BENDING FREQUENCY OF A BEAM ON AN ELASTIC FOUNDATION
Revision B

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## Introduction

Consider a free-free beam that is mounted on an elastic foundation.

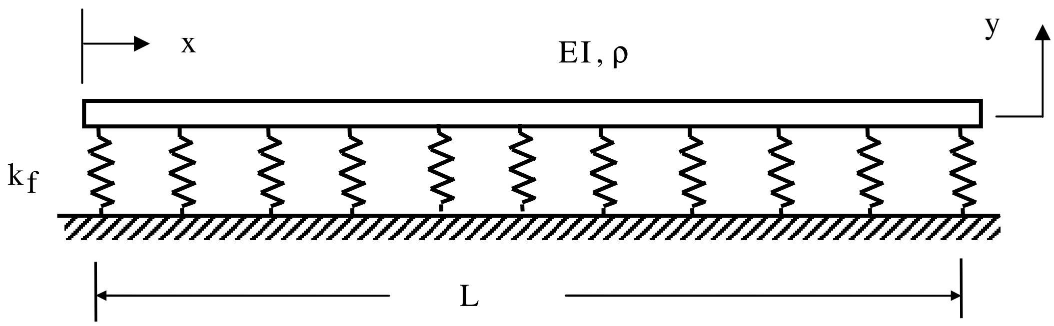



Figure 1.

The governing differential equation is

$$EI\frac{\partial^4 y}{\partial x^4} + k_f\, y = -\rho\,\frac{\partial^2 y}{\partial t^2} \tag{1}$$

where

| | |
|---|---|
| E | is the modulus of elasticity |
| I | is the area moment of inertia |
| L | is the length |
| $\rho$ | is the mass density (mass/length) |
| $k_f$ | is the foundation stiffness per length (force/length^2) |

Note that this equation neglects shear deformation and rotary inertia.

Separate the dependent variable.

$$y(x,t) = Y(x)T(t) \tag{2}$$

$$EI\frac{\partial^4[Y(x)T(t)]}{\partial x^4} + k_f\, Y(x)T(t) = -\rho\frac{\partial^2[Y(x)T(t)]}{\partial t^2} \tag{3}$$

$$T(t)\left\{EI\frac{d^4}{dx^4}Y(x) + k_f\, Y(x)\right\} = -\rho Y(x)\left\{\frac{d^2}{dt^2}T(t)\right\} \tag{4}$$

$$\left\{\frac{-1}{\rho}\right\}\frac{\left\{EI\frac{d^4}{dx^4}Y(x) + k_f\, Y(x)\right\}}{Y(x)} = \frac{\left\{\frac{d^2}{dt^2}T(t)\right\}}{T(t)} \tag{5}$$

Let c be a constant

$$\left\{\frac{-1}{\rho}\right\}\frac{\left\{EI\frac{d^4}{dx^4}Y(x) + k_f\, Y(x)\right\}}{Y(x)} = \frac{\left\{\frac{d^2}{dt^2}T(t)\right\}}{T(t)} = -c^2 \tag{6}$$

Separate the time variable.

$$\frac{\left\{\frac{d^2}{dt^2}T(t)\right\}}{T(t)} = -c^2 \tag{7}$$

$$\frac{d^2}{dt^2}T(t) + c^2\,T(t) = 0 \tag{8}$$

Propose a solution

$$T(t) = a\sin(\omega t) + b\cos(\omega t) \tag{9}$$

$$\frac{d}{dt}T(t) = a\,\omega\cos(\omega t) - b\,\omega\sin(\omega t) \tag{10}$$

$$\frac{d^2}{dt^2}T(t) = -a\omega^2\sin(\omega t) - b\omega^2\cos(\omega t) \tag{11}$$

$$-a\omega^2\sin(\omega t) - b\omega^2\cos(\omega t) + c^2[a\sin(\omega t) + b\cos(\omega t)] = 0 \tag{12}$$

$$c = \omega \tag{13}$$

Separate the spatial variable.

$$\left\{\frac{-1}{\rho}\right\}\frac{\left\{EI\frac{d^4}{dx^4}Y(x) + k_f\,Y(x)\right\}}{Y(x)} = -\omega^2 \tag{14}$$

$$EI\frac{d^4}{dx^4}Y(x) + k_f\,Y(x) - \rho\,\omega^2\,Y(x) = 0 \tag{15}$$

$$\frac{d^4}{dx^4}Y(x)+\left[\frac{1}{EI}\right]\left[k_f-\rho\,\omega^2\right]Y(x)=0 \tag{16}$$

A solution for equation (16) is

$$Y(x)=a_1\sinh(\beta x)+a_2\cosh(\beta x)+a_3\sin(\beta x)+a_4\cos(\beta x) \tag{17}$$

$$\frac{dY(x)}{dx}=a_1\beta\cosh(\beta x)+a_2\beta\sinh(\beta x)+a_3\beta\cos(\beta x)-a_4\beta\sin(\beta x) \tag{18}$$

$$\frac{d^2\,Y(x)}{dx^2}=a_1\beta^2\sinh(\beta x)+a_2\beta^2\cosh(\beta x)-a_3\beta^2\sin(\beta x)-a_4\beta^2\cos(\beta x) \tag{19}$$

$$\frac{d^3\,Y(x)}{dx^3}=a_1\beta^3\cosh(\beta x)+a_2\beta^3\sinh(\beta x)-a_3\beta^3\cos(\beta x)+a_4\beta^3\sin(\beta x) \tag{20}$$

$$\frac{d^4\,Y(x)}{dx^4}=a_1\beta^4\sinh(\beta x)+a_2\beta^4\cosh(\beta x)+a_3\beta^4\sin(\beta x)+a_4\beta^4\cos(\beta x) \tag{21}$$

By substitution,

$$\begin{aligned}&\left\{a_1\beta^4\sinh(\beta x)+a_2\beta^4\cosh(\beta x)+a_3\beta^4\sin(\beta x)+a_4\beta^4\cos(\beta x)\right\}\\&\quad+\left[\frac{1}{EI}\right]\left[k_f-\rho\,\omega^2\right]\left\{a_1\sinh(\beta x)+a_2\cosh(\beta x)+a_3\sin(\beta x)+a_4\cos(\beta x)\right\}=0\end{aligned} \tag{22}$$

$$\beta^4\{a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x)\}$$

$$+\left[\frac{1}{EI}\right]\left[k_f - \rho\omega^2\right]\{a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x)\} = 0$$

(23)

The equation is satisfied if

$$\beta^4 = \left[\frac{1}{EI}\right]\left[-k_f + \rho\omega^2\right] \tag{24}$$

$$\beta = \left\{\left[\frac{1}{EI}\right]\left[-k_f + \rho\omega^2\right]\right\}^{1/4} \tag{25}$$

Apply the boundary conditions.

$$\left.\frac{d^2Y}{dx^2}\right|_{x=0} = 0 \qquad \text{(zero bending moment)} \tag{26}$$

$$a_2 - a_4 = 0 \tag{27}$$

$$a_4 = a_2 \tag{28}$$

$$\left.\frac{d^3Y}{dx^3}\right|_{x=0} = 0 \qquad \text{(zero shear force)} \tag{29}$$

$$a_1 - a_3 = 0 \tag{30}$$

$$a_3 = a_1 \tag{31}$$

$$\frac{d^2 Y(x)}{dx^2} = a_1 \beta^2 [\sinh(\beta x) - \sin(\beta x)] + a_2 \beta^2 [\cosh(\beta x) - \cos(\beta x)] \tag{32}$$

$$\frac{d^3 Y(x)}{dx^3} = a_1 \beta^3 [\cosh(\beta x) - \cos(\beta x)] + a_2 \beta^3 [\sinh(\beta x) + \sin(\beta x)] \tag{33}$$

$$\left.\frac{d^2 Y}{dx^2}\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{34}$$

$$a_1 [\sinh(\beta L) - \sin(\beta L)] + a_2 [\cosh(\beta L) - \cos(\beta L)] = 0 \tag{35}$$

$$\left.\frac{d^3 Y}{dx^3}\right|_{x=L} = 0 \qquad \text{(zero shear force)} \tag{36}$$

$$a_1 [\cosh(\beta L) - \cos(\beta L)] + a_2 [\sinh(\beta L) + \sin(\beta L)] = 0 \tag{37}$$

Equation (35) and (37) can be arranged in matrix form.

$$\begin{bmatrix} \sinh(\beta L) - \sin(\beta L) & \cosh(\beta L) - \cos(\beta L) \\ \cosh(\beta L) - \cos(\beta L) & \sinh(\beta L) + \sin(\beta L) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{38}$$

Set the determinant equal to zero.

$$[\sinh(\beta L) - \sin(\beta L)][\sinh(\beta L) + \sin(\beta L)] - [\cosh(\beta L) - \cos(\beta L)]^2 = 0 \tag{39}$$

$$\sinh^2(\beta L) - \sin^2(\beta L) - \cosh^2(\beta L) + 2\cosh(\beta L)\cos(\beta L) - \cos^2(\beta L) = 0 \tag{40}$$

$$+ 2\cosh(\beta L)\cos(\beta L) - 2 = 0 \tag{41}$$

$$\cosh(\beta L)\cos(\beta L) - 1 = 0 \tag{42}$$

The roots can be found via the Newton-Raphson method, Reference 2. The first root is

$$\beta L = 4.73004 \tag{43}$$

$$\beta_n = \left\{ \left[\frac{1}{EI}\right]\left[-k_f + \rho\,\omega_n{}^2\right] \right\}^{1/4} \tag{44}$$

$$\beta_n^4 = \left[\frac{1}{EI}\right]\left[-k_f + \rho\omega_n{}^2\right] \tag{45}$$

$$\left[\frac{1}{EI}\right]\left[-k_f + \rho\omega_n{}^2\right] = \beta_n^4 \tag{46}$$

$$\omega_n{}^2 = \frac{1}{\rho}\left[EI\,\beta_n^4 + k_f\right] \tag{47}$$

$$\omega_n = \sqrt{\frac{1}{\rho}\left[EI\,\beta_n^4 + k_f\right]} \tag{48}$$

The fundamental bending frequency is

$$\omega_1 = \sqrt{\frac{1}{\rho}\left[ EI \left[\frac{4.73004}{L}\right]^4 + k_f \right]} \tag{49}$$

Again,

$\rho$ is the mass density (mass/length)

$k_f$ is the foundation stiffness per length (force/length^2)

In addition, there are two modes whereby the beam behaves as a rigid body as shown in Appendix A.

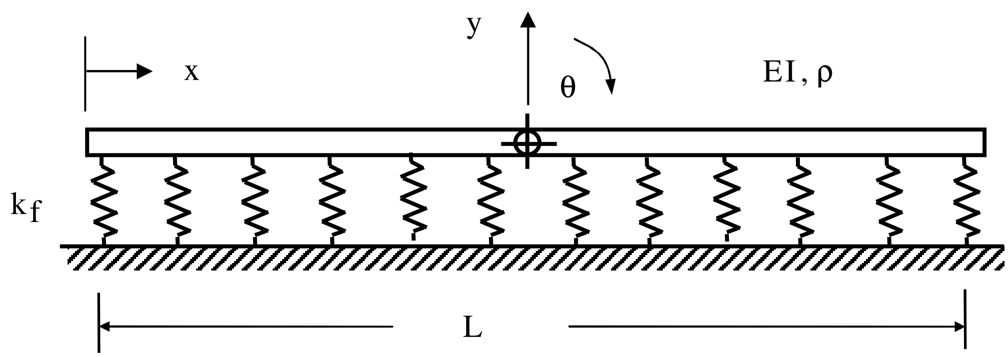
References


1. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.

2. T. Irvine, Application of the Newton-Raphson Method to Vibration Problems, Vibrationdata Publications, Rev C, 2003.

APPENDIX A

Consider the beam as behaving as a rigid mass.




Assume that the CG is at: x = L/2.

Again,

| | |
|---|---|
| $\rho$ | is the mass density (mass/length) |
| $k_f$ | is the foundation stiffness per length (force/length^2) |

The kinetic energy is

$$T = \frac{1}{2}\rho L\,\dot{y}^2 + \frac{1}{2}J\,\dot{\theta}^2 \tag{A-1}$$

The potential energy is

$$V = \frac{1}{2}k_f\left\{\int_0^L \left[y + \frac{L}{2}\theta - \theta x\right]^2 dx\right\} \tag{A-2}$$

$$V = \frac{1}{2} k_f \int_0^L \left[ \left( y + \frac{L}{2}\theta \right)^2 - 2\theta x \left( y + \frac{L}{2}\theta \right) + \theta^2 x^2 \right] dx \tag{A-3}$$

$$V = \frac{1}{2} k_f \left\{ \left( y + \frac{L}{2}\theta \right)^2 x - \theta x^2 \left( y + \frac{L}{2}\theta \right) + \frac{1}{3}\theta^2 x^3 \right\} \Bigg|_0^L \tag{A-4}$$

$$V = \frac{1}{2} k_f \left\{ \left( y + \frac{L}{2}\theta \right)^2 L - \theta L^2 \left( y + \frac{L}{2}\theta \right) + \frac{1}{3}\theta^2 L^3 \right\} \tag{A-5}$$

$$V = \frac{1}{2} k_f \, L \left\{ \left( y + \frac{L}{2}\theta \right)^2 - \theta L \left( y + \frac{L}{2}\theta \right) + \frac{1}{3}\theta^2 L^2 \right\} \tag{A-6}$$

$$V = \frac{1}{2} k_f \, L \left\{ y^2 + yL\theta + \frac{L^2}{4}\theta^2 - \theta L y - \frac{L^2}{2}\theta^2 + \frac{1}{3}\theta^2 L^2 \right\} \tag{A-7}$$

$$V = \frac{1}{2} k_f \, L \left\{ y^2 + \frac{1}{12} L^2 \theta^2 \right\} \tag{A-8}$$

The total energy of a conservative system is constant. Thus,

$$\frac{d}{dt}(T + V) = 0 \tag{A-9}$$

$$\frac{d}{dt}\left(\frac{1}{2}\rho L\,\dot{y}^2 \quad + \quad \frac{1}{2}J\,\dot{\theta}^2 + \frac{1}{2}k_f\,L\left\{y^2 + \frac{1}{12}\,L^2\,\theta^2\right\}\right) = 0 \qquad \text{(A-10)}$$

$$\rho L\,\dot{y}\ddot{y} \quad + \quad J\,\dot{\theta}\ddot{\theta} + k_f\,Ly\dot{y} + \frac{1}{12}k_f\,L^3\theta\dot{\theta} = 0 \qquad \text{(A-11)}$$

$$\{\rho\,\ddot{y} \quad + k_f\;y\}\dot{y} = 0 \qquad \text{(A-14)}$$

$$\left\{J\,\ddot{\theta} + \frac{1}{12}k_f L^3\,\theta\right\}\dot{\theta} = 0 \qquad \text{(A-15)}$$

$$\begin{bmatrix}\rho L & 0\\ 0 & J\end{bmatrix}\begin{bmatrix}\ddot{y}\\ \ddot{\theta}\end{bmatrix} + k_f\,L\begin{bmatrix}1 & 0\\ 0 & L^2/12\end{bmatrix}\begin{bmatrix}y\\ \theta\end{bmatrix} = 0 \qquad \text{(A-16)}$$

$$\left\{\;k_f\,L\begin{bmatrix}1 & 0\\ 0 & L^2/12\end{bmatrix} - \omega^2\begin{bmatrix}\rho L & 0\\ 0 & J\end{bmatrix}\right\}\begin{bmatrix}y\\ \theta\end{bmatrix} = 0 \qquad \text{(A-17)}$$

$$\omega_1 = \sqrt{\frac{k_f}{\rho}} \qquad \text{(A-18)}$$

$$\omega_2 = \sqrt{\frac{k_f\,L^3}{12J}} \qquad \text{(A-19)}$$

Note that for a uniform beam,

$$J = \frac{1}{12}\rho L^3 \tag{A-20}$$