Modal Transient Analysis of a Beam with Enforced Motion via a Ramp Invariant Digital Recursive Filtering Relationship

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Variables

M	Mass matrix
K	Stiffness matrix
F	Applied forces
F_d	Forces at driven nodes
F_f	Forces at free nodes
I	Identity matrix
Π	Transformation matrix
u	Displacement vector
u_d	Displacements at driven nodes
u_f	Displacements at free nodes

Consider a beam which is modeled via the finite element method as a multi-degree-of-freedom (MDOF) system. The homogeneous equation of motion for an MDOF system is

$$[M][\ddot{u}]+[K][u]=[0] \tag{1}$$

$$[u]=\begin{bmatrix} u_d \\ u_f \end{bmatrix} \tag{2}$$

Partition the matrices and vectors as follows

$$\begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_f \end{bmatrix} + \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \begin{bmatrix} u_d \\ u_f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

The equations of motions for enforced displacement and acceleration are given in Appendices A and B, respectively.

Create a transformation matrix such that

$$\begin{bmatrix} u_d \\ u_f \end{bmatrix} = \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} \tag{4}$$

$$\Pi = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \Pi \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

Premultiply by Π^T,

$$\Pi^T \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \Pi \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \Pi^T \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

Equation (7) can then be uncoupled using the normal modes per the method in Reference 4.

The modal transient analysis can be then be performed on the modal coordinates using the ramp invariant digital recursive filtering relationship in Reference 5 for the case where the acceleration of one or modal degrees-of-freedom have been prescribed.

The resulting modal responses are then transformed into physical responses per the method in Reference 4.

Next, the transformation in equation (4) is performed.

Finally, the physical responses are reassembled in the correct order if necessary.

References

1. T. Irvine, The Generalized Coordinate Method for Discrete Systems, Revision F, Vibrationdata, 2012.

2. T. Irvine, Modal Transient Analysis of a System Subjected to an Applied Force via Ramp Invariant Digital Recursive Filtering Relationship, Revision B, Vibrationdata, 2012.

3. T. Irvine, Transverse Vibration of a Beam via the Finite Element Method, Revision F, Vibrationdata, 2010.

4. T. Irvine, The Generalized Coordinate Method for Discrete Systems, Revision F, Vibrationdata, 2012.

5. T. Irvine, Modal Transient Analysis of a System Subjected to an Applied Force via Ramp Invariant Digital Recursive Filtering Relationship, Revision B, Vibrationdata, 2012.

APPENDIX A

Enforced Displacement

Again, the partitioned equation of motion is

$$\Pi^T \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \Pi \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \Pi^T \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \Pi^T \begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{A-1}$$

Transform the equation of motion to uncouple the mass matrix so that the resulting mass matrix is

$$\begin{bmatrix} \hat{M}_{dd} & 0 \\ 0 & \hat{M}_{ww} \end{bmatrix} \tag{A-2}$$

Apply the transformation to the mass matrix

$$\Pi^T \, M \, \Pi = \begin{bmatrix} I & T_1{}^T \\ 0 & T_2{}^T \end{bmatrix} \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \tag{A-3}$$

$$\Pi^T \, M \, \Pi = \begin{bmatrix} I & T_1{}^T \\ 0 & T_2{}^T \end{bmatrix} \begin{bmatrix} M_{dd} + M_{df} T_1 & M_{df} T_2 \\ M_{fd} + M_{ff} T_1 & M_{ff} T_2 \end{bmatrix} \tag{A-4}$$

$$\Pi^T \, M \, \Pi = \begin{bmatrix} M_{dd} + M_{df} T_1 + T_1{}^T \left(M_{fd} + M_{ff} T \right) & M_{df} T_2 + T_1{}^T M_{ff} T_2 \\ T_2{}^T \left(M_{fd} + M_{ff} T_1 \right) & T_2{}^T \left(M_{ff} T_2 \right) \end{bmatrix} \tag{A-5}$$

$$\Pi^T M \Pi = \begin{bmatrix} M_{dd} + M_{df} T_1 + T_1^T (M_{fd} + M_{ff} T_1) & (M_{df} + T_1^T M_{ff}) T_2 \\ T_2^T (M_{fd} + M_{ff} T_1) & T_2^T (M_{ff} T_2) \end{bmatrix} \tag{A-6}$$

$$\Pi^T M \Pi = \begin{bmatrix} M_{dd} + T_1^T M_{fd} + (M_{df} + T_1^T M_{ff}) T_1 & (M_{df} + T_1^T M_{ff}) T_2 \\ T_2^T (M_{fd} + M_{ff} T_1) & T_2^T (M_{ff} T_2) \end{bmatrix} \tag{A-7}$$

Let

$$T_2 = I \tag{A-8}$$

$$\Pi^T M \Pi = \begin{bmatrix} M_{dd} + T_1^T M_{fd} + (M_{df} + T_1^T M_{ff}) T_1 & (M_{df} + T_1^T M_{ff}) \\ (M_{fd} + M_{ff} T_1) & M_{ff} \end{bmatrix} = \begin{bmatrix} \hat{M}_{dd} & 0 \\ 0 & \hat{M}_{ww} \end{bmatrix} \tag{A-9}$$

$$M_{df} + T_1^T M_{ff} = 0 \tag{A-10}$$

$$T_1^T = -M_{df} M_{ff}^{-1} \tag{A-11}$$

$$T_1 = -M_{ff}^{-1} M_{fd} \tag{A-12}$$

The transformation matrix is

$$\Pi = \begin{bmatrix} I_{dd} & 0 \\ T_1 & I_{ff} \end{bmatrix} \tag{A-13}$$

$$\hat{M}_{dd} = M_{dd} + T_1^T M_{fd} + (M_{df} + T_1^T M_{ff}) T_1 \tag{A-14}$$

$$\hat{M}_{ww} = M_{ff} \tag{A-15}$$

$$\Pi^T K \Pi = \begin{bmatrix} I_{dd} & T_1{}^T \\ 0 & I_{ff} \end{bmatrix} \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \begin{bmatrix} I_{dd} & 0 \\ T_1 & I_{ff} \end{bmatrix} \tag{A-16}$$

By similarity, the transformed stiffness matrix is

$$\begin{bmatrix} \hat{k}_{dd} & \hat{k}_{dw} \\ \hat{k}_{wd} & \hat{k}_{ww} \end{bmatrix} = \begin{bmatrix} K_{dd} + T_1{}^T K_{fd} + \left(K_{df} + T_1{}^T K_{ff}\right) T_1 & \left(K_{df} + T_1{}^T K_{ff}\right) \\ \left(K_{fd} + K_{ff} T_1\right) & K_{ff} \end{bmatrix} \tag{A-17}$$

$$\begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} I_{dd} & T_1 \\ 0 & I_{ff} \end{bmatrix} \begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{A-18}$$

$$\begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} I_{dd} F_d + T_1 F_f \\ I_{ff} F_f \end{bmatrix} \tag{A-19}$$

$$\begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} F_d + T_1 F_f \\ F_f \end{bmatrix} \tag{A-20}$$

$$\begin{bmatrix} \hat{M}_{dd} & 0 \\ 0 & \hat{M}_{ww} \end{bmatrix} \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \begin{bmatrix} \hat{k}_{dd} & \hat{k}_{dw} \\ \hat{k}_{wd} & \hat{k}_{ww} \end{bmatrix} \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} \tag{A-21}$$

$$\hat{M}_{ww} \ddot{u}_w + \hat{k}_{wd} u_d + \hat{k}_{ww} u_w = \hat{F}_w \tag{A-22}$$

The equation of motion is thus

$$\hat{M}_{ww} \ddot{u}_w + \hat{k}_{ww} u_w = \hat{F}_w - \hat{k}_{wd} u_d \tag{A-23}$$

The final displacements are found via

$$\begin{bmatrix} u_d \\ u_f \end{bmatrix} = \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} \tag{A-24}$$

$$\Pi = \begin{bmatrix} I_{dd} & 0 \\ -M_{ff}{}^{-1} M_{fd} & I_{ff} \end{bmatrix} \tag{A-25}$$

APPENDIX B

Enforced Acceleration

Again, the partitioned equation of motion is

$$\Pi^T \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \Pi \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \Pi^T \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \Pi^T \begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{B-1}$$

Transform the equation of motion to uncouple the stiffness matrix so that the resulting stiffness matrix is

$$\begin{bmatrix} \hat{K}_{dd} & 0 \\ 0 & \hat{K}_{ww} \end{bmatrix} \tag{B-2}$$

$$\Pi^T \, K \, \Pi = \begin{bmatrix} I & T_1^{\,T} \\ 0 & T_2^{\,T} \end{bmatrix} \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \tag{B-3}$$

$$\Pi^T \, K \, \Pi = \begin{bmatrix} I & T_1^{\,T} \\ 0 & T_2^{\,T} \end{bmatrix} \begin{bmatrix} K_{dd} + K_{df} T_1 & K_{df} T_2 \\ K_{fd} + K_{ff} T_1 & K_{ff} T_2 \end{bmatrix} \tag{B-4}$$

$$\Pi^T \, K \, \Pi = \begin{bmatrix} K_{dd} + K_{df} T_1 + T_1^{\,T} \left(K_{fd} + K_{ff} T \right) & K_{df} T_2 + T_1^{\,T} K_{ff} T_2 \\ T_2^{\,T} \left(K_{fd} + K_{ff} T_1 \right) & T_2^{\,T} \left(K_{ff} T_2 \right) \end{bmatrix} \tag{B-5}$$

$$\Pi^T K \Pi = \begin{bmatrix} K_{dd} + K_{df} T_1 + T_1^T \left(K_{fd} + K_{ff} T_1\right) & \left(K_{df} + T_1^T K_{ff}\right) T_2 \\ T_2^T \left(K_{fd} + K_{ff} T_1\right) & T_2^T \left(K_{ff} T_2\right) \end{bmatrix} \tag{B-6}$$

$$\Pi^T K \Pi = \begin{bmatrix} K_{dd} + T_1^T K_{fd} + \left(K_{df} + T_1^T K_{ff}\right) T_1 & \left(K_{df} + T_1^T K_{ff}\right) T_2 \\ T_2^T \left(K_{fd} + K_{ff} T_1\right) & T_2^T \left(K_{ff} T_2\right) \end{bmatrix} \tag{B-7}$$

Let

$$T_2 = I \tag{B-8}$$

$$\Pi^T K \Pi = \begin{bmatrix} K_{dd} + T_1^T K_{fd} + \left(K_{df} + T_1^T K_{ff}\right) T_1 & \left(K_{df} + T_1^T K_{ff}\right) \\ \left(K_{fd} + K_{ff} T_1\right) & K_{ff} \end{bmatrix} = \begin{bmatrix} \hat{K}_{dd} & 0 \\ 0 & \hat{K}_{ww} \end{bmatrix} \tag{B-9}$$

$$K_{df} + T_1^T K_{ff} = 0 \tag{B-10}$$

$$T_1^T = -K_{df} K_{ff}^{-1} \tag{B-11}$$

$$T_1 = -K_{ff}^{-1} K_{fd} \tag{B-12}$$

$$\Pi = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \tag{B-13}$$

$$\hat{K}_{dd} = K_{dd} + T_1{}^T K_{fd} + \left(K_{df} + T_1{}^T K_{ff}\right) T_1 \tag{B-14}$$

$$\hat{K}_{ww} = K_{ff} \tag{B-15}$$

$$\Pi^T M \Pi = \begin{bmatrix} I_{dd} & T_1{}^T \\ 0 & I_{ff} \end{bmatrix} \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \begin{bmatrix} I_{dd} & 0 \\ T_1 & I_{ff} \end{bmatrix} \tag{B-16}$$

By similarity, the transformed mass matrix is

$$\begin{bmatrix} \hat{m}_{dd} & \hat{m}_{dw} \\ \hat{m}_{wd} & \hat{m}_{ww} \end{bmatrix} = \begin{bmatrix} M_{dd} + T_1{}^T M_{fd} + \left(M_{df} + T_1{}^T M_{ff}\right) T_1 & \left(M_{df} + T_1{}^T M_{ff}\right) \\ \left(M_{fd} + M_{ff} T_1\right) & M_{ff} \end{bmatrix} \tag{B-17}$$

$$\begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} I_{dd} & T_1 \\ 0 & I_{ff} \end{bmatrix} \begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{B-18}$$

$$\begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} I_{dd} F_d + T_1 F_f \\ I_{ff} F_f \end{bmatrix} \tag{B-19}$$

$$\begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} F_d + T_1 F_f \\ F_f \end{bmatrix} \tag{B-20}$$

$$\begin{bmatrix} \hat{m}_{dd} & \hat{m}_{dw} \\ \hat{m}_{wd} & \hat{m}_{ww} \end{bmatrix} \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \begin{bmatrix} \hat{K}_{dd} & 0 \\ 0 & \hat{K}_{ww} \end{bmatrix} \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \begin{bmatrix} \hat{F}_d \\ \hat{F}_w \end{bmatrix} \tag{B-21}$$

$$\hat{m}_{wd}\ddot{u}_d + \hat{m}_{ww}\ddot{u}_w + \hat{K}_{ww}u_w = \hat{F}_w \tag{B-22}$$

The equation of motion is thus

$$\hat{m}_{ww}\ddot{u}_w + \hat{K}_{ww}u_w = \hat{F}_w - \hat{m}_{wd}\ddot{u}_d \tag{B-23}$$

The final displacements are found via

$$\begin{bmatrix} u_d \\ u_f \end{bmatrix} = \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} \tag{B-24}$$

$$\Pi = \begin{bmatrix} I_{dd} & 0 \\ -K_{ff}{}^{-1}K_{fd} & I_{ff} \end{bmatrix} \tag{B-25}$$

APPENDIX C

Enforced Acceleration Example

<u>Sample Beam</u>

Consider the cantilever beam in Figure C-1.

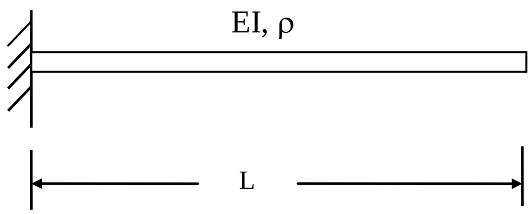

Figure C-1.

The governing differential equation for the displacement y(x,t) is

$$- EI \frac{\partial^4 y}{\partial x^4} = \rho \frac{\partial^2 y}{\partial t^2} \qquad \text{(C-1)}$$

where

E	is the modulus of elasticity
I	is the area moment of inertia
L	is the length
ρ	is the mass density (mass/length)

Note that this equation neglects shear deformation and rotary inertia.

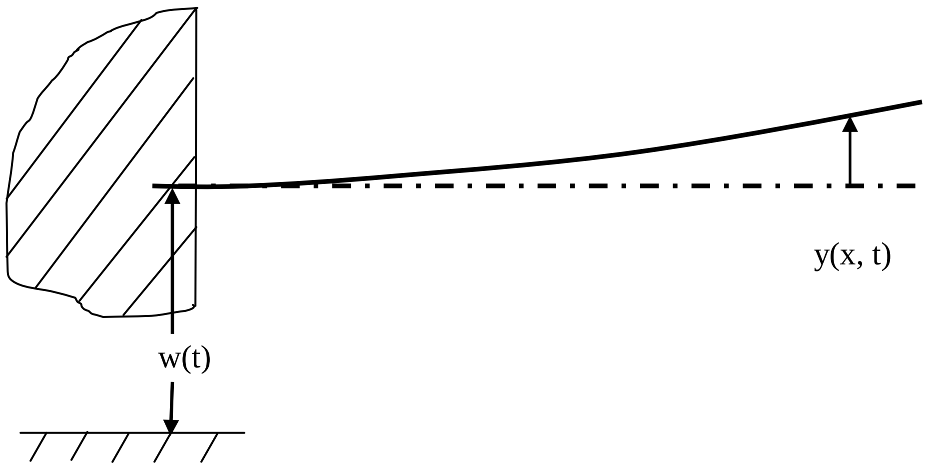

Figure C-2.

Apply base excitation to the beam as shown in Figure C-2.

There are two candidate methods for applying base excitation at the fixed end of the beam.

The first method is to add a seismic mass which is attached via a rigid link to the beam's fixed end. The seismic mass will be given a mass value which is considerably greater than the beam's own mass. Then a force will be applied to the seismic mass to yield the prescribed acceleration.

The second method is to apply the acceleration directly to the fixed end using the enforce method from the main text. This is the method used in this example.

<u>Finite Element Model</u>

Model the cantilever beam with three elements and four nodes using the mass and stiffness matrices from Reference 3. The model is shown in Figure C-3.

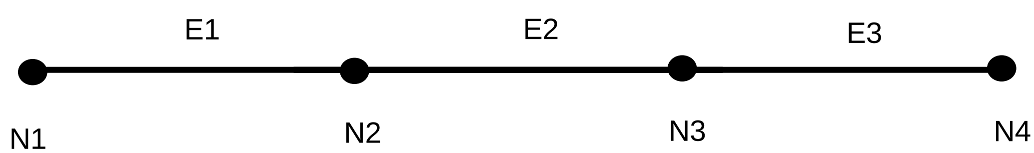

Figure C-3.

(The model is limited to three elements for brevity. A larger number of elements would normally be used.)

There is one translation and one rotational degree-of-freedom at each node.

The displacement vector for the model is:

$$[y_1 \quad \theta_1 \quad y_2 \quad \theta_2 \quad y_3 \quad \theta_3 \quad y_4 \quad \theta_4]^T$$

Set the beam properties as follows.

Cross-Section	Circular
Boundary Conditions	Fixed-Free
Material	Aluminum

Diameter	D	=	0.5 inch
Cross-Section Area	A	=	0.1963 in^2
Length	L	=	24 inch
Area Moment of Inertia	I	=	0.003068 in^4
Elastic Modulus	E	=	1.0e+07 lbf/in^2
Stiffness	EI	=	30680 lbf in^2
Mass per Volume	ρ_V	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)
Mass per Length	ρ	=	0.01963 lbm/in (5.08e-05 lbf sec^2/in^2)
Mass	ρL	=	0.471 lbm (1.22E-03 lbf sec^2/in)
Viscous Damping Ratio	ξ	=	0.05

The analysis is carried out using Matlab script: beam_base_accel_fea.m.

The unconstrained mass matrix is

1.0e-03 *

0.1511	0.1705	0.0523	-0.1008	0	0	0	0
0.1705	0.2480	0.1008	-0.1860	0	0	0	0
0.0523	0.1008	0.3023	0	0.0523	-0.1008	0	0
-0.1008	-0.1860	0	0.4961	0.1008	-0.1860	0	0
0	0	0.0523	0.1008	0.3023	0	0.0523	-0.1008
0	0	-0.1008	-0.1860	0	0.4961	0.1008	-0.1860
0	0	0	0	0.0523	0.1008	0.1511	-0.1705
0	0	0	0	-0.1008	-0.1860	-0.1705	0.2480

The unconstrained stiffness matrix is

1.0e+04 *

0.0719	0.2876	-0.0719	0.2876	0	0	0	0
0.2876	1.5340	-0.2876	0.7670	0	0	0	0
-0.0719	-0.2876	0.1438	0	-0.0719	0.2876	0	0
0.2876	0.7670	0	3.0680	-0.2876	0.7670	0	0
0	0	-0.0719	-0.2876	0.1438	0	-0.0719	0.2876
0	0	0.2876	0.7670	0	3.0680	-0.2876	0.7670
0	0	0	0	-0.0719	-0.2876	0.0719	-0.2876
0	0	0	0	0.2876	0.7670	-0.2876	1.5340

The θ_1 rotation at the left end is constrained by removing its rows and columns from the mass and stiffness matrices.

The y_1 translation at the left end is retained because it will be prescribed in a later step.

The constrained mass matrix is

1.0e-03 *

0.1511	0.0523	-0.1008	0	0	0	0
0.0523	0.3023	0	0.0523	-0.1008	0	0
-0.1008	0	0.4961	0.1008	-0.1860	0	0
0	0.0523	0.1008	0.3023	0	0.0523	-0.1008
0	-0.1008	-0.1860	0	0.4961	0.1008	-0.1860
0	0	0	0.0523	0.1008	0.1511	-0.1705
0	0	0	-0.1008	-0.1860	-0.1705	0.2480

The constrained stiffness matrix is

```
1.0e+04 *

 0.0719   -0.0719    0.2876         0         0         0         0
-0.0719    0.1438         0   -0.0719    0.2876         0         0
 0.2876         0    3.0680   -0.2876    0.7670         0         0
      0   -0.0719   -0.2876    0.1438         0   -0.0719    0.2876
      0    0.2876    0.7670         0    3.0680   -0.2876    0.7670
      0         0         0   -0.0719   -0.2876    0.0719   -0.2876
      0         0         0    0.2876    0.7670   -0.2876    1.5340
```

Create a transformation matrix such that

$$\begin{bmatrix} u_d \\ u_f \end{bmatrix} = \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} \tag{C-2}$$

$$\Pi = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \tag{C-3}$$

The transformation matrix Π is explicitly given later.

The driven displacement for the sample problem is

$$u_d = [y_1]$$

The remaining displacements are

$$u_w = [y_2 \quad \theta_2 \quad y_3 \quad \theta_3 \quad y_4 \quad \theta_4]^T$$

By substitution,

$$\begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \Pi \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{C-4}$$

For the sample problem,

```
Mdd =

   1.5115e-04



Mdf =

   1.0e-03 *

    0.0523   -0.1008         0         0         0         0


Mfd =

   1.0e-03 *

    0.0523
   -0.1008
         0
         0
         0
         0



Mff =

   1.0e-03 *

    0.3023         0    0.0523   -0.1008         0         0
         0    0.4961    0.1008   -0.1860         0         0
    0.0523    0.1008    0.3023         0    0.0523   -0.1008
   -0.1008   -0.1860         0    0.4961    0.1008   -0.1860
         0         0    0.0523    0.1008    0.1511   -0.1705
         0         0   -0.1008   -0.1860   -0.1705    0.2480
```

```
Kdd =

  719.0535


Kdf =

   1.0e+03 *

   -0.7191    2.8762         0         0         0         0


Kfd =

   1.0e+03 *

   -0.7191
    2.8762
         0
         0
         0
         0


Kff =

   1.0e+04 *

    0.1438         0   -0.0719    0.2876         0         0
         0    3.0680   -0.2876    0.7670         0         0
   -0.0719   -0.2876    0.1438         0   -0.0719    0.2876
    0.2876    0.7670         0    3.0680   -0.2876    0.7670
         0         0   -0.0719   -0.2876    0.0719   -0.2876
         0         0    0.2876    0.7670   -0.2876    1.5340
```

The transform matrix is derived as follows.

$$T_1^{T} = -K_{df} K_{ff}^{-1} \tag{C-5}$$

$$T_1 = -K_{ff}^{-1} K_{fd} \tag{C-6}$$

$$\Pi = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \tag{C-7}$$

```
invKff =

    0.0056    0.0010    0.0139    0.0010    0.0223    0.0010
    0.0010    0.0003    0.0031    0.0003    0.0052    0.0003
    0.0139    0.0031    0.0445    0.0042    0.0779    0.0042
    0.0010    0.0003    0.0042    0.0005    0.0083    0.0005
    0.0223    0.0052    0.0779    0.0083    0.1502    0.0094
    0.0010    0.0003    0.0042    0.0005    0.0094    0.0008


T1 =

    1.0000
    0.0000
    1.0000
   -0.0000
    1.0000
   -0.0000


T2 =

     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1


TT =

    1.0000         0         0         0         0         0         0
    1.0000    1.0000         0         0         0         0         0
    0.0000         0    1.0000         0         0         0         0
    1.0000         0         0    1.0000         0         0         0
   -0.0000         0         0         0    1.0000         0         0
    1.0000         0         0         0         0    1.0000         0
   -0.0000         0         0         0         0         0    1.0000
```

Premultiply by Π^T,

$$\Pi^T \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \Pi \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \Pi^T \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(C-8)}$$

$$\Pi^T \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \Pi = \begin{bmatrix} \hat{m}_{dd} & \hat{m}_{dw} \\ \hat{m}_{wd} & \hat{m}_{ww} \end{bmatrix} \quad \text{(C-9)}$$

$\hat{m}_{dd}$ =

```
    0.0012
```

$\hat{m}_{wd}$ =

```
  1.0e-03 *

   0.4069
   0.0000
   0.4069
  -0.0000
   0.2035
  -0.2713
```

$\hat{m}_{ww}$ =

```
  1.0e-03 *

   0.3023         0    0.0523   -0.1008         0         0
        0    0.4961    0.1008   -0.1860         0         0
   0.0523    0.1008    0.3023         0    0.0523   -0.1008
  -0.1008   -0.1860         0    0.4961    0.1008   -0.1860
        0         0    0.0523    0.1008    0.1511   -0.1705
        0         0   -0.1008   -0.1860   -0.1705    0.2480
```

$$\Pi^T \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \Pi = \begin{bmatrix} \hat{K}_{dd} & 0 \\ 0 & \hat{K}_{ww} \end{bmatrix} \quad \text{(C-10)}$$

$\hat{K}_{dd} = 0$

$\hat{K}_{ww} =$

1.0e+04 *

0.1438	0	-0.0719	0.2876	0	0
0	3.0680	-0.2876	0.7670	0	0
-0.0719	-0.2876	0.1438	0	-0.0719	0.2876
0.2876	0.7670	0	3.0680	-0.2876	0.7670
0	0	-0.0719	-0.2876	0.0719	-0.2876
0	0	0.2876	0.7670	-0.2876	1.5340

$$\hat{m}_{ww}\ddot{u}_w + \hat{K}_{ww}u_w = -\hat{m}_{wd}\ddot{u}_d \tag{C-11}$$

The base acceleration term is effectively a force.

The natural frequencies and mass-normalized mode shapes for the homogeneous form of equation (C-11) are

n	fn(Hz)
1	23.86
2	150
3	423.9
4	954.6
5	1796
6	3582

ModeShapes =

9.4773	33.9730	42.7950	15.5903	17.6565	15.1134
2.1577	4.2321	3.9829	24.8997	41.1090	19.3757
31.3135	24.3886	37.7361	7.2026	31.1075	23.6637
3.1225	7.1019	3.0402	25.5115	27.0399	54.4646
57.2521	57.5940	57.5163	61.8895	66.8813	125.6604
3.2837	11.4827	19.0493	32.9701	55.4204	157.0194

Again, equation (C-11) can then be uncoupled using the normal modes per the method in Reference 4.

The modal transient analysis can be then be performed on the modal coordinates using the ramp invariant digital recursive filtering relationship in Reference 5 for the case where the acceleration of one or modal degrees-of-freedom have been prescribed.

The digital recursive filtering relationship for the acceleration is

$$\ddot{x}_i = +2\exp[-\xi\omega_n T]\cos[\omega_d T]\ddot{x}_{i-1} - \exp[-2\xi\omega_n T]\ddot{x}_{i-2} + \frac{\exp(-\xi\omega_n T)\sin(\omega_d T)}{m\omega_d T}\{f_i - 2f_{i-1} + f_{i-2}\} \qquad \text{(C-12)}$$

where

$\ddot{x}_i$	is the acceleration at step i
f_i	is the force at step i
T	is the time step
ω_n	is the natural frequency in (radians/sec)
ξ	is the damping ratio
m	is the mass

Note that the mass is equal to one for the case of a system which has been decoupled using mass-normalized modes.

The damped natural frequency ω_d is

$$\omega_d = \omega_n\sqrt{1-\xi^2} \qquad \text{(C-13)}$$

The resulting modal responses are then transformed into physical responses per the method in Reference 4.

The final displacements are found via

$$\begin{bmatrix} u_d \\ u_f \end{bmatrix} = \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} \quad \text{(C-14)}$$

$$\Pi = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \quad \text{(C-15)}$$

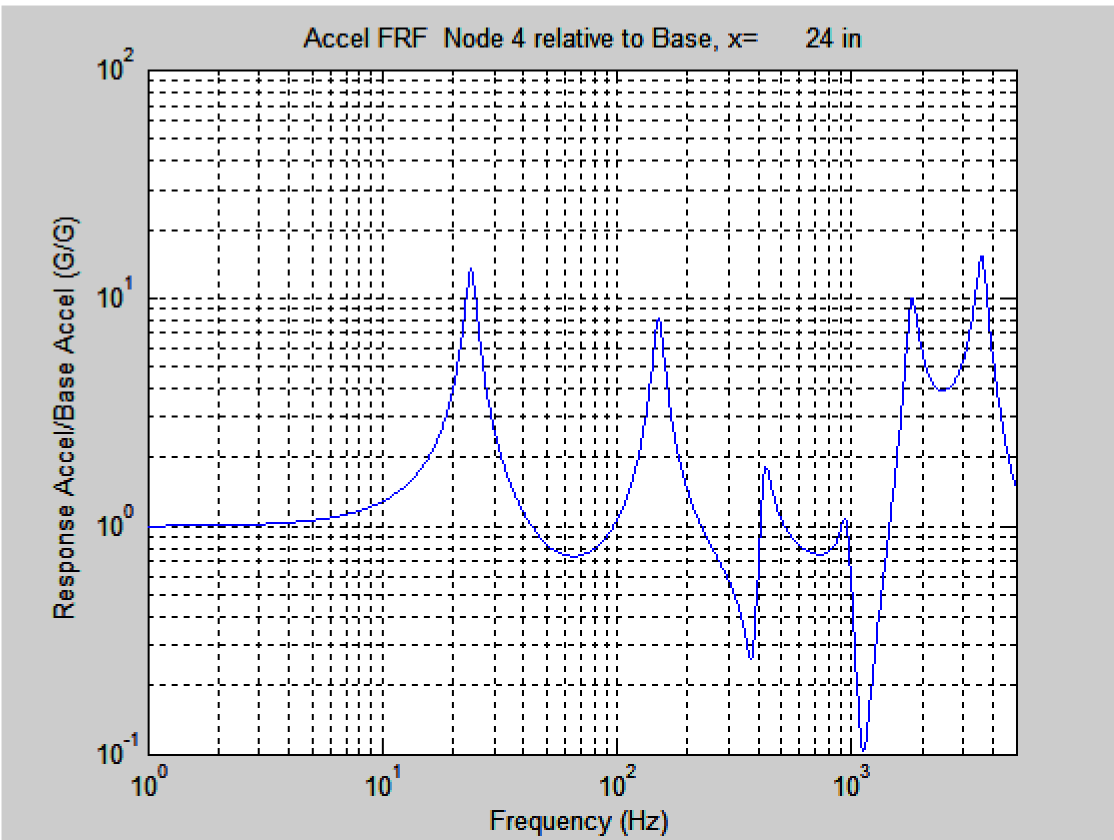

Figure C-4.

Again, each mode has 5% damping. The frequency response function is shown in Figure C-5 for reference.

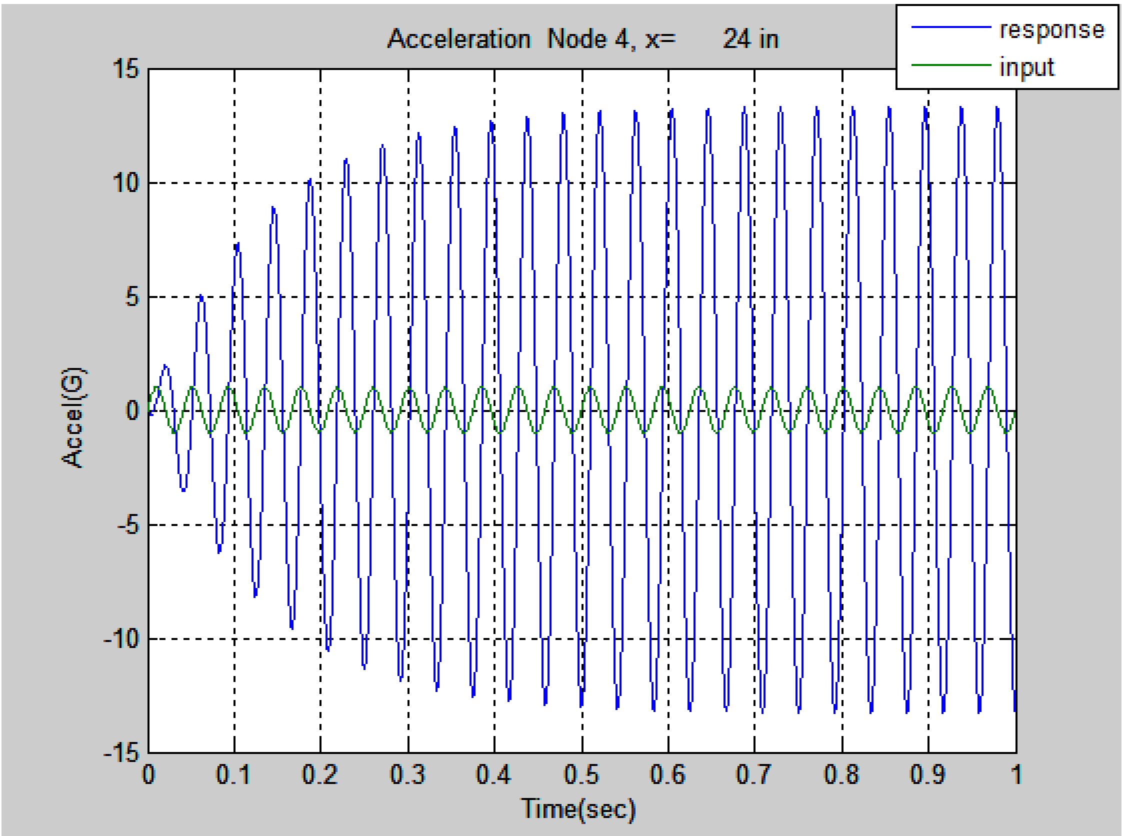

Figure C-5.

Now assume that the sample system is subjected to a 1 G, 24 Hz sine base excitation. This problem could be solved exactly using Laplace transforms. Nevertheless, the ramp invariant digital recursive relationship will be used. The resulting acceleration at the free end is given in Figure C-5.