

THE NATURAL FREQUENCY OF A RECTANGULAR PLATE WITH FIXED-FREE-FIXED-FREE BOUNDARY CONDITIONS

Revision A

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Introduction

The Rayleigh method is used in this tutorial to determine the fundamental bending frequency. A displacement function is assumed which satisfies the geometric boundary conditions. The geometric conditions are the displacement and slope conditions at the boundaries. The assumed displacement function is substituted into the strain and kinetic energy equations.

The Rayleigh method gives a natural frequency that is an upper limit of the true natural frequency. The method would give the exact natural frequency if the true displacement function were used. The true displacement function is called an eigenfunction.

Consider the rectangular plate in Figure 1. The largest dimension may be either a or b .

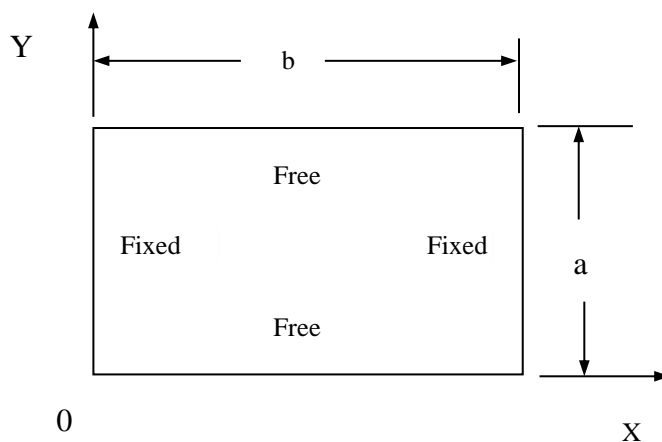


Figure 1.

Let Z represent the out-of-plane displacement. The total strain energy V of the plate is

$$V = \frac{D}{2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \left[\left(\frac{\partial^2 Z}{\partial X^2} \right)^2 + \left(\frac{\partial^2 Z}{\partial Y^2} \right)^2 + 2\mu \left(\frac{\partial^2 Z}{\partial X^2} \right) \left(\frac{\partial^2 Z}{\partial Y^2} \right) + 2(1 - \mu) \left(\frac{\partial^2 Z}{\partial X \partial Y} \right)^2 \right] dXdY \quad (1)$$

Note that the plate stiffness factor D is given by

$$D = \frac{Eh^3}{12(1 - \mu^2)} \quad (1)$$

where

- E = elastic modulus
- h = plate thickness
- μ = Poisson's ratio

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} Z^2 dXdY \quad (3)$$

where

- ρ = mass per volume
- Ω = angular natural frequency

Rayleigh's method can be applied as

$$T_{\max} = V_{\max} = \text{total energy of the system} \quad (4)$$

Fixed-Free-Fixed-Free Plate

Consider the plate in Figure 2.

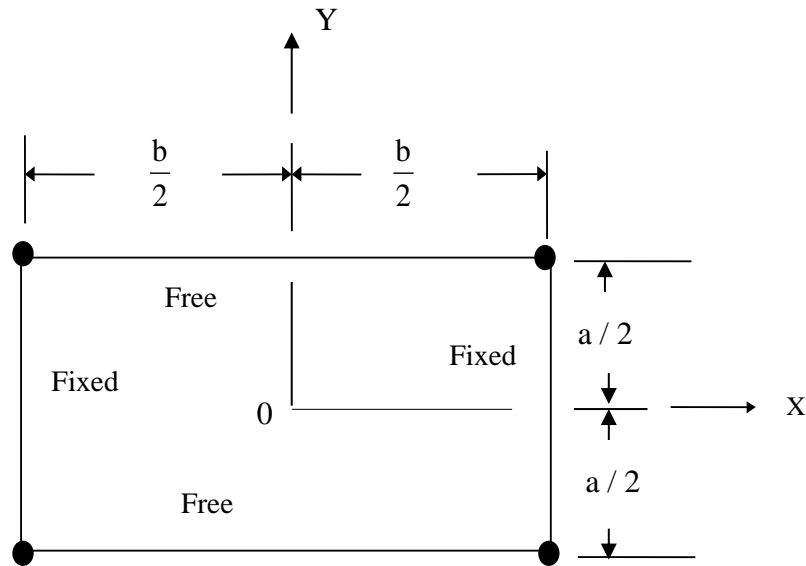


Figure 2.

Seek a displacement function $Z(x, y)$. The geometric boundary conditions are

$$Z(-b/2, y) = 0 \quad \text{and} \quad Z(+b/2, y) = 0 \quad (5)$$

$$\frac{\partial Z}{\partial x} = 0 \quad \text{at} \quad (\pm b/2, y) = 0 \quad (6)$$

M_y is the moment along the y -axis.

$$M_y = -D \left[\frac{\partial^2 Z}{\partial y^2} + \mu \frac{\partial^2 Z}{\partial x^2} \right] = 0 \quad \text{at} \quad (x, \pm a/2) \quad (7)$$

Note that the twist is

$$M_{xy} = -D \left[(1 - \mu) \frac{\partial^2 Z}{\partial x \partial y} \right] \quad (8)$$

Q_y is the shear along the y -axis.

$$Q_y = -D \frac{\partial}{\partial y} [\nabla^2 Z] = -D \frac{\partial}{\partial y} \left\{ \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] Z \right\} = 0 \text{ at } (x, \pm a/2) \quad (9)$$

The candidate displacement function is

$$Z(x, y) = V(x)[1 + \alpha \cos(\theta y)] \quad (10)$$

$$V(x) = [\cosh(\beta x) - \cos(\beta x)] - \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)} \right] [\sinh(\beta x) + \sin(\beta x)] \quad (11)$$

where

$$\beta = 4.73004 / b$$

b is the free edge length

The derivatives are

$$\frac{\partial}{\partial x} Z(x, y) = \frac{d}{dx} V(x)[1 + \alpha \cos(\theta y)] \quad (12)$$

$$\frac{\partial^2}{\partial x^2} Z(x, y) = \frac{d^2}{dx^2} V(x)[1 + \alpha \cos(\theta y)] \quad (13)$$

$$\frac{\partial}{\partial y} Z(x, y) = -V(x)[\alpha\theta \sin(\theta y)] \quad (14)$$

$$\frac{\partial^2}{\partial y^2} Z(x, y) = -V(x)[\alpha\theta^2 \cos(\theta y)] \quad (15)$$

$$\frac{\partial}{\partial x \partial y} Z(x, y) = -\frac{d}{dx} V(x)[\alpha\theta \sin(\theta y)] \quad (16)$$

The candidate displacement function satisfies the geometric boundary conditions. But it does not satisfy the moment, twist, and shear boundary conditions.

Now equate the total kinetic energy with the total strain energy per Rayleigh's method, equation (3). This is done numerically via the computer program in Appendix A. The integrals are converted to series form for this calculation.

Solve for Ω . Select α and β values to minimize Ω via trial-and-error.

The natural frequency f_n is

$$f_n \approx \frac{1}{2\pi} \Omega \quad (17)$$

A more proper equation is

$$f_n \leq \frac{1}{2\pi} \Omega \quad (18)$$

Verification

The following formula taken from Steinberg's text can be used as an approximation to check the Rayleigh natural frequency result. The variable b is the free edge length.

$$f_n \approx \frac{3.55}{b^2} \sqrt{\frac{D}{\rho h}} \quad (19)$$

The ρh term in the above equation is the mass per surface area.

Example

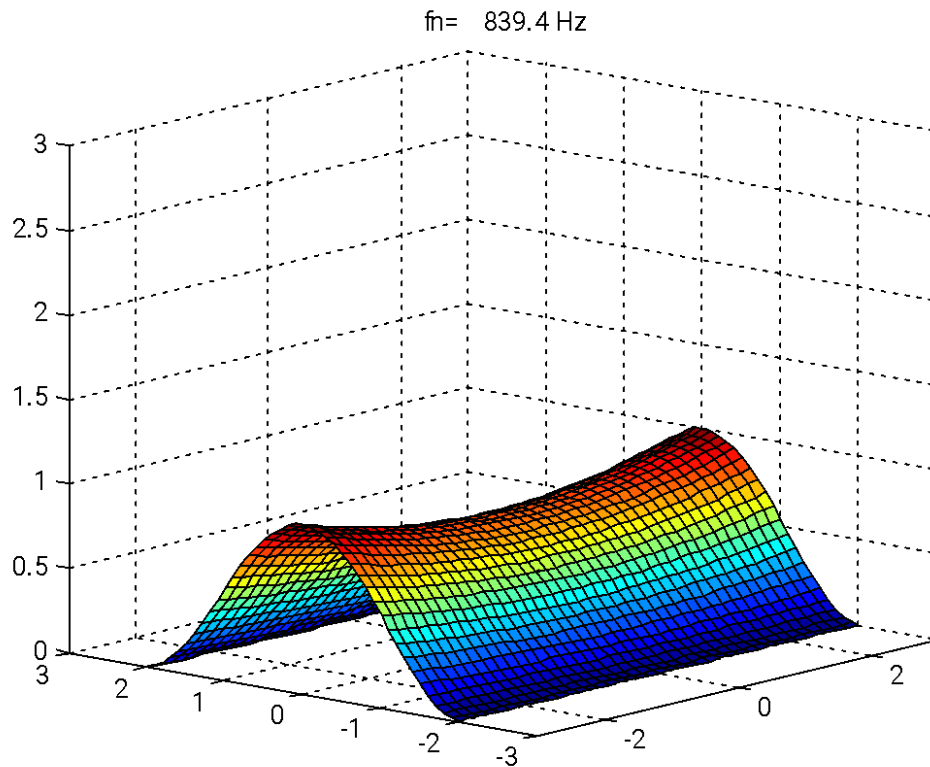


Figure 3.

A fixed-free-fixed-free aluminum plate has dimensions:

- Fixed Edge = 6 in
- Free Edge = 4 in
- Thickness = 0.063 in

The elastic modulus is $1.0 \times 10^7 \text{ lbf/in}^2$.

The mass density is 0.1 lbf/in^3 , or $0.000259 \text{ lbf sec}^2/\text{in}^4$.

The fundamental frequency is 839.4 Hz, as calculated using the trial-and-error Rayleigh method outlined above.

The expected natural frequency range per equation (19) is: $f_n \approx 833.6$ Hz.

The resulting mode shape is shown in Figure 3.

The modal displacement equation is

$$Z(x, y) = V(x)[1 + \alpha \cos(\theta y)] \quad (20)$$

$$V(x) = [\cosh(\beta x) - \cos(\beta x)] - \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)} \right] [\sinh(\beta x) + \sin(\beta x)] \quad (21)$$

where

$$\begin{aligned} \beta &= 4.73004 / b \\ b &= \text{free edge length} \\ \alpha &= -0.8577 \\ \theta &= 0.09913 \end{aligned}$$

The Rayleigh method accuracy can be improved using the Rayleigh-Ritz method.

References

1. R. Blevins, Formulas for Natural Frequency and Mode Shape, Krieger, Malabar, Florida, 1979. See Table 11-6.
2. D. Steinberg, Vibration Analysis for Electronic Equipment, Third Edition, Wiley, New York, 2000.