MASS CONDENSATION Revision A

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Introduction

The mass condensation method is taken from Reference 1. Guyan (1965) and Irons (1965) developed this method independently. The method is appropriate for the case where the inertia forces associated with some of the displacements are greater than those associated with other displacements.

The significant displacements are referred to as “master” displacements. The insignificant displacements are referred to as “slave” displacements. Note that displacements are also considered as degrees-of-freedom.

The method assumes that the elastic forces transmitted by the master displacements are much greater than the inertia forces on the slave displacements for the lower frequency modes. The slave displacements are thus assumed to move quasi-statically with the master displacements.

The procedure is to sort each displacement into these two categories. The slave displacements are then eliminated from the eigenvalue problem. This step reduces the size of the problem, but the resulting accuracy is less.

In certain structural dynamics problems, there are both translational and rotational displacements. The translation displacements can often be regarded as the masters, whereas the rotational displacements are considered as slaves.

A general rule-of-thumb is to select the slave displacements as those with a high stiffness-to-mass ratio.

The generalized eigenvalue problem in structural dynamics has the form

$$\underline{K}\,\bar{u} = \lambda\,\underline{M}\,\bar{u} \tag{1}$$

where

$\underline{K}$ is the stiffness matrix

$\underline{M}$ is the mass matrix

$\bar{u}$ is the displacement eigenvector

λ is the eigenvalue

The potential energy V is

$$V = \frac{1}{2} u^{T} \underline{K} u \tag{2}$$

The over-bar has been omitted from the u terms for brevity.

The kinetic energy T is

$$T = \frac{1}{2} \dot{u}^{T} \underline{M} \dot{u} \tag{3}$$

Divide the displacement vectors into the master vectors u_1 and slave vectors u_2.

$$u = \begin{bmatrix} u_1 \\ \cdots \\ u_2 \end{bmatrix} \tag{4}$$

Partition the mass and stiffness matrices as follows.

$$\underline{M} = \begin{bmatrix} M_{11} & \vdots & M_{12} \\ \cdots & \vdots & \cdots \\ M_{21} & \vdots & M_{22} \end{bmatrix} \tag{5}$$

$$\underline{K} = \begin{bmatrix} K_{11} & \vdots & K_{12} \\ \cdots & \vdots & \cdots \\ K_{21} & \vdots & K_{22} \end{bmatrix} \tag{6}$$

For brevity, the partition lines are omitted from the following equations.

By substitution,

$$V = \frac{1}{2}\,[u_1{}^T \quad u_2{}^T]\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{7}$$

$$T = \frac{1}{2}\,[\dot{u}_1{}^T \quad \dot{u}_2{}^T]\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \tag{8}$$

Note that

$$K_{21} = K_{12}{}^T \tag{9}$$

$$M_{21} = M_{12}{}^T \tag{10}$$

$$V = \frac{1}{2}\left\lfloor u_1{}^T \quad u_2{}^T \right\rfloor\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$V = \frac{1}{2}\left[u_1{}^T K_{11} u_1 + u_2{}^T K_{21} u_1 + u_1{}^T K_{12} u_2 + u_2{}^T K_{22} u_2 \right] \tag{11}$$

$$V = \frac{1}{2}\left[u_1{}^T K_{11} u_1 + u_2{}^T K_{21}\, u_1 + u_2{}^T K_{12}{}^T u_1 + u_2{}^T K_{22} u_2 \right] \tag{12}$$

$$V = \frac{1}{2}\left[u_1{}^T K_{11} u_1 + u_2{}^T K_{21}\, u_1 + u_2{}^T K_{21} u_1 + u_2{}^T K_{22} u_2 \right] \tag{13}$$

$$V = \frac{1}{2}\left[u_1{}^T K_{11} u_1 + 2\, u_2{}^T K_{21}\, u_1 + u_2{}^T K_{22} u_2 \right] \tag{14}$$

Take the partial derivative with respect to the slave vector.

$$\frac{\partial V}{u_2} = K_{21}u_1 + K_{22}u_2 \tag{15}$$

Assume no applied forces in the direction of the slave displacements.

$$K_{21}u_1 + K_{22}u_2 = 0 \tag{16}$$

$$K_{22}u_2 = -K_{21}u_1 \tag{17}$$

$$u_2 = -K_{22}{}^{-1}\, K_{21}u_1 \tag{18}$$

Substitute equation (18) into the displacement eigenvector (4).

$$u = \begin{bmatrix} u_1 \\ -K_{22}{}^{-1}\, K_{21}u_1 \end{bmatrix} \tag{19}$$

$$u = \begin{bmatrix} I \\ -K_{22}{}^{-1}\, K_{21} \end{bmatrix} u_1 \tag{20}$$

Define a constraint matrix C.

$$u = C\, u_1 \tag{21}$$

$$C = \begin{bmatrix} I \\ -K_{22}{}^{-1}\, K_{21} \end{bmatrix} \tag{22}$$

Matrix I is the identity matrix.

Transform the mass and stiffness matrices. The resulting matrices are the respective condensed matrices.

$$M_1 = C^T K \; C \tag{23}$$

$$K_1 = C^T K \; C \tag{24}$$

Substitute the condensed matrices into the generalized eigenvalue problem.

$$\underline{K}_1 \, \bar{u} = \lambda \, \underline{M}_1 \, \bar{u} \tag{25}$$

An example is given in Appendix A.

References

1. L. Meirovitch, Computational Methods in Structural Dynamics, Sijthoff & Noordhoff, The Netherlands, 1980.

2. T. Irvine, Transverse Vibration of a Beam via the Finite Element Method, Vibrationdata, 2004.

APPENDIX A

Example

The mass condensation method is demonstrated by an example. Consider a finite element mode of the bending vibration of beam with free-free boundary conditions.

The mass and stiffness matrices are taken from Reference 2.

The stiffness matrix is

$$\underline{K} = \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \tag{A-1}$$

The mass matrix is

$$\underline{M} = \begin{bmatrix} 156 & 22 & 54 & -13 & 0 & 0 \\ 22 & 4 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix} \tag{A-2}$$

The eigenvector is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \\ y_3 \\ h\,\theta_3 \end{bmatrix} \tag{A-3}$$

Choose three master degrees-of-freedom on the basis of a low stiffness-to-mass ratio.

Reconfiguration the rows and columns such that the eigenvector is

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ h\,\theta_1 \\ h\,\theta_2 \\ h\,\theta_3 \end{bmatrix} \tag{A-4}$$

The adjusted stiffness matrix is

$$\underline{K} = \begin{bmatrix} 12 & -12 & 0 & 6 & 6 & 0 \\ -12 & 24 & -12 & -6 & 0 & 6 \\ 0 & -12 & 12 & 0 & -6 & -6 \\ 6 & -6 & 0 & 4 & 2 & 0 \\ 6 & 0 & -6 & 2 & 8 & 2 \\ 0 & 6 & -6 & 0 & 2 & 4 \end{bmatrix} \tag{A-5}$$

Partition the stiffness matrix.

$$\underline{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{A-6}$$

$$K_{11} = \begin{bmatrix} 12 & -12 & 0 \\ -12 & 24 & -12 \\ 0 & -12 & 12 \end{bmatrix} \tag{A-7}$$

$$K_{12} = \begin{bmatrix} 6 & 6 & 0 \\ -6 & 0 & 6 \\ 0 & -6 & 0 \end{bmatrix} \tag{A-8}$$

$$K_{21} = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 0 & -6 \\ 0 & 6 & -6 \end{bmatrix} \tag{A-9}$$

$$K_{22} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 4 \end{bmatrix} \tag{A-10}$$

The adjusted mass matrix is

$$\underline{M} = \begin{bmatrix} 156 & 54 & 0 & 22 & -13 & 0 \\ 54 & 312 & 54 & 13 & 0 & -13 \\ 0 & 54 & 156 & 0 & 13 & -22 \\ 22 & 13 & 0 & 4 & -3 & 0 \\ -13 & 0 & 13 & -3 & 8 & -3 \\ 0 & -13 & -22 & 0 & -3 & 4 \end{bmatrix} \tag{A-11}$$

Partition the mass matrix.

$$\underline{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{A-12}$$

$$M_{11} = \begin{bmatrix} 156 & 54 & 0 \\ 54 & 312 & 54 \\ 0 & 54 & 156 \end{bmatrix} \tag{A-13}$$

$$M_{12} = \begin{bmatrix} 22 & -13 & 0 \\ 13 & 0 & -13 \\ 0 & 13 & -22 \end{bmatrix} \tag{A-14}$$

$$M_{21} = \begin{bmatrix} 22 & 13 & 0 \\ -13 & 0 & 13 \\ 0 & -13 & -22 \end{bmatrix} \tag{A-15}$$

$$M_{22} = \begin{bmatrix} 4 & -3 & 0 \\ -3 & 8 & -3 \\ 0 & -3 & 4 \end{bmatrix} \tag{A-16}$$

Form a constraint matrix.

$$\underline{C} = \begin{bmatrix} I \\ -K_{22}^{-1} K_{21} \end{bmatrix} \tag{A-17}$$

$$\underline{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.25 & 1.5 & -0.25 \\ -0.5 & 0 & 0.5 \\ 0.25 & -1.5 & 1.25 \end{bmatrix} \tag{A-18}$$

Calculate the reduced stiffness matrix.

$$K_1 = C^T K \; C \tag{A-19}$$

$$K_1 = \begin{bmatrix} 1.5 & -3 & 1.5 \\ -3 & 6 & -3 \\ 1.5 & -3 & 1.5 \end{bmatrix} \tag{A-20}$$

Calculate the reduced mass matrix.

$$M_1 = C^T \; K \; C \tag{A-21}$$

$$M_1 = \begin{bmatrix} 119.5 & 58.5 & -20.5 \\ 58.5 & 408 & 58.5 \\ -20.5 & 58.5 & 119.5 \end{bmatrix} \tag{A-22}$$

The reduced eigenvalue problem is

$$\underline{K}_1 \, \bar{u} = \lambda \, \underline{M}_1 \, \bar{u} \tag{A-23}$$

The eigenvalues for the reduced problems are

$$\underline{\lambda} = \begin{bmatrix} 0 \\ 0 \\ 0.0751 \end{bmatrix} \tag{A-24}$$

The eigenvalues for the full problem are

$$\underline{\lambda} = \begin{bmatrix} 0 \\ 0 \\ 0.0748 \\ 0.7329 \\ 4.5823 \\ 11.6957 \end{bmatrix} \tag{A-25}$$

The first two eigenvalues correspond to rigid-body modes.

The third eigenvalues are nearly equal, with an error of 0.4%.

Note that the natural frequencies are proportional to the square root of the respective eigenvalue. The error in the third natural frequency would be only 0.2%.