MODAL TRANSIENT ANALYSIS OF A MULTI-DEGREE-OF-FREEDOM
SYSTEM WITH ENFORCED MOTION

Revision C

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______________________________________________________________________________

Variables

| \( M \) | Mass matrix |
| \( K \) | Stiffness matrix |
| \( F \) | Applied forces |
| \( F_d \) | Forces at driven nodes |
| \( F_f \) | Forces at free nodes |
| \( I \) | Identity matrix |
| \( \Pi \) | Transformation matrix |
| \( \mathbf{u} \) | Displacement vector |
| \( \mathbf{u}_d \) | Displacements at driven nodes |
| \( \mathbf{u}_f \) | Displacements at free nodes |

The equation of motion for a multi-degree-of-freedom system is

\[ [M][\ddot{\mathbf{u}}] + [K][\mathbf{u}] = \mathbf{F} \]

(1)

\[ [\mathbf{u}] = \begin{bmatrix} \mathbf{u}_d \\ \mathbf{u}_f \end{bmatrix} \]

(2)
Partition the matrices and vectors as follows

\[
\begin{bmatrix}
M_{dd} & M_{df} \\
M_{fd} & M_{ff}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_d \\
\ddot{u}_f
\end{bmatrix}
+ \begin{bmatrix}
K_{dd} & K_{df} \\
K_{fd} & K_{ff}
\end{bmatrix}
\begin{bmatrix}
u_d \\
u_f
\end{bmatrix}
= \begin{bmatrix}
F_d \\
F_f
\end{bmatrix}
\]

(3)

The equations of motions for enforced displacement and acceleration are given in Appendices A and B, respectively.

Create a transformation matrix such that

\[
\begin{bmatrix}
u_d \\
u_f
\end{bmatrix}
= \Pi
\begin{bmatrix}
u_d \\
u_w
\end{bmatrix}
\]

(4)

\[
\Pi = \begin{bmatrix}
I & 0 \\
T_1 & T_2
\end{bmatrix}
\]

(5)

\[
\begin{bmatrix}
M_{dd} & M_{df} \\
M_{fd} & M_{ff}
\end{bmatrix}
\Pi
\begin{bmatrix}
\ddot{u}_d \\
\ddot{u}_w
\end{bmatrix}
+ \begin{bmatrix}
K_{dd} & K_{df} \\
K_{fd} & K_{ff}
\end{bmatrix}
\Pi
\begin{bmatrix}
u_d \\
u_w
\end{bmatrix}
= \begin{bmatrix}
F_d \\
F_f
\end{bmatrix}
\]

(6)

Premultiply by $\Pi^T$,

\[
\Pi^T
\begin{bmatrix}
M_{dd} & M_{df} \\
M_{fd} & M_{ff}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_d \\
\ddot{u}_w
\end{bmatrix}
+ \Pi^T
\begin{bmatrix}
K_{dd} & K_{df} \\
K_{fd} & K_{ff}
\end{bmatrix}
\Pi
\begin{bmatrix}
u_d \\
u_w
\end{bmatrix}
= \Pi^T
\begin{bmatrix}
F_d \\
F_f
\end{bmatrix}
\]

(7)
APPENDIX A
Enforced Displacement

Again, the partitioned equation of motion is

\[
\Pi^T \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \Pi \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \Pi^T \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \Pi \begin{bmatrix} \dot{u}_d \\ \dot{u}_w \end{bmatrix} = \Pi^T \begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{A-1}
\]

Transform the equation of motion to uncouple the mass matrix so that the resulting mass matrix is

\[
\begin{bmatrix} \dot{M}_{dd} & 0 \\ 0 & \dot{M}_{ww} \end{bmatrix}
\tag{A-2}
\]

Apply the transformation to the mass matrix

\[
\Pi^T \ M \ \Pi = \begin{bmatrix} I & T_1^T \\ 0 & T_2^T \end{bmatrix} \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \tag{A-3}
\]

\[
\Pi^T \ M \ \Pi = \begin{bmatrix} I & T_1^T \\ 0 & T_2^T \end{bmatrix} \begin{bmatrix} M_{dd} + M_{df} T_1 & M_{df} T_2 \\ M_{fd} + M_{ff} T_1 & M_{ff} T_2 \end{bmatrix} \tag{A-4}
\]

\[
\Pi^T \ M \ \Pi = \begin{bmatrix} M_{dd} + M_{df} T_1 + T_1^T (M_{fd} + M_{ff} T) & M_{df} T_2 + T_1^T M_{ff} T_2 \\ T_2^T (M_{fd} + M_{ff} T_1) & T_2^T (M_{ff} T_2) \end{bmatrix} \tag{A-5}
\]
\[ \Pi^T M \Pi = \begin{bmatrix} M_{dd} + M_{df}^T T_1 + T_1^T (M_{fd} + M_{ff} T_1) & (M_{df} + T_1^T M_{ff}) T_2 \\ T_2^T (M_{fd} + M_{ff} T_1) & T_2^T (M_{ff} T_2) \end{bmatrix} \]  
(A-6)

\[ \Pi^T M \Pi = \begin{bmatrix} M_{dd} + T_1^T M_{fd} + (M_{df} + T_1^T M_{ff}) T_1 & (M_{df} + T_1^T M_{ff}) T_2 \\ T_2^T (M_{fd} + M_{ff} T_1) & T_2^T (M_{ff} T_2) \end{bmatrix} \]  
(A-7)

Let

\[ T_2 = I \]  
(A-8)

\[ \Pi^T M \Pi = \begin{bmatrix} M_{dd} + T_1^T M_{fd} + (M_{df} + T_1^T M_{ff}) T_1 & (M_{df} + T_1^T M_{ff}) \\ (M_{fd} + M_{ff} T_1) & M_{ff} \end{bmatrix} = \begin{bmatrix} \dot{M}_{dd} & 0 \\ 0 & \dot{M}_{ww} \end{bmatrix} \]  
(A-9)

\[ M_{df} + T_1^T M_{ff} = 0 \]  
(A-10)

\[ T_1^T = -M_{df} M_{ff}^{-1} \]  
(A-11)

\[ T_1 = -M_{ff}^{-1} M_{fd} \]  
(A-12)

The transformation matrix is

\[ \Pi = \begin{bmatrix} I_{dd} & 0 \\ T_1 & I_{ff} \end{bmatrix} \]  
(A-13)

\[ \dot{M}_{dd} = M_{dd} + T_1^T M_{fd} + (M_{df} + T_1^T M_{ff}) T_1 \]  
(A-14)
\[ M_{ww} = M_{ff} \]  

(A-15)

\[
\Pi^T \mathbf{K} \Pi = \begin{bmatrix}
I_{dd} & T_1^T \\
0 & I_{ff}
\end{bmatrix}
\begin{bmatrix}
K_{dd} & K_{df} \\
K_{fd} & K_{ff}
\end{bmatrix}
\begin{bmatrix}
I_{dd} & 0 \\
T_1 & I_{ff}
\end{bmatrix}
\]  

(A-16)

By similarity, the transformed stiffness matrix is

\[
\begin{bmatrix}
\hat{k}_{dd} & \hat{k}_{dw} \\
\hat{k}_{wd} & \hat{k}_{ww}
\end{bmatrix} = \begin{bmatrix}
K_{dd} + T_1^T K_{fd} + \left( K_{df} + T_1^T K_{ff} \right) T_1 & \left( K_{df} + T_1^T K_{ff} \right) T_1 \\
(K_{fd} + K_{ff} T_1) & K_{ff}
\end{bmatrix}
\]  

(A-17)

\[
\begin{bmatrix}
\hat{F}_d \\
\hat{F}_w
\end{bmatrix} = \begin{bmatrix}
I_{dd} & T_1 \\
0 & I_{ff}
\end{bmatrix}
\begin{bmatrix}
F_d \\
F_f
\end{bmatrix}
\]  

(A-18)

\[
\begin{bmatrix}
\hat{F}_d \\
\hat{F}_w
\end{bmatrix} = \begin{bmatrix}
I_{dd} F_d + T_1 F_f \\
I_{ff} F_f
\end{bmatrix}
\]  

(A-19)

\[
\begin{bmatrix}
\hat{F}_d \\
\hat{F}_w
\end{bmatrix} = \begin{bmatrix}
F_d + T_1 F_f \\
F_f
\end{bmatrix}
\]  

(A-20)

\[
\begin{bmatrix}
\hat{M}_{dd} & 0 \\
0 & \hat{M}_{ww}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_d \\
\ddot{u}_w
\end{bmatrix} + \begin{bmatrix}
\hat{k}_{dd} & \hat{k}_{dw} \\
\hat{k}_{wd} & \hat{k}_{ww}
\end{bmatrix}
\begin{bmatrix}
u_d \\
u_w
\end{bmatrix} = \begin{bmatrix}
\hat{F}_d \\
\hat{F}_w
\end{bmatrix}
\]  

(A-21)

\[ \hat{M}_{ww} \dddot{u}_w + \hat{k}_{ww} \dot{u}_d + \hat{k}_{ww} u_w = \hat{F}_w \]  

(A-22)
The equation of motion is thus

\[ \dot{M}_{ww} \ddot{u}_w + \dot{k}_{ww} u_w = \dot{F}_w - \dot{k}_{wd} u_d \]  

(A-23)

The final displacement are found via

\[
\begin{bmatrix}
  \dot{u}_d \\
  \dot{u}_f
\end{bmatrix} = \Pi
\begin{bmatrix}
  u_d \\
  u_w
\end{bmatrix}
\]  

(A-24)

\[
\Pi = \begin{bmatrix}
  I_{dd} & 0 \\
  -M_{ff}^{-1} M_{fd} & I_{ff}
\end{bmatrix}
\]  

(A-25)
APPENDIX B

Enforced Acceleration

Again, the partitioned equation of motion is

\[
\Pi^T \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \Pi \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \Pi^T \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \Pi^T \begin{bmatrix} F_d \\ F_w \end{bmatrix}
\]  

(B-1)

Transform the equation of motion to uncouple the stiffness matrix so that the resulting stiffness matrix is

\[
\begin{bmatrix} \ddot{K}_{dd} & 0 \\ 0 & \ddot{K}_{ww} \end{bmatrix}
\]

(B-2)

\[
\Pi^T \Pi = \begin{bmatrix} I & T_1^T \\ 0 & T_2^T \end{bmatrix} \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix}
\]

(B-3)

\[
\Pi^T \Pi = \begin{bmatrix} I & T_1^T \\ 0 & T_2^T \end{bmatrix} \begin{bmatrix} K_{dd} + K_{df}T_1 & K_{df}T_2 \\ K_{fd} + K_{ff}T_1 & K_{ff}T_2 \end{bmatrix}
\]

(B-4)

\[
\Pi^T \Pi = \begin{bmatrix} \left( K_{dd} + K_{df}T_1 + T_1^T(K_{fd} + K_{ff}T) \right) & \left( K_{df}T_2 + T_1^TK_{ff}T_2 \right) \\ T_2^T(K_{fd} + K_{ff}T_1) & T_2^T(K_{ff}T_2) \end{bmatrix}
\]

(B-5)
\[ \Pi^T K \Pi = \begin{bmatrix} K_{dd} + K_{df} T_1 + T_1^T (K_{fd} + K_{ff} T_1) & \left(K_{df} + T_1^T K_{ff}\right) T_2 \\ T_2^T (K_{fd} + K_{ff} T_1) & T_2^T (K_{ff} T_2) \end{bmatrix} \]  

(B-6)

\[ \Pi^T K \Pi = \begin{bmatrix} K_{dd} + T_1^T K_{fd} + \left(K_{df} + T_1^T K_{ff}\right) \Gamma_1 & \left(K_{df} + T_1^T K_{ff}\right) T_2 \\ T_2^T (K_{fd} + K_{ff} T_1) & T_2^T (K_{ff} T_2) \end{bmatrix} \]  

(B-7)

Let

\[ T_2 = \mathbf{I} \]  

(B-8)

\[ \Pi^T K \Pi = \begin{bmatrix} K_{dd} + T_1^T K_{fd} + \left(K_{df} + T_1^T K_{ff}\right) \Gamma_1 & \left(K_{df} + T_1^T K_{ff}\right) T_2 \\ (K_{fd} + K_{ff} T_1) & K_{ff} \end{bmatrix} = \begin{bmatrix} \hat{K}_{dd} & 0 \\ 0 & \hat{K}_{ww} \end{bmatrix} \]  

(B-9)

\[ K_{df} + T_1^T K_{ff} = 0 \]  

(B-10)

\[ T_1^T = -K_{df} K_{ff}^{-1} \]  

(B-11)

\[ T_1 = -K_{ff}^{-1} K_{fd} \]  

(B-12)

\[ \Pi = \begin{bmatrix} I_{dd} & 0 \\ T_1 & I_{ff} \end{bmatrix} \]  

(B-13)
\[ \hat{K}_{dd} = K_{dd} + T_1^T K_{fd} + \left( K_{df} + T_1^T K_{ff} \right) T_1 \]  \hfill (B-14)

\[ \hat{K}_{ww} = K_{ff} \]  \hfill (B-15)

\[ \Pi^T M \Pi = \begin{bmatrix} I_{dd} & T_1^T & M_{dd} & M_{df} & I_{dd} & 0 \\ 0 & I_{ff} & M_{fd} & M_{ff} & T_1 & I_{ff} \end{bmatrix} \]  \hfill (B-16)

By similarity, the transformed mass matrix is

\[ \begin{bmatrix} \hat{m}_{dd} & \hat{m}_{dw} \\ \hat{m}_{wd} & \hat{m}_{ww} \end{bmatrix} = \begin{bmatrix} M_{dd} + T_1^T M_{fd} + \left( M_{df} + T_1^T M_{ff} \right) T_1 \\ (M_{fd} + M_{ff} T_1) \end{bmatrix} \begin{bmatrix} I_{dd} & T_1^T & M_{dd} & M_{df} & I_{dd} & 0 \\ 0 & I_{ff} & M_{fd} & M_{ff} & T_1 & I_{ff} \end{bmatrix} \]  \hfill (B-17)

\[ \begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} I_{dd} & T_1 \\ 0 & I_{ff} \end{bmatrix} \begin{bmatrix} F_d \\ F_f \end{bmatrix} \]  \hfill (B-18)

\[ \begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} I_{dd} F_d + T_1 F_f \\ I_{ff} F_f \end{bmatrix} \]  \hfill (B-19)

\[ \begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} F_d + T_1 F_f \\ F_f \end{bmatrix} \]  \hfill (B-20)

\[ \begin{bmatrix} \hat{\dot{m}}_{dd} & \hat{\dot{m}}_{dw} \\ \hat{\dot{m}}_{wd} & \hat{\dot{m}}_{ww} \end{bmatrix} \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \begin{bmatrix} \hat{K}_{dd} & 0 \\ 0 & \hat{K}_{ww} \end{bmatrix} \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} \]  \hfill (B-21)
\[ \dot{m}_{wd} \ddot{u}_d + \dot{m}_{ww} \ddot{u}_w + \dot{K}_{ww} u_w = \dot{F}_w \]  \hspace{1cm} (B-22)

The equation of motion is thus

\[ \dot{m}_{ww} \ddot{u}_w + \dot{K}_{ww} u_w = \dot{F}_w - \dot{m}_{wd} \ddot{u}_d \]  \hspace{1cm} (B-23)

The final displacement are found via

\[
\begin{bmatrix}
    u_d \\
    u_f
\end{bmatrix} = \Pi
\begin{bmatrix}
    u_d \\
    u_w
\end{bmatrix}
\]

\hspace{1cm} (B-24)

\[
\Pi = \begin{bmatrix}
    I_{dd} & 0 \\
    -K_{ff}^{-1}K_{fd} & I_{ff}
\end{bmatrix}
\]

\hspace{1cm} (B-25)
The diagram of a sample system is shown in Figure 1.

```
<table>
<thead>
<tr>
<th>k</th>
<th>10e+07</th>
<th>m</th>
<th>65,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>k1</td>
<td>10e+07</td>
<td>m1</td>
<td>65,000</td>
</tr>
<tr>
<td>k2</td>
<td>8e+07</td>
<td>m2</td>
<td>65,000</td>
</tr>
<tr>
<td>k3</td>
<td>6e+07</td>
<td>m3</td>
<td>65,000</td>
</tr>
<tr>
<td>k4</td>
<td>8e+07</td>
<td>m4</td>
<td>60,000</td>
</tr>
<tr>
<td>k5</td>
<td>6e+07</td>
<td>m5</td>
<td>45,000</td>
</tr>
</tbody>
</table>
```

English units:
- stiffness (lbf/in), mass(lbf sec^2/in), force(lbf)
- Assume modal damping of 5% for all modes.

Figure C-1.
The equation of motion is

\[
\begin{bmatrix}
  m_1 & 0 & 0 & 0 & 0 \\
  0 & m_2 & 0 & 0 & 0 \\
  0 & 0 & m_3 & 0 & 0 \\
  0 & 0 & 0 & m_4 & 0 \\
  0 & 0 & 0 & 0 & m_5
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\ddot{x}_3 \\
\ddot{x}_4 \\
\ddot{x}_5
\end{bmatrix}
= \begin{bmatrix}
  k_1 + k_2 & -k_2 & 0 & 0 & 0 \\
  -k_2 & k_2 + k_3 & -k_3 & 0 & 0 \\
  0 & -k_3 & k_3 + k_4 & -k_4 & 0 \\
  0 & 0 & -k_4 & k_4 + k_5 & -k_5 \\
  0 & 0 & 0 & -k_5 & k_5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
+ \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5
\end{bmatrix}
\]  

(C-1)

Now set all of the applied forces \( f_i \) to zero.

Drive mass 4 with acceleration as follows:

\[
a_4(t) = (386 \text{ in/sec}^2) \sin \left[ 2\pi \left(4 \text{ Hz}\right) t \right], \quad 0 \leq t \leq 3 \text{ sec}
\]  

(C-2)

Set the sample rate at 200 samples/sec.

The results are shown in Figure C-2, as calculated by Matlab script: enforced_acceleration.m.
Figure C-2.

```matlab
>> enforced_acceleration

enforced_acceleration.m  ver 2.1  March 9, 2011
by Tom Irvine

Enter the units system
1=English  2=metric
1
Assume symmetric mass and stiffness matrices.
Select input mass unit
1=lbm  2=lbf sec^2/in
2
stiffness unit = lbf/in

Select file input method
1=file preloaded into Matlab
2=Excel file
1

Mass Matrix
Enter the matrix name:  mass_5dof
```
Stiffness Matrix
Enter the matrix name: stiff_5dof

Input Matrices

\[
\text{mass} =
\begin{bmatrix}
65000 & 0 & 0 & 0 & 0 \\
0 & 65000 & 0 & 0 & 0 \\
0 & 0 & 65000 & 0 & 0 \\
0 & 0 & 0 & 60000 & 0 \\
0 & 0 & 0 & 0 & 45000 \\
\end{bmatrix}
\]

\[
\text{stiff} =
\begin{bmatrix}
200000000 & -100000000 & 0 & 0 & 0 \\
-100000000 & 180000000 & -80000000 & 0 & 0 \\
0 & -80000000 & 160000000 & -80000000 & 0 \\
0 & 0 & -80000000 & 140000000 & -60000000 \\
0 & 0 & 0 & -60000000 & 60000000 \\
\end{bmatrix}
\]

Select modal damping input method
1 = uniform damping for all modes
2 = damping vector
1

Enter damping ratio
0.05

number of dofs = 5

Enter the starting time (sec)
0

Enter the end time (sec)
10

Enter the sample rate (samples/sec)
200

Enter the number of dofs with enforced acceleration. (maximum = 4)
1

Each input file must have two columns: time & acceleration

Enter the first dof
4

The input file must have: time(sec) & accel
Enter the applied acceleration input matrix name for this dof.
sine
MT =

\[ 1.0 \times 10^5 * \]
\[
\begin{bmatrix}
1.5495 & 0.1444 & 0.2889 & 0.4694 & 0.4500 \\
0.1444 & 0.6500 & 0 & 0 & 0 \\
0.2889 & 0 & 0.6500 & 0 & 0 \\
0.4694 & 0 & 0 & 0.6500 & 0 \\
0.4500 & 0 & 0 & 0 & 0.4500
\end{bmatrix}
\]

KT =

\[ 1.0 \times 10^8 * \]
\[
\begin{bmatrix}
0.2222 & 0.0000 & -0.0000 & 0 & -0.0000 \\
0.0000 & 2.0000 & -1.0000 & 0 & 0 \\
-0.0000 & -1.0000 & 1.8000 & -0.8000 & 0 \\
0 & 0 & -0.8000 & 1.6000 & 0 \\
-0.0000 & 0 & 0 & 0 & 0.6000
\end{bmatrix}
\]

Natural Frequencies

<table>
<thead>
<tr>
<th>No.</th>
<th>f(\text{Hz})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>1.8283</td>
</tr>
<tr>
<td>2.</td>
<td>4.9465</td>
</tr>
<tr>
<td>3.</td>
<td>7.4613</td>
</tr>
<tr>
<td>4.</td>
<td>9.7491</td>
</tr>
<tr>
<td>5.</td>
<td>11.171</td>
</tr>
</tbody>
</table>

Modes Shapes (column format)

ModeShapes =

\[
\begin{bmatrix}
0.0023 & -0.0008 & -0.0016 & -0.0027 & -0.0011 \\
0.0002 & 0.0018 & 0.0023 & -0.0013 & 0.0024 \\
0.0002 & 0.0026 & 0.0018 & 0.0020 & -0.0021 \\
0.0002 & 0.0018 & -0.0008 & 0.0036 & 0.0026 \\
0.0003 & -0.0020 & 0.0040 & 0.0041 & 0.0015
\end{bmatrix}
\]

Mwd =

\[ 1.0 \times 10^4 * \]
\[
\begin{bmatrix}
1.4444 \\
2.8889 \\
4.6944 \\
4.5000
\end{bmatrix}
\]
\[
M_{ww} =
\begin{bmatrix}
65000 & 0 & 0 & 0 \\
0 & 65000 & 0 & 0 \\
0 & 0 & 65000 & 0 \\
0 & 0 & 0 & 45000
\end{bmatrix}
\]

\[
K_{ww} =
\begin{bmatrix}
200000000 & -100000000 & 0 & 0 \\
-100000000 & 180000000 & -80000000 & 0 \\
0 & -80000000 & 160000000 & 0 \\
0 & 0 & 0 & 60000000
\end{bmatrix}
\]

Natural Frequencies
No.      f(Hz)
1.        4.5268
2.        5.8115
3.        8.2749
4.        11.021

Modes Shapes (column format)

\[
\text{ModeShapes} =
\begin{bmatrix}
0.0019 & 0 & 0.0024 & 0.0024 \\
0.0028 & 0 & 0.0006 & -0.0027 \\
0.0021 & 0 & -0.0030 & 0.0014 \\
0 & 0.0047 & 0 & 0
\end{bmatrix}
\]

Participation Factors

\[
\text{part} =
\begin{bmatrix}
435.2223 \\
212.1320 \\
0.9398 \\
74.7038
\end{bmatrix}
\]
APPENDIX D

Enforced Displacement Example

Figure D-1.

Consider the spring-mass system from Appendix C.

Drive mass 2 with displacement as follows:

\[ d_2(t) = (1 \text{ inch}) \sin \left[ 2\pi (3 \text{ Hz}) \cdot t \right], \quad 0 \leq t \leq 3 \text{ sec} \]  

(D-1)

The results are shown in Figure D-1, as calculated by Matlab script: enforced_displacement.m.
>> enforced_displacement

enforced_displacement.m ver 1.5 March 9, 2011
by Tom Irvine

Enter the units system
1=English  2=metric
1
Assume symmetric mass and stiffness matrices.
Select input mass unit
1=lbm  2=lbf sec^2/in
2
stiffness unit = lbf/in

Select file input method
1=file preloaded into Matlab
2=Excel file
1

Mass Matrix
Enter the matrix name: mass_5dof

Stiffness Matrix
Enter the matrix name: stiff_5dof

Input Matrices

mass =

   65000       0       0       0       0
     0   65000       0       0       0
     0       0   65000       0       0
     0       0       0   60000       0
     0       0       0       0   45000

stiff =

  200000000 -100000000       0       0       0
-100000000  180000000 -80000000       0       0
     0 -80000000  160000000 -80000000       0
     0       0 -80000000  140000000 -60000000
     0       0       0 -60000000   60000000

Select modal damping input method
1=uniform damping for all modes
2=damping vector
1

Enter damping ratio
0.05

number of dofs =5
Enter the starting time (sec)
0

Enter the end time (sec)
5

Enter the sample rate (samples/sec)
200

Enter the number of dofs with enforced displacement. (maximum = 4)
1

Each input file must have two columns: time & displacement

Enter the first dof
2

The input file must have: time(sec) & disp(inch)
Enter the applied displacement input matrix name for this dof.
sine

MT =

\[
\begin{bmatrix}
65000 & 0 & 0 & 0 & 0 \\
0 & 65000 & 0 & 0 & 0 \\
0 & 0 & 65000 & 0 & 0 \\
0 & 0 & 0 & 60000 & 0 \\
0 & 0 & 0 & 0 & 45000 \\
\end{bmatrix}
\]

KT =

\[
\begin{bmatrix}
180000000 & -100000000 & -80000000 & 0 & 0 \\
-100000000 & 200000000 & 0 & 0 & 0 \\
-80000000 & 0 & 160000000 & -80000000 & 0 \\
0 & 0 & -80000000 & 140000000 & -60000000 \\
0 & 0 & 0 & -60000000 & 60000000 \\
\end{bmatrix}
\]

Natural Frequencies
No. f(Hz)
1. 1.8283
2. 4.9465
3. 7.4613
4. 9.7491
5. 11.171
Modes Shapes (column format)

ModeShapes =

\[
\begin{bmatrix}
0.0013 & 0.0023 & 0.0011 & -0.0008 & -0.0025 \\
0.0007 & 0.0017 & 0.0020 & 0.0019 & 0.0021 \\
0.0019 & 0.0013 & -0.0020 & -0.0017 & 0.0018 \\
0.0023 & -0.0008 & -0.0016 & 0.0027 & -0.0011 \\
0.0026 & -0.0027 & 0.0024 & -0.0015 & 0.0004
\end{bmatrix}
\]

Kwd =

\[
\begin{bmatrix}
-100000000 \\
-80000000 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Kww =

\[
\begin{bmatrix}
200000000 & 0 & 0 & 0 & 0 \\
0 & 160000000 & -80000000 & 0 & 0 \\
0 & -80000000 & 140000000 & -60000000 & 0 \\
0 & 0 & -60000000 & 60000000 & 0
\end{bmatrix}
\]

Mww =

\[
\begin{bmatrix}
65000 & 0 & 0 & 0 & 0 \\
0 & 65000 & 0 & 0 & 0 \\
0 & 0 & 60000 & 0 & 0 \\
0 & 0 & 0 & 45000 & 0
\end{bmatrix}
\]

Natural Frequencies

<table>
<thead>
<tr>
<th>No.</th>
<th>f(Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>2.7278</td>
</tr>
<tr>
<td>2.</td>
<td>6.9133</td>
</tr>
<tr>
<td>3.</td>
<td>8.8283</td>
</tr>
<tr>
<td>4.</td>
<td>9.9997</td>
</tr>
</tbody>
</table>

Modes Shapes (column format)

ModeShapes =

\[
\begin{bmatrix}
0 & 0 & 0.0039 & 0 \\
0.0014 & 0.0027 & 0 & 0.0024 \\
0.0025 & 0.0013 & 0 & -0.0029 \\
0.0033 & -0.0031 & 0 & 0.0015
\end{bmatrix}
\]
Participation Factors

part =

392.7638
115.3874
254.9510
49.2174