THE MODE ACCELERATION METHOD
Revision A

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January 20, 2014

The following derivation is based on Reference 1.

Variables

$x(t)$	is the displacement vector
M	is the mass matrix
C	is the damping matrix
K	is the stiffness matrix
$P(t)$	is the applied load vector
F_i	is the modal force
$\{\phi_i\}$	is the eigenvector matrix
ω_i	is the natural frequency
ξ_i	is the damping ratio
$\eta(t)$	is the modal displacement

Structural dynamic problems can be solved by modal superposition methods. Two particular methods are the mode displacement method and the mode acceleration method.

Furthermore, reduction techniques such as mass condensation are often used to reduce the size of a dynamics problem, thereby eliminated higher modes.

The mode acceleration method uses a pseudo-static solution to account for the flexibility of modes which are eliminated in the mode displacement method.

The equations of motion for an n-degree-of-freedom system are

$$[M]\{\ddot{x}(t)\}+[C]\{\dot{x}(t)\}+[K]\{x(t)\}=\{P(t)\} \quad (1a)$$

The corresponding generalized eigenvalue problem for the undamped system is

$$[K]\{\phi_i\} = \omega_i^2 [M]\{\phi_i\} \quad , \ i=1, 2,\ldots, n \tag{1b}$$

Perform a transformation to modal coordinates.

$$\{x(t)\} = \sum_{i=1}^{n} \{\phi_i\}\{\eta(t)\} \tag{2}$$

By substitution,

$$[M]\{\phi_i\}\{\ddot{\eta}(t)\} + [C]\{\phi_i\}\{\dot{\eta}(t)\} + [K]\{\phi_i\}\{\eta(t)\} = \{P(t)\} \tag{3}$$

Premultiply $\{\phi_i\}^T$

$$\{\phi_i\}^T [M]\{\phi_i\}\{\ddot{\eta}(t)\} + \{\phi_i\}^T [C]\{\phi_i\}\{\dot{\eta}(t)\} + \{\phi_i\}^T [K]\{\phi_i\}\{\eta(t)\} = \{\phi_i\}^T \{P(t)\} \tag{4}$$

Normalize the eigenvectors such that

$$\{\phi_i\}^T [M]\{\phi_i\} = 1 \tag{5}$$

Let

$$F_i = \{\phi_i\}^T P(t) \tag{6}$$

The transformation also yields a set of n uncoupled equation

$$\ddot{\eta}_i + 2\xi_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i = F_i \quad , \ i=1, 2,\ldots, n \tag{7}$$

The modal displacement can be represented as

$$\eta_i = \frac{F_i}{\omega_i^2} - \left(\frac{2\xi_i}{\omega_i}\right)\dot{\eta}_i - \left(\frac{1}{\omega_i^2}\right)\ddot{\eta}_i \tag{8}$$

Now assume that a reduction method has been performed such that there are m degrees-of-freedom, where m < n. The displacement for the reduced system is thus

$$\{x(t)\} = \sum_{i=1}^{m} \{\phi_i\} \eta_i(t) \tag{9}$$

$$\{x(t)\} = \sum_{i=1}^{m} \{\phi_i\} \left\{ \frac{F_i}{\omega_i^2} - \left(\frac{2\xi_i}{\omega_i}\right)\dot{\eta}_i - \left(\frac{1}{\omega_i^2}\right)\ddot{\eta}_i \right\} \tag{10}$$

$$\{x(t)\} = \sum_{i=1}^{m} \{\phi_i\} \left\{ \frac{F_i}{\omega_i^2} \right\} - \sum_{i=1}^{m} \{\phi_i\} \left\{ \left(\frac{2\xi_i}{\omega_i}\right)\dot{\eta}_i + \left(\frac{1}{\omega_i^2}\right)\ddot{\eta}_i \right\} \tag{11}$$

Greater accuracy is achieved by replacing $\sum_{i=1}^{m} \{\phi_i\} \left\{ \frac{F_i}{\omega_i^2} \right\}$ with the pseudo-static term

$[K]^{-1}\{P(t)\}$, where the pseudo-static term contains all n degrees-of-freedom.

$$\{x(t)\} = [K]^{-1}\{P(t)\} - \sum_{i=1}^{m} \{\phi_i\} \left[\left(\frac{2\xi_i}{\omega_i}\right)\dot{\eta}_i(t) + \left(\frac{1}{\omega_i^2}\right)\ddot{\eta}_i(t) \right] \tag{12}$$

The summation in equation (12) represents a dynamic correction to the pseudo-static response.

Note that the value m may be less than the total degrees-of-freedom n if a reduction technique such as mass condensation was used. On the other hand, the pseudo-static term immediately to the right of the equal sign contains all of the degrees-of-freedom.

Also note that $[K]^{-1}$ does not exist for certain ungrounded structures such as a free-free beam. Pseudo-inverse methods are thus required.

Reference

1. Cornwell, Craig, & Johnson; On the Application of the Mode-Acceleration Method to Structural Engineering Problems, Earthquake Engineering and Structural Dynamics, Vol. 11, 1983.

APPENDIX A

Example

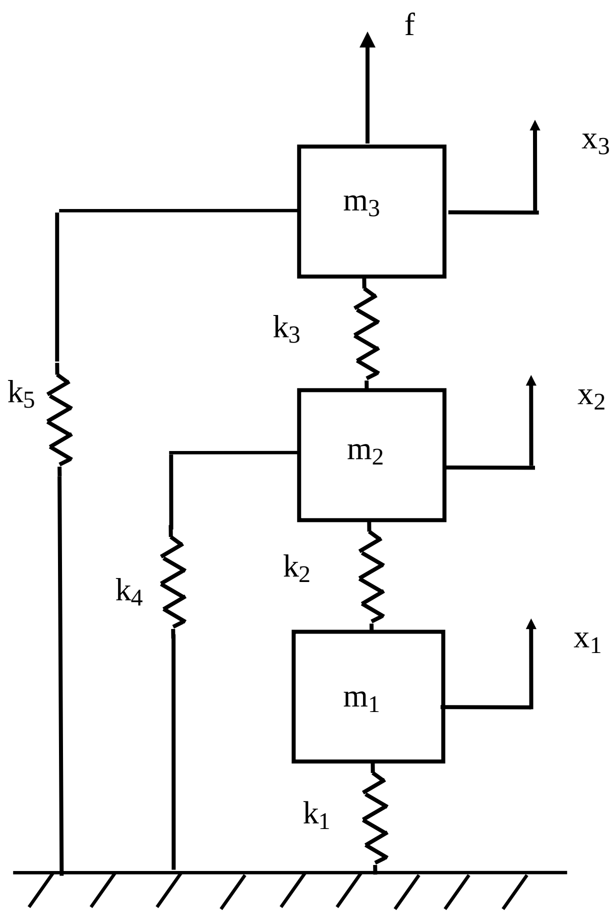

Figure A-1.

A three-degree-of-freedom system is shown in Figure A-1. First, determine the displacement using the full mode set. Then solve for two modes only. Finally solve using mode acceleration with two modes via equation (12). Compare the results at mass 3.

The parameters are

m1	0.0895	lbf sec^2/in
m2	0.0887	lbf sec^2/in
m3	0.0770	lbf sec^2/in
k1	1.8522e+04	lbf/in
k2	0.2157e+04	lbf/in
k3	0.2270e+04	lbf/in
k4	1.9429e+04	lbf/in
k5	1.7072e+04	lbf/in

The damping is 0.05 for all modes.

The force applied to mass 3 is: 1 lbf, 70 Hz sine function, 0.2 second duration.

The mass matrix is

$$\begin{bmatrix} 0.0895 & 0 & 0 \\ 0 & 0.0887 & 0 \\ 0 & 0 & 0.0770 \end{bmatrix} \text{ lbf sec}^2/\text{in} \tag{A-1}$$

The stiffness matrix is

$$\begin{bmatrix} 20679 & -2157 & 0 \\ -2157 & 23856 & -2270 \\ 0 & -2270 & 19342 \end{bmatrix} \text{ lbf/in} \tag{A-2}$$

The natural frequencies are

$$\begin{bmatrix} 73.639 & 78.277 & 86.476 \end{bmatrix} \text{ Hz} \tag{A-3}$$

The mode shapes are

$$\begin{bmatrix} 2.5194 & -1.9462 & 1.0190 \\ 1.7737 & 0.8756 & -2.7131 \\ 1.4090 & 2.7751 & 1.8167 \end{bmatrix} \tag{A-4}$$

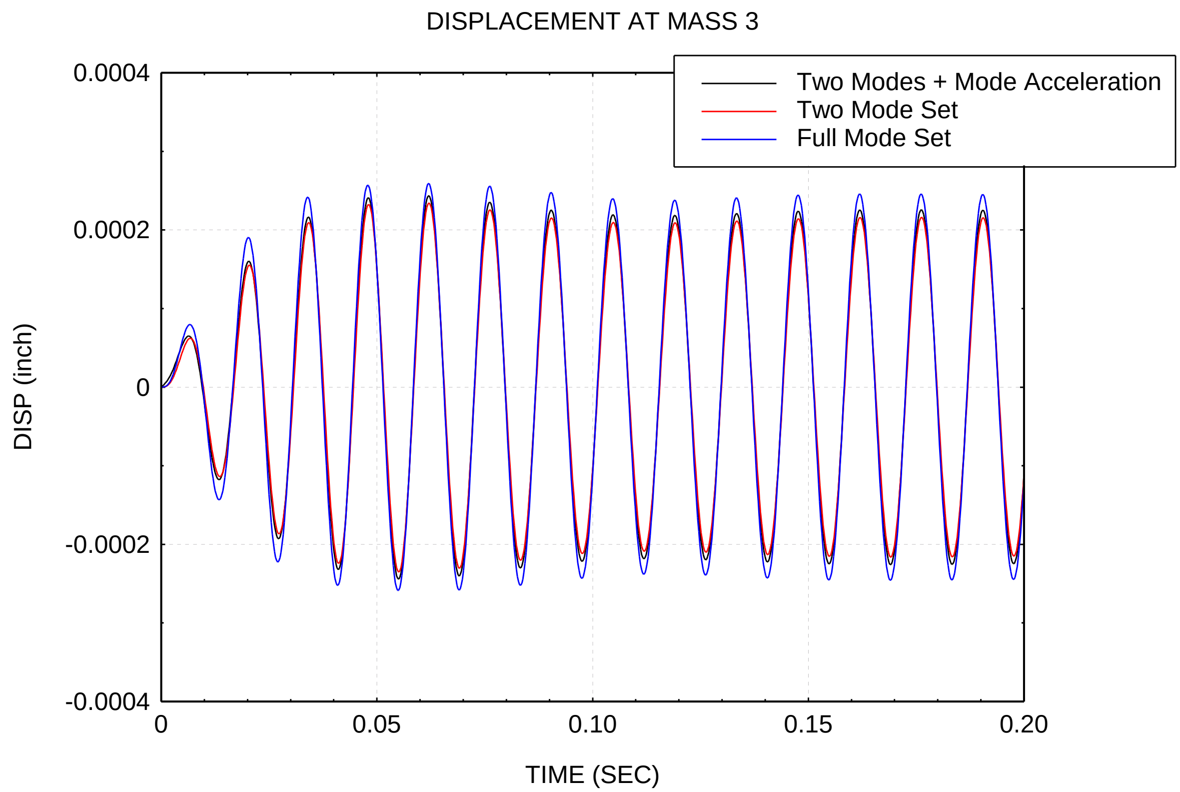

Figure A-2.

The displacements are calculated using: mdof_modal_arbit_force_newmark_MA.m

The Mode Acceleration method improves the "Two Modes" results. Further test cases are needed to evaluate the method.