The Haar Wavelet Transform

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Introduction

The Haar wavelet is a nonlinear sequence of discontinuous rescaled "square-shaped" functions which together form a wavelet family or basis. Wavelet analysis is similar to Fourier analysis in that it allows a signal over an interval to be represented in terms of an orthonormal function basis.

An advantage of wavelets over Fourier transforms is temporal resolution. The wavelet captures both frequency and location in time. (But a waterfall FFT can also achieve some measure of temporal resolution.)

Wavelets can be used for signal compression and filtering.

Haar Functions

Consider a vector with four points. Four wavelet basis functions are needed, as shown graphically in Figure 1.

The first is simply a constant level as shown in the upper left hand corner. The second is a square-shaped function in the upper right corner. The two remaining functions can be thought of as square-shaped functions which are compressed and shifted.

Note that the Euclidean space inner product (dot product) between any two pair of basis vectors is zero. The base functions are thus mutually orthogonal. The functions can also be normalized to have a length of one, thus rendering them as orthonormal functions.

The sequence for a set of eight points is shown in Figure 2.

The idea is that a discrete waveform can be decomposed into a series of orthonormal wavelets where each wavelet is multiplied by the proper scale factor or coefficient. The coefficients are found via a transformation process.

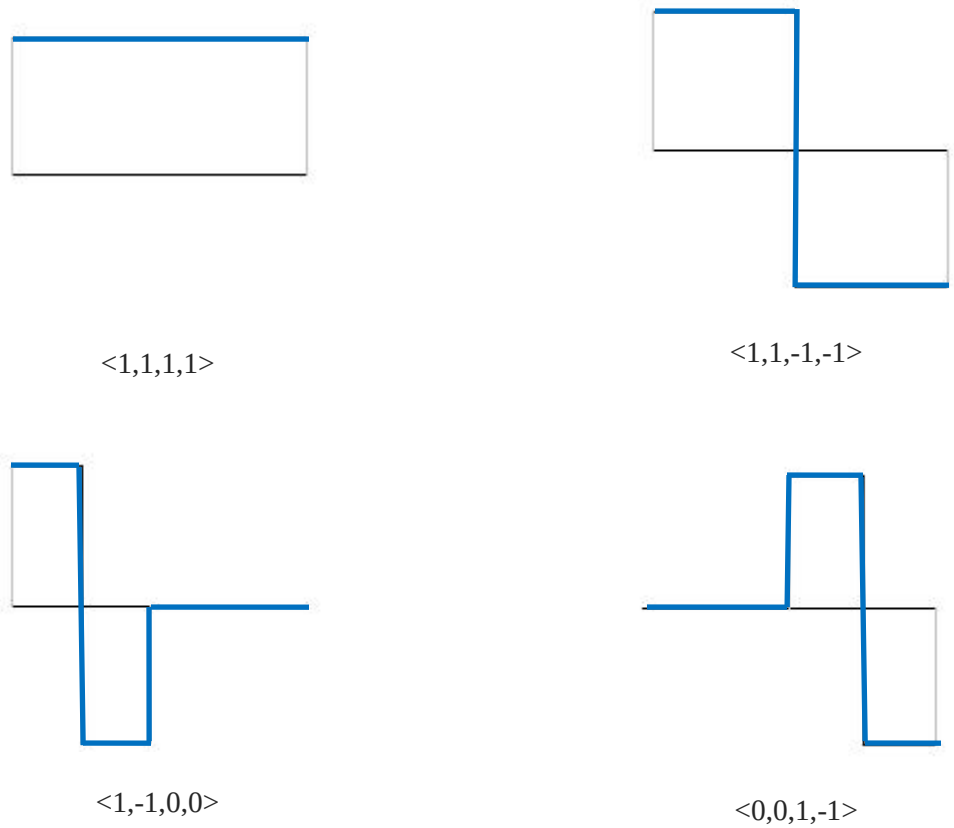

Figure 1. Haar Base Functions for Four Data Points

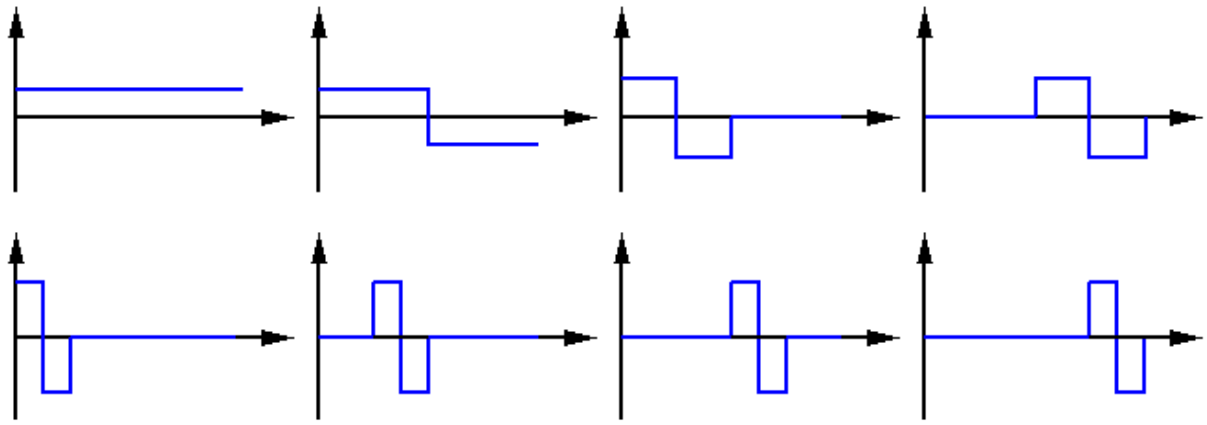

Figure 2. Haar Base Functions for Eight Data Points

<u>One Dimensional Discrete Wavelet Transform</u>

Assume that that a time history x has 2^n points where n is an integer. Calculate the Haar Transform y.

$$y = H\,x \tag{1}$$

H is the wavelet transform matrix. It contains the orthonormal functions in row format.

The transformation matrix for two data points is

$$H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2}$$

The transformation matrix for four data points is

$$H_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \tag{3}$$

The transformation matrix for eight data points is

$$H_8 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \tag{4}$$

The transformation has the following properties

$$H^{-1} = H^{T} \tag{5}$$

$$H^{T} H = I \tag{6}$$

The corresponding inverse transformation is thus

$$x = H^{-1} y = H^{T} y \tag{7}$$

Example 1

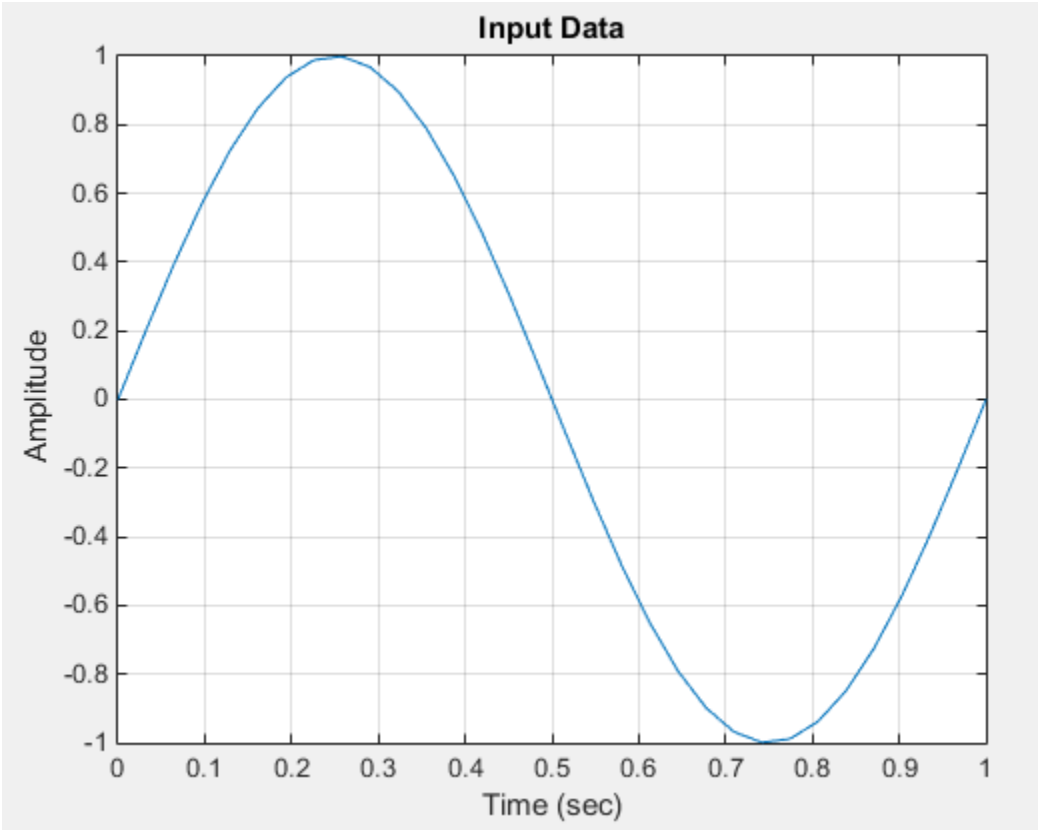

Figure 3. Sine Function, 1 Hz, 1 sec, 32 Points

One period of a sine function is shown in Figure 3.

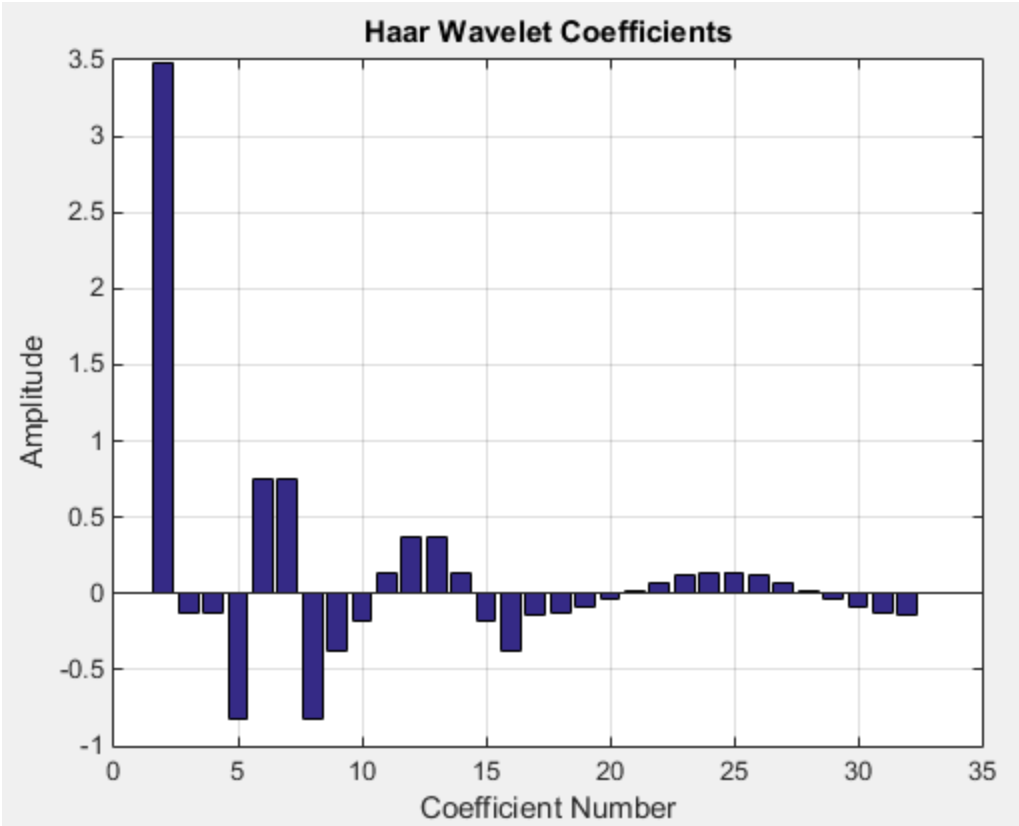

Figure 4. Sine Function Coefficients

The wavelet coefficients are shown in Figure 4.

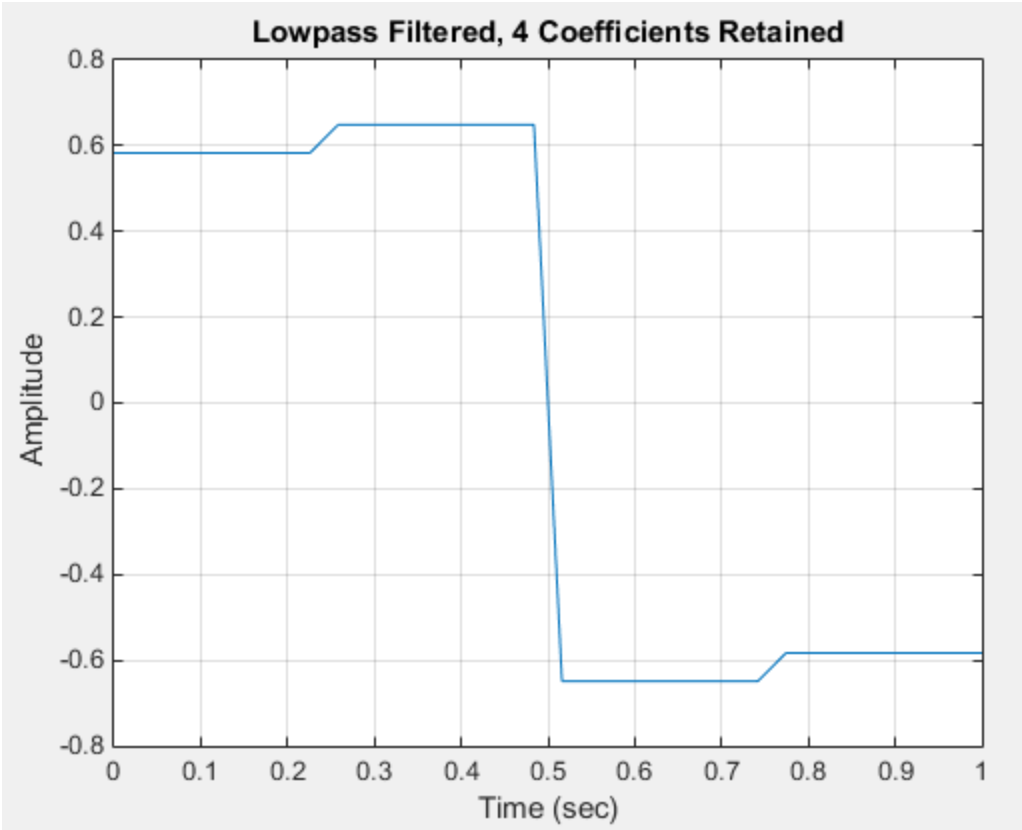

Figure 5. Four Coefficient Reconstruction

The inverse transform is taken by retaining the first four coefficients and setting the others equal to zero.

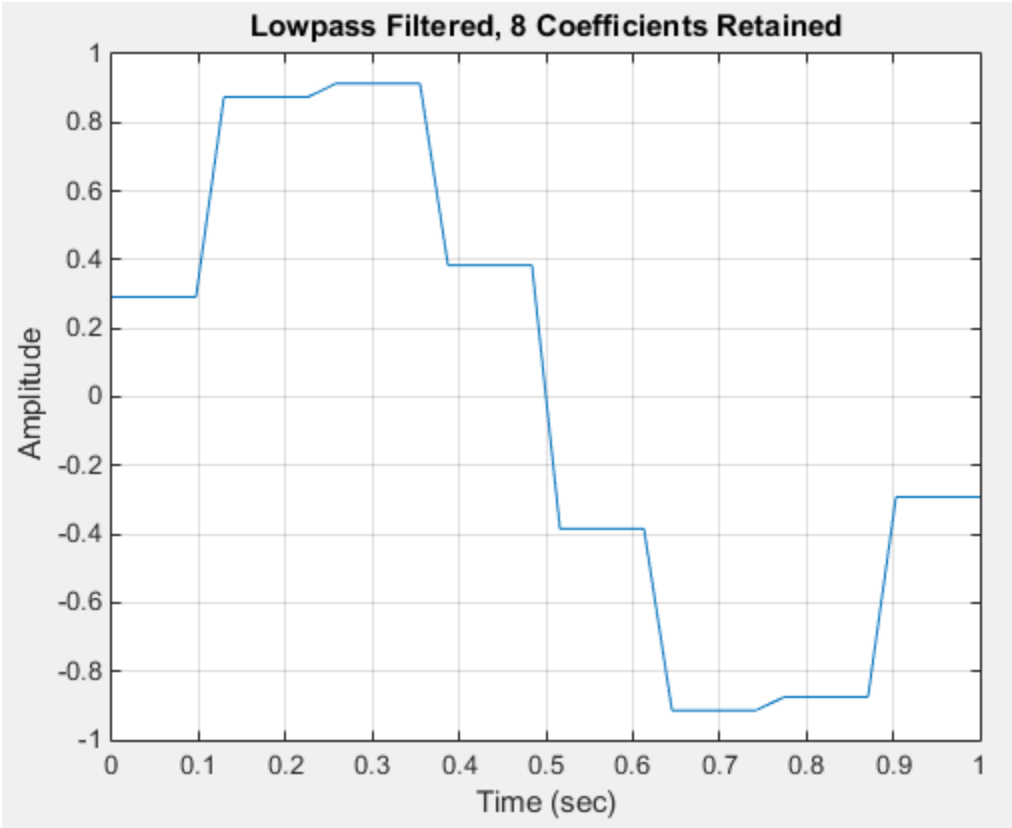

Figure 6. Eight Coefficient Reconstruction

The inverse transform is taken by retaining the first eight coefficients and setting the others equal to zero.

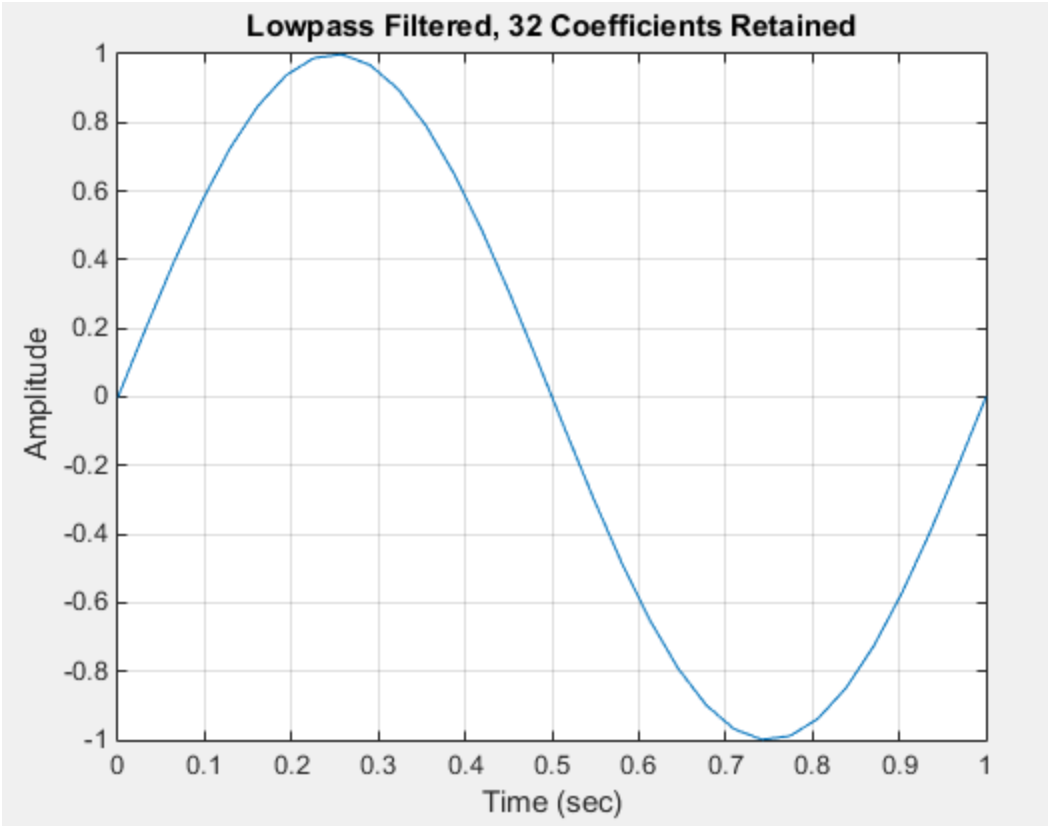

Figure 7. All Coefficients Retained

The inverse transform in Figure 8 is the original waveform. The signal is essentially unfiltered.

Example 2

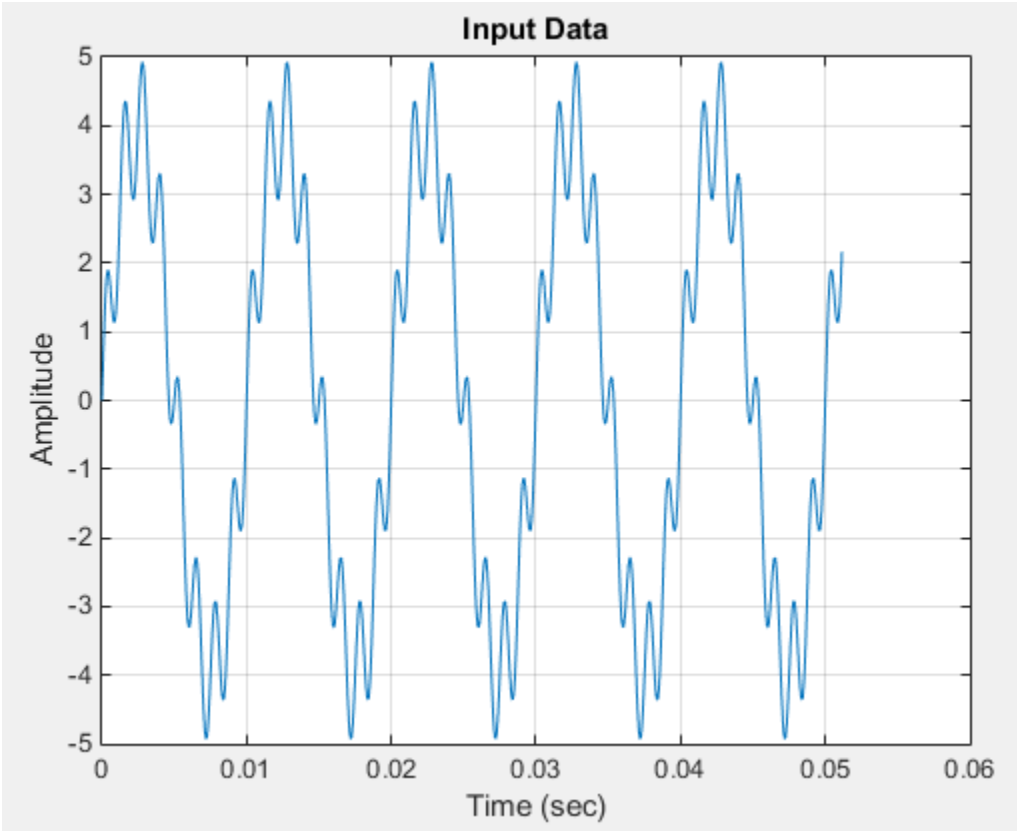

Figure 8. Summation of Two Sine Functions

The frequencies are 100 and 1000 Hz, with amplitudes of 4 and 1, respectively.

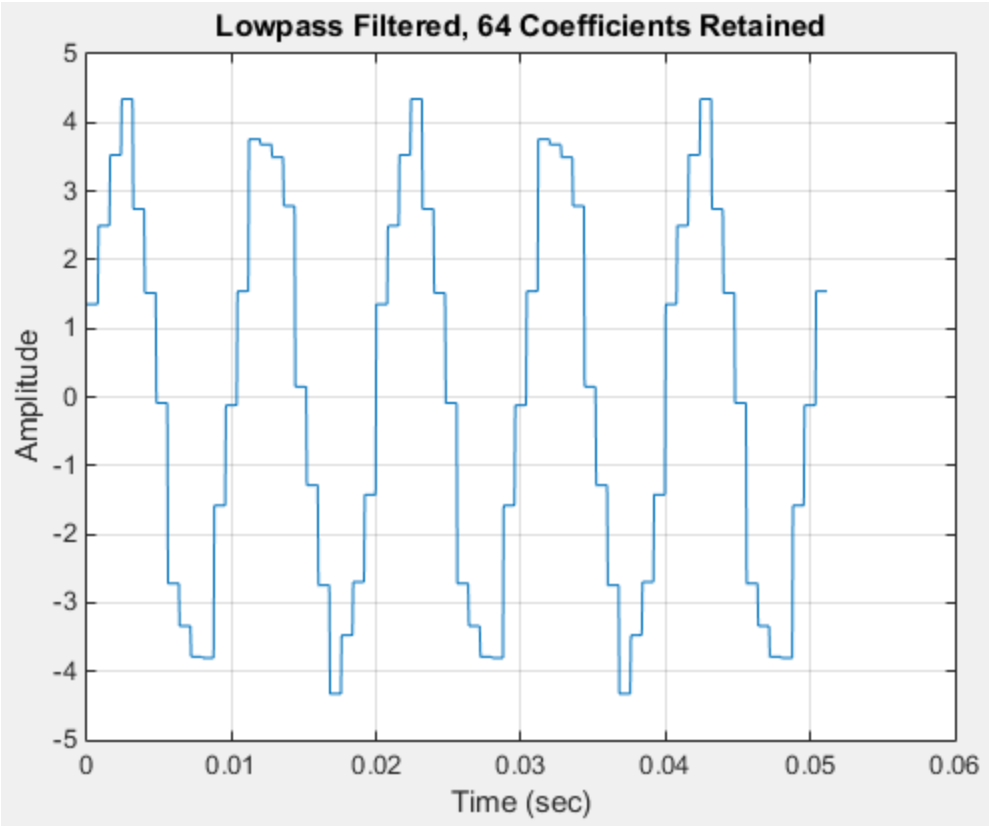

Figure 9. Lowpass Filter of Two Sine Summation

The 100 Hz component is retained, although with a stair step shape. The 1000 Hz sinusoid has been mostly filtered out.