EFFECTIVE MODAL MASS & MODAL PARTICIPATION FACTORS
Revision I

By Tom Irvine
Email: tom@vibrationdata.com

December 11, 2015

Introduction

The effective modal mass provides a method for judging the "significance" of a vibration mode.

Modes with relatively high effective masses can be readily excited by base excitation. On the other hand, modes with low effective masses cannot be readily excited in this manner.

Consider a modal transient or frequency response function analysis via the finite element method. Also consider that the system is a multi-degree-of-freedom system. For brevity, only a limited number of modes should be included in the analysis.

How many modes should be included in the analysis? Perhaps the number should be enough so that the total effective modal mass of the model is at least 90% of the actual mass.

Definitions

The equation definitions in this section are taken from Reference 1.

Consider a discrete dynamic system governed by the following equation

$$M\ddot{\bar{x}} + K\bar{x} = \bar{F} \tag{1}$$

where

- M is the mass matrix
- K is the stiffness matrix
- $\ddot{\bar{x}}$ is the acceleration vector
- $\bar{x}$ is the displacement vector
- $\bar{F}$ is the forcing function or base excitation function

A solution to the homogeneous form of equation (1) can be found in terms of eigenvalues and eigenvectors. The eigenvectors represent vibration modes.

Let ϕ be the eigenvector matrix.

The system's generalized mass matrix $\hat{m}$ is given by

$$\hat{m} = \phi^{T} M \phi \tag{2}$$

Let $\bar{r}$ be the influence vector which represents the displacements of the masses resulting from static application of a unit ground displacement. The influence vector induces a rigid body motion in all modes.

Define a coefficient vector $\bar{L}$ as

$$\bar{L} = \phi^{T} M \bar{r} \tag{3}$$

The modal participation factor matrix Γ_i for mode i is

$$\Gamma_i = \frac{\bar{L}_i}{\hat{m}_{ii}} \tag{4}$$

The effective modal mass $m_{eff,i}$ for mode i is

$$m_{eff,i} = \frac{\bar{L}_i^{\,2}}{\hat{m}_{ii}} \tag{5}$$

Note that $\hat{m}_{ii} = 1$ for each index if the eigenvectors have been normalized with respect to the mass matrix.

Furthermore, the off-diagonal modal mass ($\hat{m}_{ij}, i \neq j$) terms are zero regardless of the normalization and even if the physical mass matrix M has distributed mass. This is due to the orthogonality of the eigenvectors. The off-diagonal modal mass terms do not appear in equation (5), however. An example for a system with distributed mass is shown in Appendix F.

Example

Consider the two-degree-of-freedom system shown in Figure 1, with the parameters shown in Table 1.

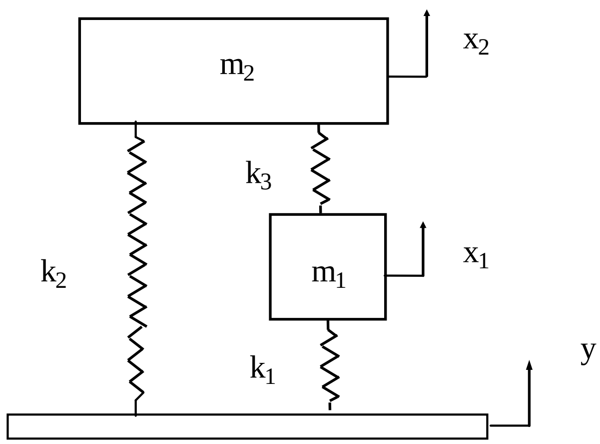

Figure 1.

Table 1. Parameters	
Variable	Value
m_1	2.0 kg
m_2	1.0 kg
k_1	1000 N/m
k_2	2000 N/m
k_3	3000 N/m

The homogeneous equation of motion is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} k_1+k_3 & -k_3 \\ -k_3 & k_2+k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

The mass matrix is

$$M=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\text{kg} \tag{7}$$

The stiffness matrix is

$$K = \begin{bmatrix} 4000 & -3000 \\ -3000 & 5000 \end{bmatrix} \ N/m \tag{8}$$

The eigenvalues and eigenvectors can be found using the method in Reference 2.

The eigenvalues are the roots of the following equation.

$$\det \left[K - \omega^2 M \right] = 0 \tag{9}$$

The eigenvalues are

$$\omega_1{}^2 = [901.9 \ \mathrm{rad/sec}]^2 \tag{10}$$

$$\omega_1 = 30.03 \ \ \mathrm{rad/sec} \tag{11}$$

$$f_1 = 4.78 \ \mathrm{Hz} \tag{12}$$

$$\omega_2{}^2 = [6098 \ \mathrm{rad/sec}]^2 \tag{13}$$

$$\omega_2 = 78.09 \ \mathrm{rad/sec} \tag{14}$$

$$f_2 = 12.4 \ \mathrm{Hz} \tag{15}$$

The eigenvector matrix is

$$\phi = \begin{bmatrix} 0.6280 & -0.3251 \\ 0.4597 & 0.8881 \end{bmatrix} \tag{16}$$

The eigenvectors were previously normalized so that the generalized mass is the identity matrix.

$$\hat{m} = \phi^{T} M \phi \tag{17}$$

$$\hat{m} = \begin{bmatrix} 0.6280 & 0.4597 \\ -0.3251 & 0.8881 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.6280 & -0.3251 \\ 0.4597 & 0.8881 \end{bmatrix} \tag{18}$$

$$\hat{m} = \begin{bmatrix} 0.6280 & 0.4597 \\ -0.3251 & 0.8881 \end{bmatrix} \begin{bmatrix} 1.2560 & -0.6502 \\ 0.4597 & 0.8881 \end{bmatrix} \tag{19}$$

$$\hat{m} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{20}$$

Again, $\bar{r}$ is the influence vector which represents the displacements of the masses resulting from static application of a unit ground displacement. For this example, each mass simply has the same static displacement as the ground displacement.

$$\bar{r} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{21}$$

The coefficient vector $\overline{L}$ is

$$\overline{L} = \phi^{T} M \, \bar{r} \tag{22}$$

$$\overline{L} = \begin{bmatrix} 0.6280 & 0.4597 \\ -0.3251 & 0.8881 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{23}$$

$$\bar{L} = \begin{bmatrix} 0.6280 & 0.4597 \\ -0.3251 & 0.8881 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{24}$$

$$\bar{L} = \begin{bmatrix} 1.7157 \\ -0.2379 \end{bmatrix} \text{ kg} \tag{25}$$

The modal participation factor Γ_i for mode i is

$$\Gamma_i = \frac{\bar{L}_i}{\hat{m}_{ii}} \tag{26}$$

The modal participation vector is thus

$$\Gamma = \begin{bmatrix} 1.7157 \\ -0.2379 \end{bmatrix} \tag{27}$$

The coefficient vector $\bar{L}$ and the modal participation vector Γ are identical in this example because the generalized mass matrix is the identity matrix.

The effective modal mass $m_{eff,i}$ for mode i is

$$m_{eff,i} = \frac{\bar{L}_i^{\;2}}{\hat{m}_{ii}} \tag{28}$$

For mode 1,

$$m_{eff,1} = \frac{[1.7157 \text{ kg}]^2}{1 \text{ kg}} \tag{29}$$

$$m_{eff,1} = 2.944 \text{ kg} \tag{30}$$

For mode 2,

$$m_{eff,2} = \frac{[-0.2379 \text{ kg}]^2}{1 \text{ kg}} \tag{31}$$

$$m_{eff,2} = 0.056 \text{ kg} \tag{32}$$

Note that

$$m_{eff,1} + m_{eff,2} = 2.944 \text{ kg} + 0.056 \text{ kg} \tag{33}$$

$$m_{eff,1} + m_{eff,2} = 3 \text{ kg} \tag{34}$$

Thus, the sum of the effective masses equals the total system mass.

Also, note that the first mode has a much higher effective mass than the second mode.

Thus, the first mode can be readily excited by base excitation. On the other hand, the second mode is negligible in this sense.

From another viewpoint, the center of gravity of the first mode experiences a significant translation when the first mode is excited.

On the other hand, the center of gravity of the second mode remains nearly stationary when the second mode is excited.

Each degree-of-freedom in the previous example was a translation in the X-axis. This characteristic simplified the effective modal mass calculation.

In general, a system will have at least one translation degree-of-freedom in each of three orthogonal axes. Likewise, it will have at least one rotational degree-of-freedom about each of three orthogonal axes. The effective modal mass calculation for a general system is shown by the example in Appendix A. The example is from a real-world problem.

Aside

An alternate definition of the participation factor is given in Appendix B.

References

1. M. Papadrakakis, N. Lagaros, V. Plevris; Optimum Design of Structures under Seismic Loading, European Congress on Computational Methods in Applied Sciences and Engineering, Barcelona, 2000.

2. T. Irvine, The Generalized Coordinate Method For Discrete Systems, Vibrationdata, 2000.

3. W. Thomson, Theory of Vibration with Applications 2nd Edition, Prentice Hall, New Jersey, 1981.

4. T. Irvine, Bending Frequencies of Beams, Rods, and Pipes, Rev M, Vibrationdata, 2010.

5. T. Irvine, Rod Response to Longitudinal Base Excitation, Steady-State and Transient, Rev B, Vibrationdata, 2009.

6. T. Irvine, Longitudinal Vibration of a Rod via the Finite Element Method, Revision B, Vibrationdata, 2008.

APPENDIX A

<u>Equation of Motion, Isolated Avionics Component</u>

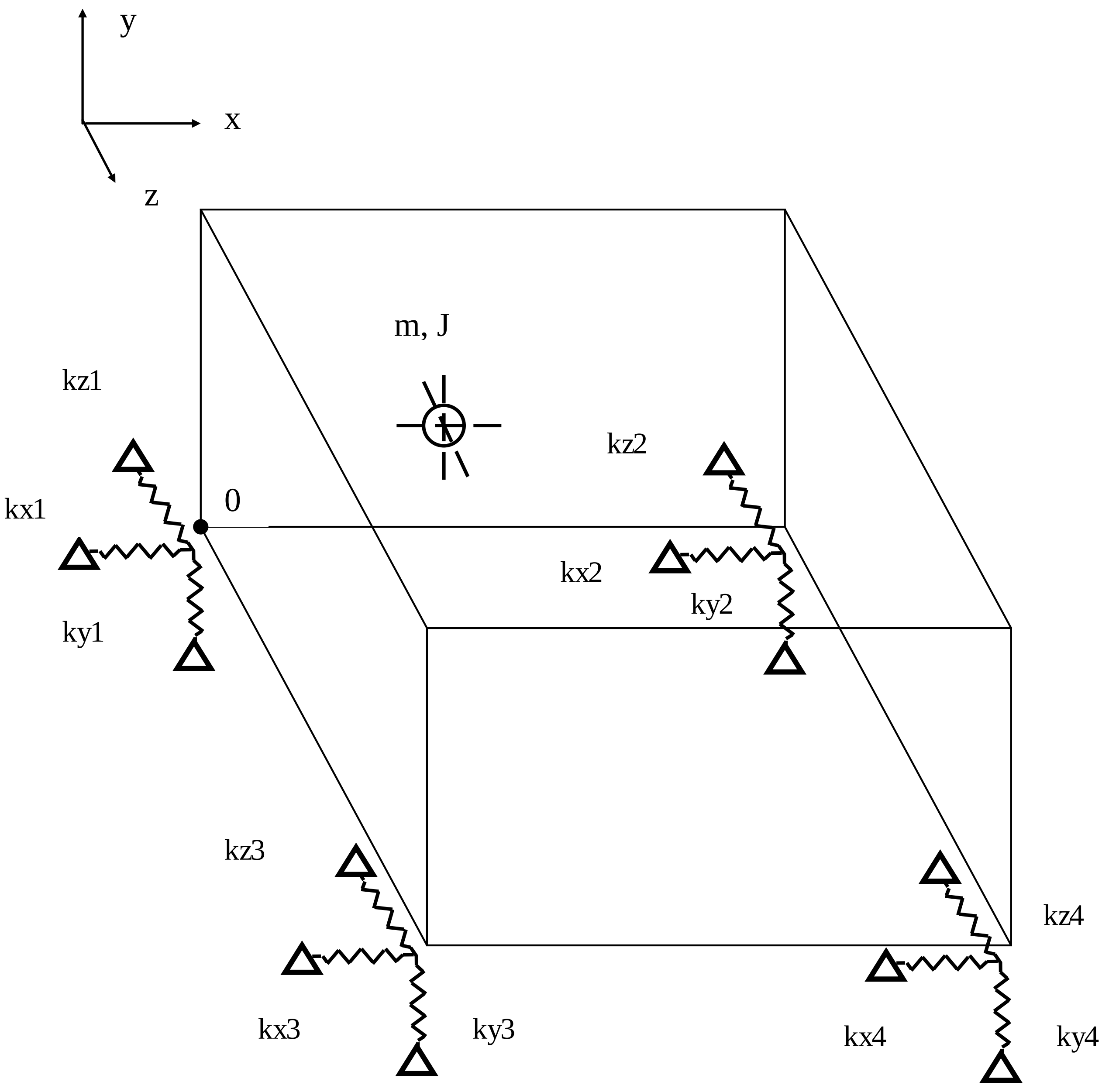

Figure A-1. Isolated Avionics Component Model

The mass and inertia are represented at a point with the circle symbol. Each isolator is modeled by three orthogonal DOF springs. The springs are mounted at each corner. The springs are shown with an offset from the corners for clarity. The triangles indicate fixed constraints. “0” indicates the origin.

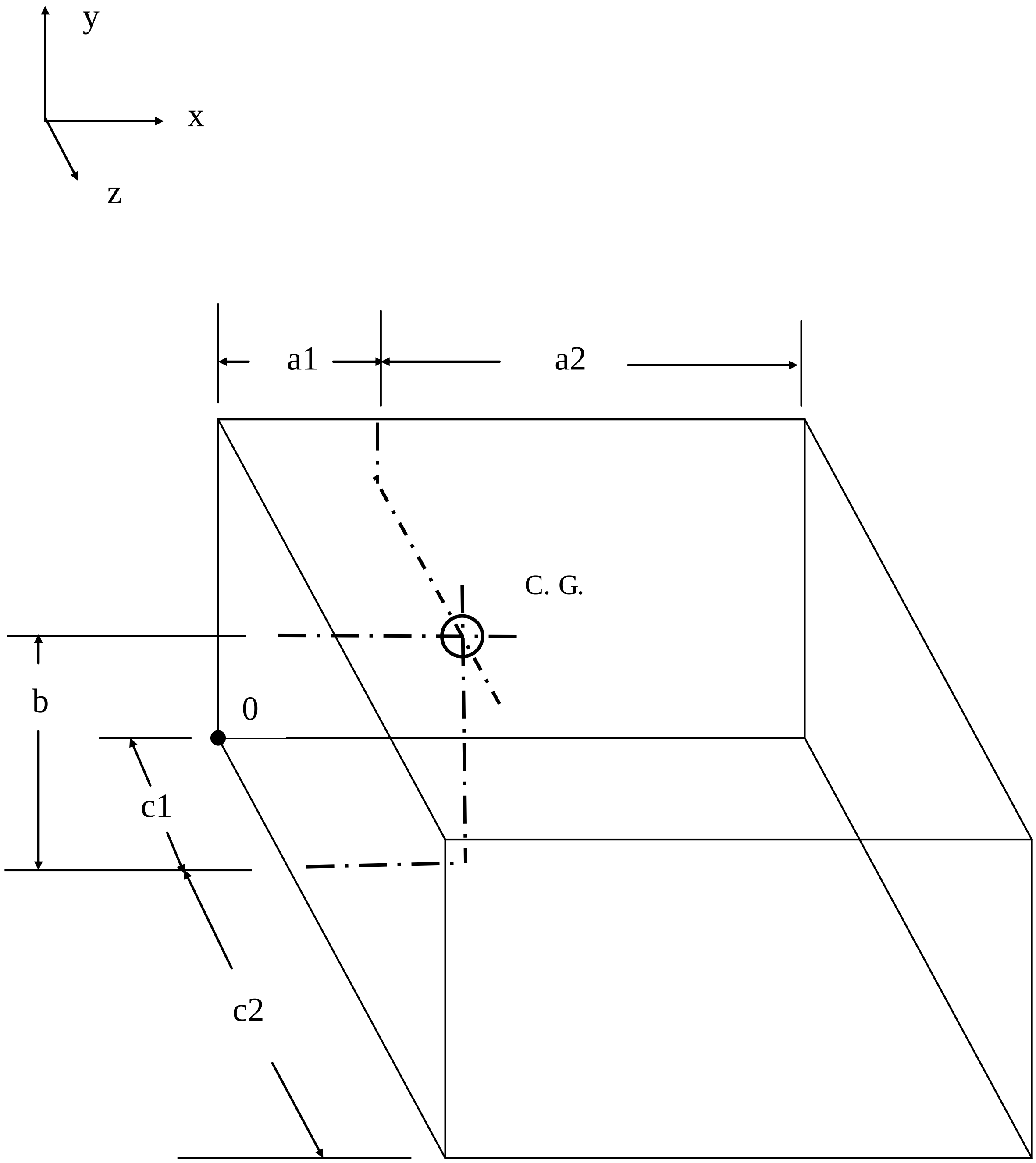

Figure A-2. Isolated Avionics Component Model with Dimensions

All dimensions are positive as long as the C.G. is “inside the box.” At least one dimension will be negative otherwise.

The mass and stiffness matrices are shown in upper triangular form due to symmetry.

$$\underline{M} = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ & m & 0 & 0 & 0 & 0 \\ & & m & 0 & 0 & 0 \\ & & & J_x & 0 & 0 \\ & & & & J_y & 0 \\ & & & & & J_z \end{bmatrix} \tag{A-1}$$

$$\underline{K} = \begin{bmatrix} 4k_x & 0 & 0 & 0 & 2k_x(-c_1+c_2) & 4k_x b \\ & 4k_y & 0 & 2k_y(c_1-c_2) & 0 & 2k_y(-a_1+a_2) \\ & & 4k_z & -4k_z b & 2k_z(a_1-a_2) & 0 \\ & & & 4k_z b^2+2k_y\left(c_1^2+c_2^2\right) & 2k_z(-a_1+a_2)b & k_y(-a_1+a_2)(c_1-c_2) \\ & & & & 2k_x\left(c_1^2+c_2^2\right)+2k_z\left(a_1^2+a_2^2\right) & 2k_x(-c_1+c_2)b \\ & & & & & 4k_x b^2+2k_y\left(a_1^2+a_2^2\right) \end{bmatrix} \tag{A-2}$$

The equation of motion is

$$\underline{M}\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\\ \ddot{\alpha}\\ \ddot{\beta}\\ \ddot{\theta}\end{bmatrix}+\underline{K}\begin{bmatrix}x\\ y\\ z\\ \alpha\\ \beta\\ \theta\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \tag{A-3}$$

The variables α, β and θ represent rotations about the X, Y, and Z axes, respectively.

Example

A mass is mounted to a surface with four isolators. The system has the following properties.

M	=	4.28 lbm
Jx	=	44.9 lbm in^2
Jy	=	39.9 lbm in^2
Jz	=	18.8 lbm in^2
kx	=	80 lbf/in
ky	=	80 lbf/in
kz	=	80 lbf/in
a1	=	6.18 in
a2	=	-2.68 in
b	=	3.85 in
c1	=	3. in
c2	=	3. in

Let $\bar{r}$ be the influence matrix which represents the displacements of the masses resulting from static application of unit ground displacements and rotations. *The influence matrix for this example is the identity matrix provided that the C.G is the reference point.*

$$\bar{r} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A-4}$$

The coefficient matrix $\bar{L}$ is

$$\bar{L} = \phi^T M \, \bar{r} \tag{A-5}$$

The modal participation factor matrix Γ_i for mode i at dof j is

$$\Gamma_{ij} = \frac{\bar{L}_{ij}}{\hat{m}_{ii}} \tag{A-6}$$

Each $\hat{m}_{ii}$ coefficient is 1 if the eigenvectors have been normalized with respect to the mass matrix.

The effective modal mass $m_{eff,i}$ vector for mode i and dof j is

$$m_{eff,ij} = \frac{\left[\bar{L}_{ij}\right]^2}{\hat{m}_{ii}} \tag{A-7}$$

The natural frequency results for the sample problem are calculated using the program: six_dof_iso.m.

The results are given in the next pages.

```
six_dof_iso.m   ver 1.2  March 31, 2005

 by Tom Irvine   Email: tomirvine@aol.com

 This program finds the eigenvalues and eigenvectors for a
 six-degree-of-freedom system.

 Refer to six_dof_isolated.pdf for a diagram.

 The equation of motion is:   M (d^2x/dt^2) + K x = 0

 Enter m (lbm)
 4.28
 Enter Jx (lbm in^2)
 44.9
 Enter Jy (lbm in^2)
 39.9
 Enter Jz (lbm in^2)
 18.8

 Note that the stiffness values are for individual springs

 Enter kx (lbf/in)
 80

 Enter ky (lbf/in)
 80

 Enter kz (lbf/in)
 80

 Enter a1 (in)
 6.18

 Enter a2 (in)
 -2.68

 Enter b (in)
 3.85

 Enter c1 (in)
 3
```

```
Enter c2 (in)
3


 The mass matrix is

m =

    0.0111         0         0         0         0         0
         0    0.0111         0         0         0         0
         0         0    0.0111         0         0         0
         0         0         0    0.1163         0         0
         0         0         0         0    0.1034         0
         0         0         0         0         0    0.0487


 The stiffness matrix is

k =

  1.0e+004 *

    0.0320         0         0         0         0    0.1232
         0    0.0320         0         0         0   -0.1418
         0         0    0.0320   -0.1232    0.1418         0
         0         0   -0.1232    0.7623   -0.5458         0
         0         0    0.1418   -0.5458    1.0140         0
    0.1232   -0.1418         0         0         0    1.2003


 Eigenvalues

lambda =

  1.0e+005 *

    0.0213    0.0570    0.2886    0.2980    1.5699    2.7318
```

Natural Frequencies =

1. 7.338 Hz
2. 12.02 Hz
3. 27.04 Hz
4. 27.47 Hz
5. 63.06 Hz
6. 83.19 Hz

Modes Shapes (rows represent modes)

	x	y	z	alpha	beta	theta
1.	5.91	-6.81	0	0	0	-1.42
2.	0	0	8.69	0.954	-0.744	0
3.	7.17	6.23	0	0	0	0
4.	0	0	1.04	-2.26	-1.95	0
5.	0	0	-3.69	1.61	-2.3	0
6.	1.96	-2.25	0	0	0	4.3

Participation Factors (rows represent modes)

	x	y	z	alpha	beta	theta
1.	0.0656	-0.0755	0	0	0	-0.0693
2.	0	0	0.0963	0.111	-0.0769	0
3.	0.0795	0.0691	0	0	0	0
4.	0	0	0.0115	-0.263	-0.202	0
5.	0	0	-0.0409	0.187	-0.238	0
6.	0.0217	-0.025	0	0	0	0.21

Effective Modal Mass (rows represent modes)

	x	y	z	alpha	beta	theta
1.	0.0043	0.00569	0	0	0	0.0048
2.	0	0	0.00928	0.0123	0.00592	0
3.	0.00632	0.00477	0	0	0	0
4.	0	0	0.000133	0.069	0.0408	0
5.	0	0	0.00168	0.035	0.0566	0
6.	0.000471	0.000623	0	0	0	0.0439

Total Modal Mass

0.0111	0.0111	0.0111	0.116	0.103	0.0487

APPENDIX B

Modal Participation Factor for Applied Force

The following definition is taken from Reference 3. Note that the mode shape functions are unscaled. Hence, the participation factor is unscaled.

Consider a beam of length L loaded by a distributed force p(x,t).

Consider that the loading per unit length is separable in the form

$$p(x,t) = \frac{P_o}{L} p(x) f(t) \tag{B-1}$$

The modal participation factor Γ_i for mode i is defined as

$$\Gamma_i = \frac{1}{L} \int_0^L p(x) \phi_i(x) dx \tag{B-2}$$

where

$\phi_i(x)$ is the normal mode shape for mode i

APPENDIX C

Modal Participation Factor for a Beam

Let

$Y_n(x)$	=	mass-normalized eigenvectors
$m(x)$	=	mass per length

The participation factor is

$$\Gamma_n = \int_0^L m(x)\, Y_n(x) dx \tag{C-1}$$

The effective modal mass is

$$m_{eff,\,n} = \frac{\left[\int_0^L m(x)\, Y_n(x) dx \right]^2}{\int_0^L m(x) [Y_n(x)]^2\, dx} \tag{C-2}$$

The eigenvectors should be normalized such that

$$\int_0^L m(x) [Y_n(x)]^2\, dx = 1 \tag{C-3}$$

Thus,

$$m_{eff,\,n} = [\Gamma_n]^2 = \left[\int_0^L m(x)\, Y_n(x) dx \right]^2 \tag{C-4}$$

APPENDIX D

Effective Modal Mass Values for Bernoulli-Euler Beams

The results are calculated using formulas from Reference 4. The variables are

E = is the modulus of elasticity

I = is the area moment of inertia

L = is the length

ρ = is (mass/length)

Table D-1. Bending Vibration, Beam Simply-Supported at Both Ends

Mode	Natural Frequency ω_n	Participation Factor	Effective Modal Mass
1	$\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	$\frac{2}{\pi}\sqrt{2\rho L}$	$\frac{8}{\pi^2}\rho L$
2	$4\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	0	0
3	$9\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	$\frac{2}{3\pi}\sqrt{2\rho L}$	$\frac{8}{9\pi^2}\rho L$
4	$16\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	0	0
5	$25\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	$\frac{2}{5\pi}\sqrt{2\rho L}$	$\frac{8}{25\pi^2}\rho L$
6	$36\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	0	0
7	$49\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	$\frac{2}{7\pi}\sqrt{2\rho L}$	$\frac{8}{49\pi^2}\rho L$

95% of the total mass is accounted for using the first seven modes.

Table D-2. Bending Vibration, Fixed-Free Beam			
Mode	Natural Frequency ω_n	Participation Factor	Effective Modal Mass
1	$\left[\frac{1.87510}{L}\right]^2 \sqrt{\frac{EI}{\rho}}$	$0.7830 \sqrt{\rho L}$	$0.6131\ \rho L$
2	$\left[\frac{4.69409}{L}\right]^2 \sqrt{\frac{EI}{\rho}}$	$0.4339 \sqrt{\rho L}$	$0.1883\ \rho L$
3	$\left[\frac{5\pi}{2L}\right]^2 \sqrt{\frac{EI}{\rho}}$	$0.2544 \sqrt{\rho L}$	$0.06474\ \rho L$
4	$\left[\frac{7\pi}{2L}\right]^2 \sqrt{\frac{EI}{\rho}}$	$0.1818 \sqrt{\rho L}$	$0.03306\ \rho L$

90% of the total mass is accounted for using the first four modes.

APPENDIX E

Rod, Longitudinal Vibration, Classical Solution

The results are taken from Reference 5.

Table E-1. Longitudinal Vibration of a Rod, Fixed-Free			
Mode	Natural Frequency ω_n	Participation Factor	Effective Modal Mass
1	$0.5\ \pi\ c\ /\ L$	$\frac{2}{\pi}\sqrt{2\rho L}$	$\frac{8}{\pi^2}\rho L$
2	$1.5\ \pi\ c\ /\ L$	$\frac{2}{3\pi}\sqrt{2\rho L}$	$\frac{8}{9\pi^2}\rho L$
3	$2.5\ \pi\ c\ /\ L$	$\frac{2}{5\pi}\sqrt{2\rho L}$	$\frac{8}{25\pi^2}\rho L$

The longitudinal wave speed c is

$$c = \sqrt{\frac{E}{\rho}} \tag{E-1}$$

93% of the total mass is accounted for by using the first three modes.

APPENDIX F

This example shows a system with distributed or consistent mass matrix.

Rod, Longitudinal Vibration, Finite Element Method

Consider an aluminum rod with 1 inch diameter and 48 inch length. The rod has fixed-free boundary conditions.

A finite element model of the rod is shown in Figure F-1. It consists of four elements and five nodes. Each element has an equal length.

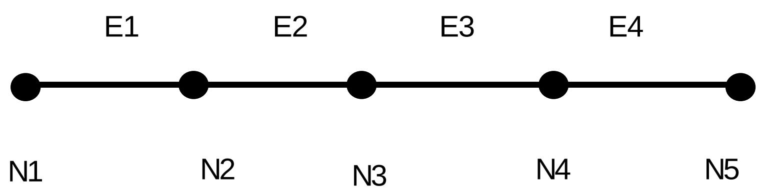

Figure F-1.

The boundary conditions are

$$U(0) = 0 \qquad \text{(Fixed end)} \qquad \text{(F-1)}$$

$$\left.\frac{dU}{dx}\right|_{x=L} = 0 \qquad \text{(Free end)} \qquad \text{(F-2)}$$

The natural frequencies and modes are determined using the finite element method in Reference 6.

The resulting eigenvalue problem for the constrained system has the following mass and stiffness matrices as calculated via Matlab script: rod_FEA.m.

Mass =

```
    0.0016    0.0004         0         0
    0.0004    0.0016    0.0004         0
         0    0.0004    0.0016    0.0004
         0         0    0.0004    0.0008
```

Stiffness =

```
  1.0e+006 *

    1.3090   -0.6545         0         0
   -0.6545    1.3090   -0.6545         0
         0   -0.6545    1.3090   -0.6545
         0         0   -0.6545    0.6545
```

The natural frequencies are

```
n         fn(Hz)
1         1029.9
2         3248.8
3         5901.6
4         8534.3
```

The mass-normalized eigenvectors in column format are

```
    5.5471   14.8349   18.0062   -9.1435
   10.2496   11.3542  -13.7813   16.8950
   13.3918   -6.1448   -7.4584  -22.0744
   14.4952  -16.0572   19.4897   23.8931
```

Let $\bar{r}$ be the influence vector which represents the displacements of the masses resulting from static application of a unit ground displacement.

The influence vector for the sample problem is

```
r̄ =
      1
      1
      1
      1
```

The coefficient vector $\bar{L}$ is

$$\bar{L} = \phi^T M \bar{r} \tag{F-3}$$

where

ϕ^T	=	transposed eigenvector matrix
M	=	mass matrix

The coefficient vector for the sample problem is

```
L̄ =

    0.0867
    0.0233
    0.0086
   -0.0021
```

The modal participation factor matrix Γ_i for mode i is

$$\Gamma_i = \frac{\bar{L}_i}{\hat{m}_{ii}} \tag{F-4}$$

Note that $\hat{m}_{ii} = 1$ for each index since the eigenvectors have been previously normalized with respect to the mass matrix.

Thus, for the sample problem,

$$\Gamma_i = \bar{L}_i \tag{F-5}$$

The effective modal mass $m_{eff,i}$ for mode i is

$$m_{eff,i} = \frac{\bar{L}_i^{\,2}}{\hat{m}_{ii}} \tag{F-6}$$

Again, the eigenvectors are mass normalized.

Thus

$$m_{eff,i} = \bar{L}_i^{\,2} \tag{F-7}$$

The effective modal mass for the sample problem is

m_{eff} =

```
0.0075
0.0005
0.0001
0.0000
```

The model's total modal mass is 0.0081 lbf sec^2/in. This is equivalent to 3.14 lbm.

The true mass or the rod is 3.77 lbm.

Thus, the four-element model accounts for 83% of the true mass. This percentage can be increased by using a larger number of elements with corresponding shorter lengths.

APPENDIX G

Two-degree-of-freedom System, Static Coupling

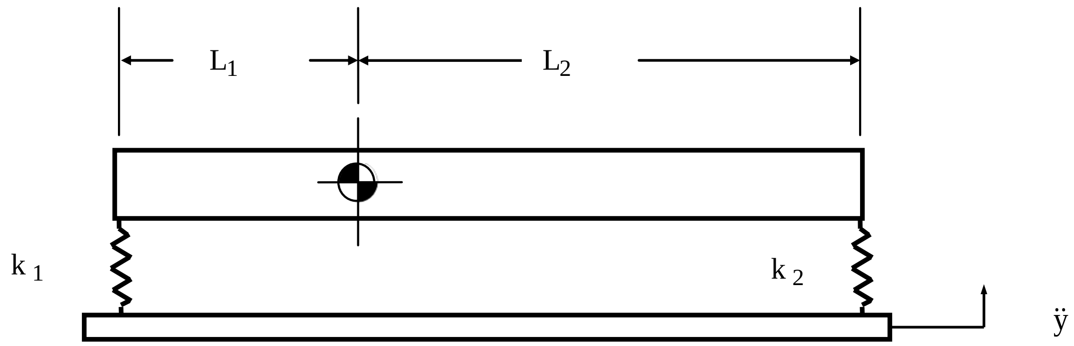

Figure G-1.

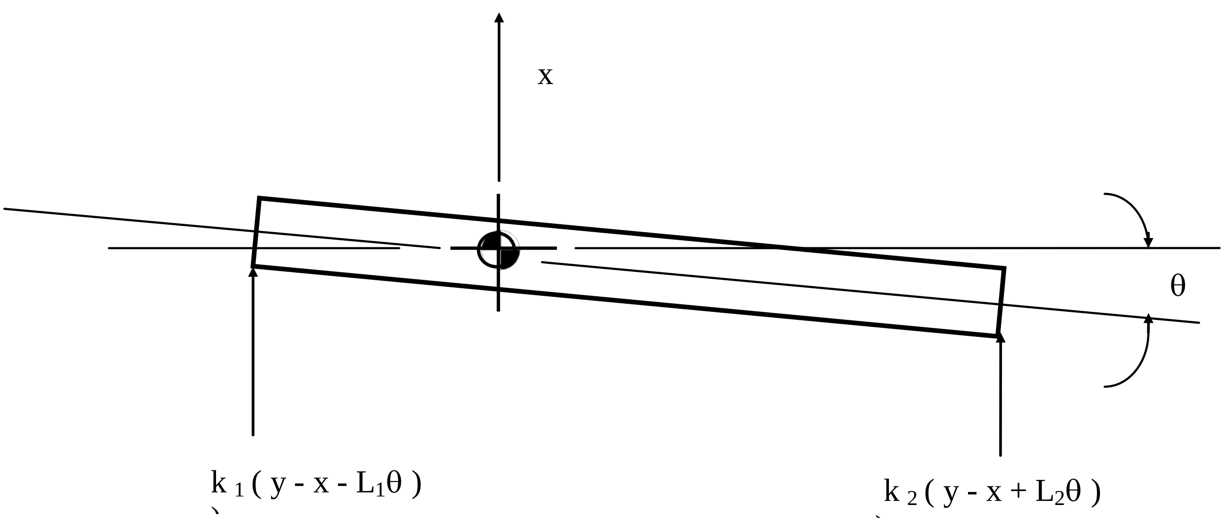

Figure G-2.

The free-body diagram is given in Figure G-2.

The system has a CG offset if $L_1 \neq L_2$.

The system is statically coupled if $k_1 L_1 \neq k_2 L_2$.

The rotation is positive in the clockwise direction.

The variables are

y	is the base displacement
x	is the translation of the CG
θ	is the rotation about the CG
m	is the mass
J	is the polar mass moment of inertia
k_i	is the stiffness for spring i
z_i	is the relative displacement for spring i

Sign Convention:

Translation: upward in vertical axis is positive.
Rotation: clockwise is positive.

Sum the forces in the vertical direction

$$\sum F = m\ddot{x} \tag{G-1}$$

$$m\ddot{x} = k_1(y - x - L_1\theta) + k_2(y - x + L_2\theta) \tag{G-2}$$

$$m\ddot{x} + k_1(-y + x + L_1\theta) + k_2(-y + x - L_2\theta) = 0 \tag{G-3}$$

$$m\ddot{x} - k_1 y + k_1 x + k_1 L_1 \theta - k_2 y + k_2 x - k_2 L_2 \theta = 0 \tag{G-4}$$

$$m\ddot{x} + (k_1 + k_2)x + (k_1 L_1 - k_2 L_2)\theta = (k_1 + k_2)\, y \tag{G-5}$$

Sum the moments about the center of mass.

$$\sum M = J\ddot{\theta} \tag{G-6}$$

$$J\ddot{\theta} = +k_1 L_1 (y - x - L_1\theta) - k_2 L_2 (y - x + L_2\theta) \tag{G-7}$$

$$J\ddot{\theta} + k_1 L_1 (-y + x + L_1\theta) + k_2 L_2 (y - x + L_2\theta) = 0 \tag{G-8}$$

$$J\ddot{\theta} - k_1 y + k_1 L_1 x + k_1 L_1^2 \theta + k_2 L_2 y - k_2 L_2 x + k_2 L_2^2 \theta = 0 \tag{G-9}$$

$$J\ddot{\theta} + [k_1 L_1 - k_2 L_2]x + \left[k_1 L_1^2 + k_2 L_2^2\right]\theta = (k_1 L_1 - k_2 L_2)y \tag{G-10}$$

The equations of motion are

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ k_1 L_1 - k_2 L_2 \end{bmatrix} y$$

(G-11)

The pseudo-static problem is

$$\begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ k_1 L_1 - k_2 L_2 \end{bmatrix} y \tag{G-12}$$

Solve for the influence vector r by applying a unit displacement.

$$\begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ k_1 L_1 - k_2 L_2 \end{bmatrix} \tag{G-13}$$

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{G-14}$$

Define a relative displacement z.

$$z = x - y \tag{G-15}$$

$$x = z + y \tag{G-16}$$

$$\begin{aligned} &\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{z} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} m\ddot{y} \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix} \begin{bmatrix} z \\ \theta \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} k_1 + k_2 \\ k_1 L_1 - k_2 L_2 \end{bmatrix} [\, y \,] \end{aligned} \tag{G-17}$$

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{z} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix} \begin{bmatrix} z \\ \theta \end{bmatrix} = - \begin{bmatrix} m\ddot{y} \\ 0 \end{bmatrix}$$

(G-18)

The equation is more formally

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{z} \\ \ddot{\theta} \end{bmatrix}+\begin{bmatrix} k_1+k_2 & k_1L_1-k_2L_2 \\ k_1L_1-k_2L_2 & k_1L_1^2+k_2L_2^2 \end{bmatrix}\begin{bmatrix} z \\ \theta \end{bmatrix}=-\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\ddot{y}$$

(G-19)

Solve for the eigenvalues and mass-normalized eigenvectors matrix ϕ using the homogeneous problem form of equation (G-9).

Define modal coordinates

$$\begin{bmatrix} z \\ \theta \end{bmatrix}=[\phi]\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{G-20}$$

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}[\phi]\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix}+\begin{bmatrix} k_1+k_2 & k_1L_1-k_2L_2 \\ k_1L_1-k_2L_2 & k_1L_1^2+k_2L_2^2 \end{bmatrix}[\phi]\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}=-\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\ddot{y}$$

(G-21)

Then premultiply by the transpose of the eigenvector matrix ϕ^T.

$$\left[\phi^T\right]\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}[\phi]\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix}+\left[\phi^T\right]\begin{bmatrix} k_1+k_2 & k_1L_1-k_2L_2 \\ k_1L_1-k_2L_2 & k_1L_1^2+k_2L_2^2 \end{bmatrix}[\phi]\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$=-\left[\phi^T\right]\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\ddot{y}$$

(G-22)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = -\left[\phi^T\right]\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\ddot{y} \tag{G-23}$$

The participation factor vector is

$$\Gamma = \phi^T \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \tag{G-24}$$

$$\Gamma = \phi^T \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \phi^T \begin{bmatrix} m \\ 0 \end{bmatrix} \tag{G-25}$$

<u>Example</u>

Consider the system in Figure G-1. Assign the following values. The values are based on a slender rod, aluminum, diameter =1 inch, total length=24 inch.

Table G-1. Parameters	
Variable	Value
m	18.9 lbm
J	907 lbm in^2
k_1	20,000 lbf/in
k_2	20,000 lbf/in
L_1	8 in
L_2	16 in

The following parameters were calculated for the sample system via a Matlab script.

```
The mass matrix is
```

```
m =

    0.0490         0
         0    2.3497


 The stiffness matrix is

k =

       40000      160000
      160000     6400000


  Natural Frequencies =

     133.8 Hz
     267.9 Hz

  Modes Shapes (column format) =

         -4.4            1.029
       0.1486           0.6352

 Participation Factors =

    0.2156
    0.0504

 Effective Modal Mass

    0.0465
    0.0025
```

The total modal mass is 0.0490 lbf sec^2/in, equivalent to18.9 lbm.

APPENDIX H

Two-degree-of-freedom System, Static & Dynamic Coupling

Repeat the example in Appendix G, but use the left end as the coordinate reference point.

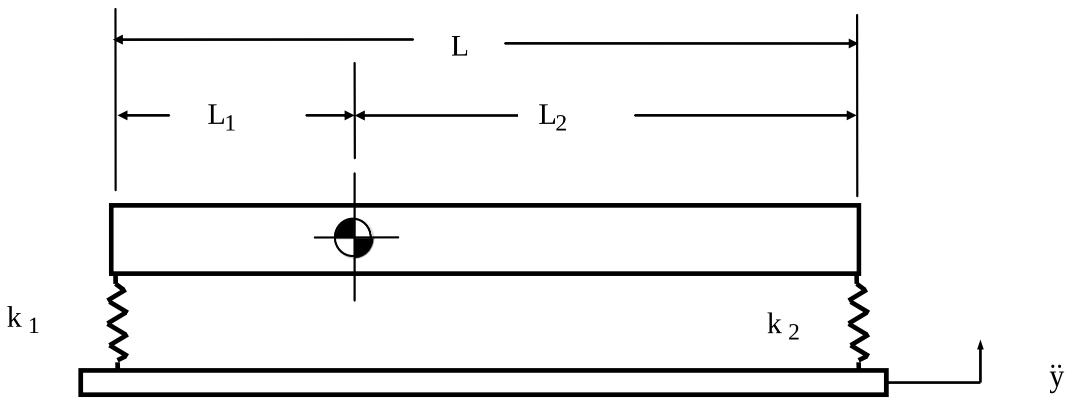

Figure H-1.

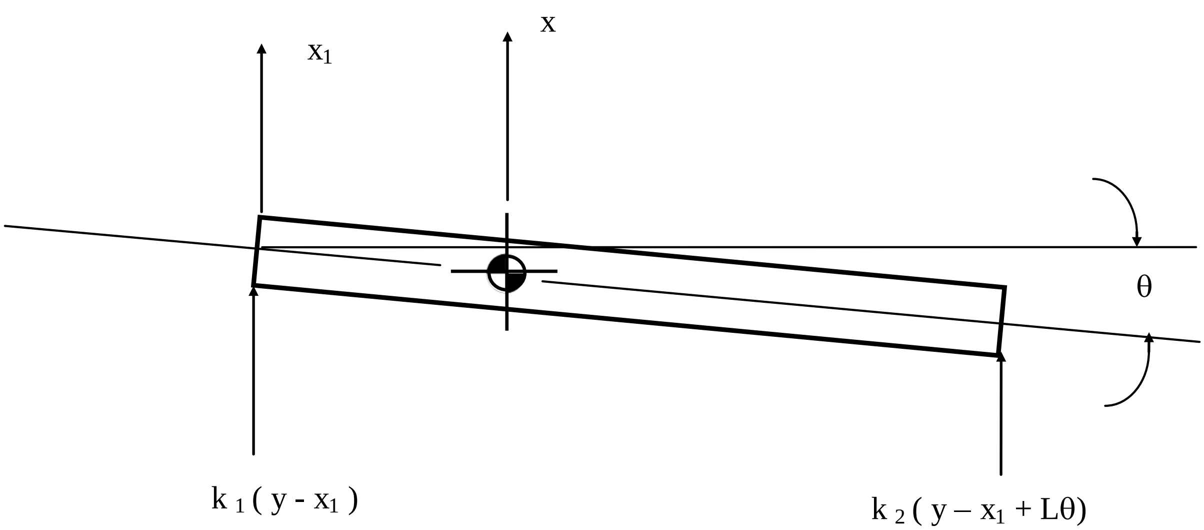

Figure H-2.

The free-body diagram is given in Figure H-2. Again, the displacement and rotation are referenced to the left end.

Sign Convention:

Translation: upward in vertical axis is positive.
Rotation: clockwise is positive.

Sum the forces in the vertical direction

$$\sum F = m\ddot{x} \tag{H-1}$$

$$m\ddot{x} = k_1(y - x_1) + k_2(y - x_1 + L\theta) \tag{H-2}$$

$$m\ddot{x} + k_1(-y + x_1) + k_2(-y + x_1 - L\theta) = 0 \tag{H-3}$$

$$m\ddot{x} - k_1 y + k_1 x_1 - k_2 y + k_2 x_1 - k_2 L\theta = 0 \tag{H-4}$$

$$m\ddot{x} + (k_1 + k_2)x_1 - k_2 L\theta = (k_1 + k_2)\,y \tag{H-5}$$

$$x = x_1 - L_1\theta \tag{H-6}$$

$$m\left(\ddot{x}_1 - L_1\ddot{\theta}\right) + (k_1 + k_2)x_1 - k_2 L\theta = (k_1 + k_2)\,y \tag{H-7}$$

$$m\ddot{x}_1 - mL_1\ddot{\theta} + (k_1 + k_2)x_1 - k_2 L\theta = (k_1 + k_2)\,y \tag{H-8}$$

Sum the moments about the left end.

$$\sum M_1 = J_1\ddot{\theta} \tag{H-9}$$

$$J_1\ddot{\theta} = -k_2 L(y - x_1 + L\theta) - mL_1(\ddot{x} - \ddot{x}_1) \tag{H-10}$$

$$J_1\ddot{\theta} + mL_1(\ddot{x} - \ddot{x}_1) + k_2 L(y - x_1 + L\theta) = 0 \tag{H-11}$$

$$J_1\ddot{\theta} + mL_1(\ddot{x} - \ddot{x}_1) + k_2 Ly - k_2 Lx_1 + k_2 L^2\theta = 0 \tag{H-12}$$

$$J_1\ddot{\theta} + mL_1(\ddot{x} - \ddot{x}_1) - k_2 Lx_1 + k_2 L^2\theta = -k_2 Ly \tag{H-13}$$

$$x = x_1 - L_1\theta \tag{H-14}$$

$$J_1\ddot{\theta} + mL_1\left(\ddot{x}_1 - L_1\ddot{\theta} - \ddot{x}_1\right) - k_2Lx_1 + k_2L^2\theta = -k_2Ly \tag{H-15}$$

$$J_1\ddot{\theta} - mL_1\ddot{x}_1 - k_2Lx_1 + k_2L^2\theta = -k_2Ly \tag{H-16}$$

The equations of motion are

$$\begin{bmatrix} m & -mL_1 \\ -mL_1 & J_1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2L \\ -k_2L & k_2L^2 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ -k_2L \end{bmatrix} y \tag{H-17}$$

Note that

$$J_1 = J + mL_1^2 \tag{H-18}$$

$$\begin{bmatrix} m & -mL_1 \\ -mL_1 & J + mL_1^2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2L \\ -k_2L & k_2L^2 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ -k_2L \end{bmatrix} y \tag{H-19}$$

The pseudo-static problem is

$$\begin{bmatrix} k_1 + k_2 & -k_2L \\ -k_2L & k_2L^2 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ -k_2L \end{bmatrix} y \tag{H-20}$$

Solve for the influence vector r by applying a unit displacement.

$$\begin{bmatrix} k_1 + k_2 & -k_2L \\ -k_2L & k_2L^2 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ -k_2L \end{bmatrix} \tag{H-21}$$

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{H-22}$$

The influence coefficient vector is the same as that in Appendix G.

The natural frequencies are obtained via a Matlab script. The results are:

```
Natural Frequencies
 No.       f(Hz)
1.         133.79
2.         267.93

  Modes Shapes (column format)

ModeShapes =

    5.5889    4.0527
    0.1486    0.6352


 Participation Factors =

    0.2155
  -0.05039

 Effective Modal Mass =

   0.04642
  0.002539
```

The total modal mass is 0.0490 lbf sec^2/in, equivalent to18.9 lbm.