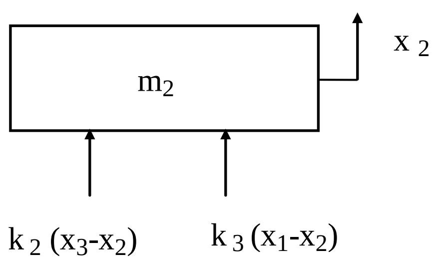
APPENDIX C



$$m_1\ddot{x}_1 = k_1(x_3 - x_1) + k_3(x_2 - x_1)$$

$$m_1\ddot{x}_1 + k_1(-x_3 + x_1) + k_3(-x_2 + x_1) = 0$$

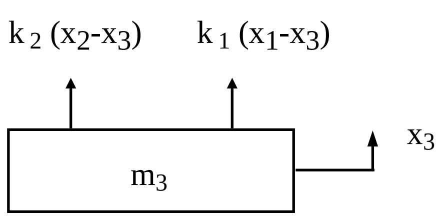
$$m_1\ddot{x}_1 + (k_1 + k_3)x_1 - k_3x_2 - k_1x_3 = 0$$



$$m_2\ddot{x}_2 = k_2(x_3 - x_2) + k_3(x_1 - x_2)$$

$$m_2\ddot{x}_2 + k_2(-x_3 + x_2) + k_3(-x_1 + x_2) = 0$$

$$m_2\ddot{x}_2 - k_3x_1 + (k_2 + k_3)x_2 - k_2x_3 = 0$$



$$m_3\ddot{x}_3 = k_1(x_1 - x_3) + k_2(x_2 - x_3)$$

$$m_3\ddot{x}_3 + k_1(-x_1 + x_3) + k_2(-x_2 + x_3) = 0$$

$$m_3\ddot{x}_3 - k_1x_1 - k_2x_2 + (k_1 + k_2)x_3 = 0$$