Frequency Response Function Substructuring Revision A

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Introduction

Frequency Response Function based substructuring (FBS) predicts the dynamic behavior of a rigidlycoupled system on the basis of the free interface FRFs of the uncoupled components. This method is the frequency-domain analog of time-domain component mode synthesis (CMS).

Theory

The following derivation is taken from Reference 1.

The two subsystems in Figure 1 are considered as independent of one another prior to coupling. The degrees-of-freedom (dof) of each subsystem are classified as either coupling or internal dofs.



Figure 1.

The equation of motion for each subsystem in the frequency domain for the complex displacement $\{X_n\}$ is

$$\left\{\mathbf{X}\right\}_{\mathbf{n}} = \left[\mathbf{H}\right]_{\mathbf{n}\mathbf{n}} \left\{\mathbf{F}\right\}_{\mathbf{n}} \tag{1}$$

where

$$\begin{array}{ll} \left[H\right]_{nn} & \text{ is the complex admittance matrix (displacement/force)} \\ \left\{F\right\}_n & \text{ is the applied force vector} \end{array}$$

The subscript n represents the total number of dof for each subsystem.

$$n = a + c$$
 for subsystem A (2)

$$n = b + c$$
 for subsystem B (3)

Note that the number of c dof is the same for both subsystems.

Subsystem A can be partitioned as

$$\begin{cases} \left\{ X^{A} \right\}_{a} \\ \left\{ X^{A} \right\}_{c} \end{cases} = \begin{bmatrix} \left[H^{A} \right]_{aa} & \left[H^{A} \right]_{ac} \\ \left[H^{A} \right]_{ca} & \left[H^{A} \right]_{cc} \end{bmatrix} \\ \begin{cases} \left\{ F^{A} \right\}_{a} \\ \left\{ F^{A} \right\}_{c} \end{cases} \end{cases}$$

$$(4)$$

Subsystem B can be partitioned as

$$\begin{cases} \left\{ X^{B} \right\}_{b} \\ \left\{ X^{B} \right\}_{c} \end{cases}_{n} = \begin{bmatrix} \left[H^{B} \right]_{bb} & \left[H^{B} \right]_{bc} \\ \left[H^{B} \right]_{cb} & \left[H^{B} \right]_{cc} \end{bmatrix} \begin{bmatrix} \left\{ F^{B} \right\}_{b} \\ \left\{ F^{B} \right\}_{c} \end{bmatrix}_{n} \end{cases}$$
(5)

Let the superscript S represent the combined system where subsystems A and B are rigidily connected at dof C.

The rigid connection of the two systems at the c dofs requires that

$$\left\{ X^{A} \right\}_{c} = \left\{ X^{B} \right\}_{c} = \left\{ X^{S} \right\}_{c} \tag{6}$$

The force equilibrium requires

$$\left\{ \mathbf{F}^{\mathbf{A}} \right\}_{\mathbf{C}} = \left\{ \mathbf{F}^{\mathbf{B}} \right\}_{\mathbf{C}} = \left\{ \mathbf{F}^{\mathbf{S}} \right\}_{\mathbf{C}}$$
(7)

The FRFs of the system can be represented as

$$\begin{cases} \left\{ \mathbf{X}^{\mathbf{S}} \right\}_{\mathbf{a}} \\ \left\{ \mathbf{X}^{\mathbf{S}} \right\}_{\mathbf{c}} \\ \left\{ \mathbf{X}^{\mathbf{S}} \right\}_{\mathbf{c}} \\ \left\{ \mathbf{X}^{\mathbf{S}} \right\}_{\mathbf{b}} \\ \mathbf{H}^{\mathbf{S}} \right\}_{\mathbf{a}} & \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{a}\mathbf{c}} & \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{a}\mathbf{b}} \\ \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{c}\mathbf{a}} & \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{c}\mathbf{c}} & \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{c}\mathbf{b}} \\ \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{c}\mathbf{a}} & \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{c}\mathbf{c}} & \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{c}\mathbf{b}} \\ \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{b}\mathbf{a}} & \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{b}\mathbf{c}} & \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{b}\mathbf{b}} \\ \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{b}\mathbf{a}} & \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{b}\mathbf{c}} & \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{b}\mathbf{b}} \\ \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{b}\mathbf{b}} & \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{b}\mathbf{b}} \\ \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right\}_{\mathbf{b}\mathbf{b}} & \begin{bmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{b}\mathbf{b}} \\ \end{bmatrix}_{\mathbf{b}\mathbf{b}} \begin{pmatrix} \mathbf{H}^{\mathbf{S}} \right]_{\mathbf{b}\mathbf{b}} \end{pmatrix}_{\mathbf{b}}$$
(8)

The connection dof from the partitioned equation (5) for subsystems A and B are

$$\left\{ X^{A} \right\}_{c} = \left[H^{A} \right]_{ca} \left\{ F^{A} \right\}_{a} + \left[H^{A} \right]_{cc} \left\{ F^{A} \right\}_{c}$$

$$\tag{9}$$

$$\left\{ X^{B} \right\}_{c} = \left[H^{B} \right]_{cb} \left\{ F^{B} \right\}_{b} + \left[H^{B} \right]_{cc} \left\{ F^{B} \right\}_{c}$$
(10)

Now let $\langle \widetilde{F}^A \rangle_c$ and $\langle \widetilde{F}^B \rangle_c$ be the internal transmitted forces at the interfaces for subsystems A and B respectively.

Note that for the fully coupled system

$$\left\{ \mathbf{F}^{\mathbf{S}} \right\}_{\mathbf{C}} = \left\{ \mathbf{\widetilde{F}}^{\mathbf{A}} \right\}_{\mathbf{C}} + \left\{ \mathbf{\widetilde{F}}^{\mathbf{B}} \right\}_{\mathbf{C}}$$
(11)

$$\left\{ \widetilde{\mathbf{F}}^{\mathbf{A}} \right\}_{\mathbf{C}} = \left\{ \mathbf{F}^{\mathbf{S}} \right\}_{\mathbf{C}} - \left\{ \widetilde{\mathbf{F}}^{\mathbf{B}} \right\}_{\mathbf{C}}$$
(12)

$$\left\{ \widetilde{\mathbf{F}}^{\mathbf{B}} \right\}_{\mathbf{C}} = \left\{ \mathbf{F}^{\mathbf{S}} \right\}_{\mathbf{C}} - \left\{ \widetilde{\mathbf{F}}^{\mathbf{A}} \right\}_{\mathbf{C}}$$
(13)

Substitute these transmitted forces into equations (9) and (10) in preparation for coupling these equations.

$$\left\{ X^{A} \right\}_{c} = \left[H^{A} \right]_{ca} \left\{ F^{A} \right\}_{a} + \left[H^{A} \right]_{cc} \left\{ \widetilde{F}^{A} \right\}_{c}$$
(14)

$$\left\{ X^{B} \right\}_{C} = \left[H^{B} \right]_{Cb} \left\{ F^{B} \right\}_{b} + \left[H^{B} \right]_{Cc} \left\{ \widetilde{F}^{B} \right\}_{c}$$
(15)

Set equations (14) and (15) equal to one another using equation (6).

$$\left[H^{A}\right]_{ca}\left\{F^{A}\right\}_{a} + \left[H^{A}\right]_{cc}\left\{\widetilde{F}^{A}\right\}_{c} = \left[H^{B}\right]_{cb}\left\{F^{B}\right\}_{b} + \left[H^{B}\right]_{cc}\left\{\widetilde{F}^{B}\right\}_{c}$$
(16)

$$\left[H^{A}\right]_{cc}\left\{\widetilde{F}^{A}\right\}_{c} = \left[H^{B}\right]_{cb}\left\{F^{B}\right\}_{b} - \left[H^{A}\right]_{ca}\left\{F^{A}\right\}_{a} + \left[H^{B}\right]_{cc}\left\{\widetilde{F}^{B}\right\}_{c}$$
(17)

Substitute equation (13) into (17).

$$\left[H^{A}\right]_{cc}\left\{\widetilde{F}^{A}\right\}_{c} = \left[H^{B}\right]_{cb}\left\{F^{B}\right\}_{b} - \left[H^{A}\right]_{ca}\left\{F^{A}\right\}_{a} + \left[H^{B}\right]_{cc}\left\{F^{S}\right\}_{c} - \left\{\widetilde{F}^{A}\right\}_{c}\right\}$$
(18)

$$\left[H^{A} \right]_{cc} + \left[H^{B} \right]_{cc} \left[F^{A} \right]_{c} = \left[H^{B} \right]_{cb} \left[F^{B} \right]_{b} - \left[H^{A} \right]_{ca} \left[F^{A} \right]_{a} + \left[H^{B} \right]_{cc} \left[F^{S} \right]_{c}$$
(19)

$$\left\{ \widetilde{\mathbf{F}}^{\mathbf{A}} \right\}_{\mathbf{C}} = \left\{ \mathbf{H}^{\mathbf{A}} \right\}_{\mathbf{C}\mathbf{C}} + \left[\mathbf{H}^{\mathbf{B}} \right]_{\mathbf{C}\mathbf{C}} \left\}^{-1} \left\{ \mathbf{H}^{\mathbf{B}} \right\}_{\mathbf{C}\mathbf{b}} \left\{ \mathbf{F}^{\mathbf{B}} \right\}_{\mathbf{b}} - \left[\mathbf{H}^{\mathbf{A}} \right]_{\mathbf{C}\mathbf{a}} \left\{ \mathbf{F}^{\mathbf{A}} \right\}_{\mathbf{a}} + \left[\mathbf{H}^{\mathbf{B}} \right]_{\mathbf{C}\mathbf{c}} \left\{ \mathbf{F}^{\mathbf{S}} \right\}_{\mathbf{c}} \right\}$$
(20)

Now derive coupled FRF equations. Derive $\left[H^{S}\right]_{aa}$ as an example.

Recall from equation for the coupled system (8).

$$\left\{ X^{S} \right\}_{a} = \left[H^{S} \right]_{aa} \left\{ F^{S} \right\}_{a} + \left[H^{S} \right]_{ac} \left\{ F^{S} \right\}_{c} + \left[H^{S} \right]_{ab} \left\{ F^{S} \right\}_{b}$$
 (21)

4

Recall from equation (4) for the uncoupled subsystem A.

$$\left\{ X^{A} \right\}_{a} = \left[H^{A} \right]_{aa} \left\{ F^{A} \right\}_{a} + \left[H^{A} \right]_{ac} \left\{ F^{A} \right\}_{c}$$
(22)

For the coupled system, equation (22) becomes

$$\left\{ X^{S} \right\}_{a} = \left[H^{A} \right]_{aa} \left\{ F^{S} \right\}_{a} + \left[H^{A} \right]_{ac} \left\{ \widetilde{F}^{A} \right\}_{c}$$

$$\tag{23}$$

Substitute equation (21) into equation (23).

$$\begin{bmatrix} H^{S} \\ aa \\ F^{S} \\ a \\ F^{S} \\ a \\ F^{S} \\ c \\ F^{S} \\ c \\ F^{S} \\ c \\ F^{S} \\ b \\ F$$

$$\left[H^{S}\right]_{aa}\left\{F^{S}\right\}_{a} = -\left[H^{S}\right]_{ac}\left\{F^{S}\right\}_{c} - \left[H^{S}\right]_{ab}\left\{F^{S}\right\}_{b} + \left[H^{A}\right]_{aa}\left\{F^{S}\right\}_{a} + \left[H^{A}\right]_{ac}\left\{\widetilde{F}^{A}\right\}_{c}$$
(25)

Substitute equation (20) into equation (25).

$$\begin{split} \left[H^{S} \right]_{aa} \left\{ F^{S} \right\}_{a} &= - \left[H^{S} \right]_{ac} \left\{ F^{S} \right\}_{c} - \left[H^{S} \right]_{ab} \left\{ F^{S} \right\}_{b} + \left[H^{A} \right]_{aa} \left\{ F^{S} \right\}_{a} \\ &+ \left[H^{A} \right]_{ac} \left\{ \left\{ H^{A} \right\}_{cc} + \left[H^{B} \right]_{cc} \right\}^{-1} \left\{ H^{B} \right]_{cb} \left\{ F^{B} \right\}_{b} - \left[H^{A} \right]_{ca} \left\{ F^{A} \right\}_{a} + \left[H^{B} \right]_{cc} \left\{ F^{S} \right\}_{c} \right\} \end{split}$$

$$(26)$$

$$\{\mathbf{F}^{\mathbf{S}}\}_{\mathbf{c}} = 0, \quad \{\mathbf{F}^{\mathbf{S}}\}_{\mathbf{b}} = 0, \quad \{\mathbf{F}^{\mathbf{B}}\}_{\mathbf{b}} = 0$$
(27)

$$\left[H^{S} \right]_{aa} \left\{ F^{S} \right\}_{a} = \left[H^{A} \right]_{aa} \left\{ F^{S} \right\}_{a} - \left[H^{A} \right]_{ac} \left\{ \left[H^{A} \right]_{cc} + \left[H^{B} \right]_{cc} \right\}^{-1} \left\{ \left[H^{A} \right]_{ca} \left\{ F^{A} \right\}_{a} \right\}$$
(28)

Note that at the system level $\{F^A\}_a$ can be replaced by $\{F^S\}_a$.

Thus,

$$\begin{bmatrix} \mathbf{H}^{\mathbf{S}} \end{bmatrix}_{\mathbf{a}a} \left\{ \mathbf{F}^{\mathbf{S}} \right\}_{\mathbf{a}} = \begin{bmatrix} \mathbf{H}^{\mathbf{A}} \end{bmatrix}_{\mathbf{a}a} \left\{ \mathbf{F}^{\mathbf{S}} \right\}_{\mathbf{a}} - \begin{bmatrix} \mathbf{H}^{\mathbf{A}} \end{bmatrix}_{\mathbf{a}c} \left\{ \begin{bmatrix} \mathbf{H}^{\mathbf{A}} \end{bmatrix}_{\mathbf{c}c} + \begin{bmatrix} \mathbf{H}^{\mathbf{B}} \end{bmatrix}_{\mathbf{c}c} \right\}^{-1} \left\{ \begin{bmatrix} \mathbf{H}^{\mathbf{A}} \end{bmatrix}_{\mathbf{c}a} \left\{ \mathbf{F}^{\mathbf{S}} \right\}_{\mathbf{a}} \right\}$$
(29)

Divide each side of equation (29) by $\{F^S\}_a$.

$$\begin{bmatrix} \mathbf{H}^{\mathbf{S}} \end{bmatrix}_{\mathbf{a}\mathbf{a}} = \begin{bmatrix} \mathbf{H}^{\mathbf{A}} \end{bmatrix}_{\mathbf{a}\mathbf{c}} - \begin{bmatrix} \mathbf{H}^{\mathbf{A}} \end{bmatrix}_{\mathbf{a}\mathbf{c}} \left\{ \begin{bmatrix} \mathbf{H}^{\mathbf{A}} \end{bmatrix}_{\mathbf{c}\mathbf{c}} + \begin{bmatrix} \mathbf{H}^{\mathbf{B}} \end{bmatrix}_{\mathbf{c}\mathbf{c}} \right\}^{-1} \begin{bmatrix} \mathbf{H}^{\mathbf{A}} \end{bmatrix}_{\mathbf{c}\mathbf{a}}$$
(30)

An individual admittance function $\left[h^{S}\right]_{ij}$ can be accessed via

$$\left[h^{S}\right]_{ij} = \left[h^{A}\right]_{ij} - \left[H^{A}\right]_{ic} \left\{\left[H^{A}\right]_{cc} + \left[H^{B}\right]_{cc}\right\}^{-1} \left[H^{A}\right]_{cj}$$

$$(31)$$

Conclusions

A practical diagram is given in Appendix A which shows the meaning of equation (31).

Examples are given in Appendices B & C. The results show that equation (31) gives an accurate FRF at the system level for non-interface degrees-of-freedom.

The results actually showed that equation (31) gave slightly higher peak magnitude values at the system modal frequencies. Briefly, the examples made an incorrect assumption that the uniform subsystem modal damping would be the same as the combined system modal damping. This is explained further in Reference 2. So the observed difference at the system modal frequencies does not detract from equation (31).

References

- Dr. Peter Avitabile, Impedance Modeling & Frequency Based Substructuring, Modal Analysis and Controls Laboratory, University of Massachusetts Lowell, PowerPoint Presentation. http://faculty.uml.edu/pavitabile/22.515/Fabio_fbs_thesis.pdf
- 2. T. Irvine, Notes on Damping in FRF Substructuring, Vibrationdata, 2014.
- 3. T. Irvine, Calculating Transfer Functions from Normal Modes, Revision E, Vibrationdata, 2013.

APPENDIX A

<u>Diagram</u>



Figure A-1.

This figure is taken from Reference 1.

APPENDIX B

Example 1

The combined system S is



Figure B-1.

| m1 | 4 lbm |
|----|---------------|
| m2 | 2 lbm |
| k1 | 10,000 lbf/in |
| k2 | 10,000 lbf/in |
| k3 | 500 lbf/in |

5% damping for all modes

The analysis goal is to calculate the admittance at dof 2 relative to a force at this same location at the system level using only subsystem admittance functions. The transfer functions are calculated using the method in Reference 3.

Subsystem A is



Figure B-2.





Results



Figure B-4.

The magnitude unit is (in/lbf) in this and all of the following admittance plots.

Note that HA21=HA12 by reciprocity.





Note that the two subsystems are independent.



Figure B-6.

The direct (blue) curve was calculated from the full system.

The indirect (green) curve was calculated from the subsystem FRFs via equation (31).

The indirect curve is slightly higher than the FRF at the system natural frequencies.

A half-power-bandwidth type curve-fit was performed on each FRF curve. The curve-fit plots are omtted for brevity. The resulting damping for the direct curve was 5% at the fundamental frequency. The indirect curve has 4% damping at this same frequency, which is reasonable per Reference 2.

Originally, the direct curve was thought to be "exact." But futher investigation showed that the system level damping should have been lower than the subsystem damping due to the "damping physics" in Reference 2. So the indirect curve from equation (31) appears to be the more accurate of the two curves.

APPENDIX C

Example 2

The combined system S is



Figure C-1.

| m1 | 4 lbm |
|----|---------------|
| m2 | 2 lbm |
| k1 | 10,000 lbf/in |
| k2 | 10,000 lbf/in |

5% damping for all modes

The following analysis uses accelerance rather than admittance because system A has a rigid-body mode which causes the admittance functions to approach infinity as the excitation frequency decreases toward zero. Note that accelerance FRFs may be substituted in place of the admittance FRFs equation (31).

The analysis goal is to calculate the accelerance at dof 2 relative to a force at this same location at the system level using only subsystem accelerance functions. The transfer functions are calculated using the method in Reference 2.

Subsystem A is



m1a=1 lbm

Figure C-2.





Results



Figure C-4.

The magnitude unit is (G/lbf) in this and all of the following accelerance plots.

The fundamental frequency was zero but was set at 1.0e-04 Hz to form stable accelerance FRFs Note that HA21=HA12 by reciprocity.

Note that the two subsystems are independent.

The direct (blue) curve was calculated from the full system.

The indirect (green) curve was calculated from the subsystem FRFs via equation (31).

The indirect curve has a slightly higher FRF at the system natural frequencies, as was the case in the example in Figure B-6.

Again, the system level damping should have been lower than the subsystem damping per Reference 2.