THE NATURAL FREQUENCIES OF MULTISPAN BEAMS
Revision C

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Introduction

This tutorial is based on the analytical methods for multispan beams presented in References 1 through 3.

The approach for finding the natural frequency of a beam on multiple supports is to consider the section between each pair of supports as a separate beam with its origin at the left support of each section. The equations for adjacent beams must be reconciled at their common boundary.

The governing equation for the lateral or transverse vibration of a beam is

$$-EI\frac{\partial^4 y}{\partial x^4} = \rho\frac{\partial^2 y}{\partial t^2} \tag{1}$$

where

E	=	Mass
I	=	viscous damping coefficient
ρ	=	mass per length
y	=	lateral displacement

Separation of variables yields the following spatial equation.

$$\frac{d^4}{dx^4}Y(x) - c^2\left\{\frac{\rho}{EI}\right\}Y(x) = 0 \tag{2}$$

Separation of variables yields the following spatial equation.

$$Y(x) = a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x) \tag{3}$$

$$\frac{dY(x)}{dx} = a_1\beta\cosh(\beta x) + a_2\beta\sinh(\beta x) + a_3\beta\cos(\beta x) - a_4\beta\sin(\beta x) \quad (4)$$

$$\frac{d^2 Y(x)}{dx^2} = a_1\beta^2\sinh(\beta x) + a_2\beta^2\cosh(\beta x) - a_3\beta^2\sin(\beta x) - a_4\beta^2\cos(\beta x) \quad (5)$$

$$\frac{d^3 Y(x)}{dx^3} = a_1\beta^3\cosh(\beta x) + a_2\beta^3\sinh(\beta x) - a_3\beta^3\cos(\beta x) + a_4\beta^3\sin(\beta x) \quad (6)$$

The constant value c is equation 2 is related to the other variables as follows.

$$c^2 = \beta_n{}^4\left[\frac{EI}{\rho}\right] \quad (7)$$

The subscript n denotes that there are multiple roots, or natural frequencies, which satisfy the governing equation. The constant c is actually the natural frequency, with dimensions of radian per time.

References

1. C. Harris, editor; Shock and Vibration Handbook, 4th edition; W. Stokey, "Vibration of Systems Having Distributed Mass and Elasticity," McGraw-Hill, New York, 1988.

2. R. Blevins, Formulas for Natural Frequency and Mode Shape, Krieger, Malabar, Florida, 1979.

3. W. Seto, Mechanical Vibrations, McGraw-Hill, New York, 1964.

4. T. Irvine, Application of the Newton-Raphson Method to Vibration Problems, Vibrationdata.com Publications, 1999.

APPENDIX A

Clamped-Pinned-Clamped Beam

Consider the beam in Figure A-1. For the case of two segments, the x-coordinates can be defined as starting from opposite ends.

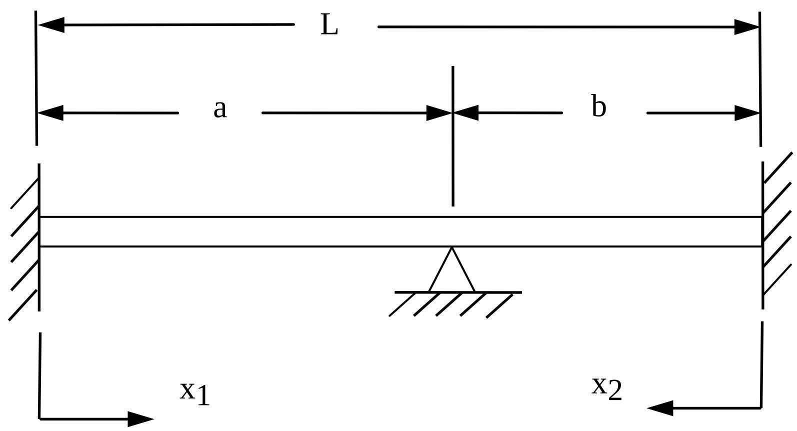

Figure A-1.

The boundary conditions at the fixed end for the first segment are

$$y_1\big|_{x_1=0} = 0 \qquad \text{(zero displacement)} \tag{A-1}$$

$$\left.\frac{dy_1}{dx_1}\right|_{x_1=0} = 0 \qquad \text{(slope)} \tag{A-2}$$

The boundary conditions at the intermediate pinned location for the first segment are

$$y_1\big|_{x_1=a} = 0 \qquad \text{(zero displacement)} \tag{A-3}$$

The boundary conditions at the intermediate pinned location for the second segment is

$$y_2 \big|_{x_2=b} = 0 \quad \text{(zero displacement)} \tag{A-4}$$

An additional constraint at the intermediate pinned location is

$$\left.\frac{dy_1}{dx_1}\right|_{x_1=a} = -\left.\frac{dy_2}{dx_2}\right|_{x_2=b} \quad \text{(equal slope but opposite polarity)} \tag{A-5}$$

$$\left.\frac{d^2y_1}{dx_1{}^2}\right|_{x_1=a} = -\left.\frac{d^2y_2}{dx_2{}^2}\right|_{x_2=b} \quad \text{(equal bending moment)} \tag{A-6}$$

The boundary conditions at the fixed end of the second segment are

$$y_2 \big|_{x_2=0} = 0 \quad \text{(zero displacement)} \tag{A-7}$$

$$\left.\frac{dy_2}{dx_2}\right|_{x_2=0} = 0 \quad \text{(slope)} \tag{A-8}$$

Eight constraints are thus defined.

Consider the first segment.

$$Y_1(x_1) = a_1 \sinh(\beta x_1) + a_2 \cosh(\beta x_1) + a_3 \sin(\beta x_1) + a_4 \cos(\beta x_1) \tag{A-9}$$

Apply boundary condition (A-1).

$$a_2 + a_4 = 0 \tag{A-10}$$

$$a_4 = -a_2 \tag{A-11}$$

The displacement equation thus simplifies to

$$Y_1(x_1) = a_1 \sinh(\beta x_1) + a_2[\cosh(\beta x_1) - \cos(\beta x_1)] + a_3 \sin(\beta x_1) \qquad \text{(A-12)}$$

The first derivative is

$$\frac{d}{dx_1} Y_1(x_1) = a_1\beta \cosh(\beta x_1) + a_2\beta[\sinh(\beta x_1) + \sin(\beta x_1)] + a_3\beta \cos(\beta x_1) \qquad \text{(A-13)}$$

Apply boundary condition (A-2).

$$a_1\beta + a_3\beta = 0 \qquad \text{(A-14)}$$

The beta term must be nonzero for a nontrivial solution. Thus

$$a_3 = -a_1 \qquad \text{(A-15)}$$

The displacement equation is thus

$$Y_1(x_1) = a_1[\sinh(\beta x_1) - \sin(\beta x_1)] + a_2[\cosh(\beta x_1) - \cos(\beta x_1)] \qquad \text{(A-16)}$$

Apply boundary condition (A-3).

$$a_1[\sinh(\beta a) - \sin(\beta a)] + a_2[\cosh(\beta a) - \cos(\beta a)] = 0 \qquad \text{(A-17)}$$

$$a_2 = -a_1 \frac{[\sinh(\beta a) - \sin(\beta a)]}{[\cosh(\beta a) - \cos(\beta a)]} \qquad \text{(A-18)}$$

The displacement equation is thus

$$Y_1(x_1) = a_1\left\{[\sinh(\beta x_1) - \sin(\beta x_1)] - \frac{[\sinh(\beta a) - \sin(\beta a)]}{[\cosh(\beta a) - \cos(\beta a)]}[\cosh(\beta x_1) - \cos(\beta x_1)]\right\}$$

(A-19)

The first derivative is

$$\frac{d}{dx_1}Y_1(x_1) = a_1\beta\left\{\left[\cosh(\beta x_1) - \cos(\beta x_1)\right] - \frac{\left[\sinh(\beta a) - \sin(\beta a)\right]}{\left[\cosh(\beta a) - \cos(\beta a)\right]}\left[\sinh(\beta x_1) + \sin(\beta x_1)\right]\right\}$$

(A-20)

The second derivative is

$$\frac{d^2}{dx_1{}^2}Y_1(x_1) = a_1\beta^2\left\{\left[\sinh(\beta x_1) + \sin(\beta x_1)\right] - \frac{\left[\sinh(\beta a) - \sin(\beta a)\right]}{\left[\cosh(\beta a) - \cos(\beta a)\right]}\left[\cosh(\beta x_1) + \cos(\beta x_1)\right]\right\}$$

(A-21)

The displacement equation is thus

$$\begin{aligned} Y_1(x_1) = {} & \frac{a_1}{\left[\cosh(\beta a) - \cos(\beta a)\right]}\left\{\left[\cosh(\beta a) - \cos(\beta a)\right]\left[\sinh(\beta x_1) - \sin(\beta x_1)\right]\right\} \\ & + \frac{-a_1}{\left[\cosh(\beta a) - \cos(\beta a)\right]}\left\{\left[\sinh(\beta a) - \sin(\beta a)\right]\left[\cosh(\beta x_1) - \cos(\beta x_1)\right]\right\} \end{aligned}$$

(A-22)

The first derivative is

$$\begin{aligned} \frac{d}{dx_1}Y_1(x_1) = {} & \frac{a_1\beta}{\left[\cosh(\beta a) - \cos(\beta a)\right]}\left\{\left[\cosh(\beta a) - \cos(\beta a)\right]\left[\cosh(\beta x_1) - \cos(\beta x_1)\right]\right\} \\ & + \frac{-a_1\beta}{\left[\cosh(\beta a) - \cos(\beta a)\right]}\left\{\left[\sinh(\beta a) - \sin(\beta a)\right]\left[\sinh(\beta x_1) + \sin(\beta x_1)\right]\right\} \end{aligned}$$

(A-23)

The second derivative is

$$\frac{d^2}{dx_1^2} Y_1(x_1) = \frac{a_1\beta^2}{[\cosh(\beta a) - \cos(\beta a)]} \{[\cosh(\beta a) - \cos(\beta a)][\sinh(\beta x_1) + \sin(\beta x_1)]\}$$
$$+ \frac{-a_1\beta^2}{[\cosh(\beta a) - \cos(\beta a)]} \{[\sinh(\beta a) - \sin(\beta a)][\cosh(\beta x_1) + \cos(\beta x_1)]\}$$

(A-24)

Evaluate the derivatives at the intermediate boundary.

$$\left.\frac{d}{dx_1} Y_1\right|_{x_1=a} = \frac{a_1\beta}{[\cosh(\beta a) - \cos(\beta a)]} \{[\cosh(\beta a) - \cos(\beta a)][\cosh(\beta a) - \cos(\beta a)]\}$$
$$+ \frac{-a_1\beta}{[\cosh(\beta a) - \cos(\beta a)]} \{[\sinh(\beta a) - \sin(\beta a)][\sinh(\beta a) + \sin(\beta a)]\}$$

(A-25)

$$\left.\frac{d}{dx_1} Y_1\right|_{x_1=a} = \frac{a_1\beta}{[\cosh(\beta a) - \cos(\beta a)]} \left\{\cosh^2(\beta a) - 2\cosh(\beta a)\cos(\beta a) + \cos^2(\beta a)\right\}$$
$$+ \frac{-a_1\beta}{[\cosh(\beta a) - \cos(\beta a)]} \left\{\sinh^2(\beta a) - \sin^2(\beta a)\right\}$$

(A-26)

$$\left.\frac{d}{dx_1} Y_1\right|_{x_1=a} = \frac{a_1\beta[2 - 2\cosh(\beta a)\cos(\beta a)]}{[\cosh(\beta a) - \cos(\beta a)]} \quad \text{(A-27)}$$

$$\left.\frac{d}{dx_1} Y_1\right|_{x_1=a} = \frac{2\beta[1 - \cosh(\beta a)\cos(\beta a)]}{[\cosh(\beta a) - \cos(\beta a)]} a_1 \quad \text{(A-28)}$$

$$
\left.\frac{d^2}{dx_1^{\,2}}Y_1\right|_{x_1=a} = \frac{a_1\beta^2}{[\cosh(\beta a)-\cos(\beta a)]}\{[\cosh(\beta a)-\cos(\beta a)][\sinh(\beta a)+\sin(\beta a)]\}
+\frac{-a_1\beta^2}{[\cosh(\beta a)-\cos(\beta a)]}\{[\sinh(\beta a)-\sin(\beta a)][\cosh(\beta a)+\cos(\beta a)]\}
\tag{A-29}
$$

$$
\left.\frac{d^2}{dx_1^{\,2}}Y_1\right|_{x_1=a} =
$$

$$
\frac{a_1\beta^2\{\cosh(\beta a)\sinh(\beta a)-\cos(\beta a)\sinh(\beta a)+\sin(\beta a)\cosh(\beta a)-\cos(\beta a)\sin(\beta a)\}}{[\cosh(\beta a)-\cos(\beta a)]}
-\frac{a_1\beta^2\{\cosh(\beta a)\sinh(\beta a)-\sin(\beta a)\cosh(\beta a)+\cos(\beta a)\sinh(\beta a)-\cos(\beta a)\sin(\beta a)\}}{[\cosh(\beta a)-\cos(\beta a)]}
\tag{A-30}
$$

$$
\left.\frac{d^2}{dx_1^{\,2}}Y_1\right|_{x_1=a} = \frac{2\beta^2\{\sin(\beta a)\cosh(\beta a)-\cos(\beta a)\sinh(\beta a)\}}{[\cosh(\beta a)-\cos(\beta a)]}a_1 \tag{A-31}
$$

Consider the second segment.

$$
Y_2(x_2) = b_1\sinh(\beta x_2)+b_2\cosh(\beta x_2)+b_3\sin(\beta x_2)+b_4\cos(\beta x_2) \tag{A-32}
$$

The first and second segments are conceptually similar. Therefore, the derivative terms for the second segment can be written directly.

$$
\left.\frac{d}{dx_2}Y_2\right|_{x_2=b} = \frac{2\beta[1-\cosh(\beta b)\cos(\beta b)]}{[\cosh(\beta b)-\cos(\beta b)]}\,b_1 \tag{A-33}
$$

$$\left.\frac{d^2}{dx_2^{\,2}} Y_2\right|_{x_2=b} = \frac{2\beta^2\{\sin(\beta b)\cosh(\beta b)-\cos(\beta b)\sinh(\beta b)\}}{[\cosh(\beta b)-\cos(\beta b)]} b_1 \tag{A-34}$$

Apply boundary condition (A-5) to equations (A-28) and (A-33).

$$\frac{2\beta[1-\cosh(\beta a)\cos(\beta a)]}{[\cosh(\beta a)-\cos(\beta a)]} a_1 = -\frac{2\beta[1-\cosh(\beta b)\cos(\beta b)]}{[\cosh(\beta b)-\cos(\beta b)]} b_1 \tag{A-35}$$

$$\frac{[1-\cosh(\beta a)\cos(\beta a)]}{[\cosh(\beta a)-\cos(\beta a)]} a_1 + \frac{[1-\cosh(\beta b)\cos(\beta b)]}{[\cosh(\beta b)-\cos(\beta b)]} b_1 = 0 \tag{A-36}$$

$$b_1 = -\frac{[\cosh(\beta b)-\cos(\beta b)][1-\cosh(\beta a)\cos(\beta a)]}{[1-\cosh(\beta b)\cos(\beta b)][\cosh(\beta a)-\cos(\beta a)]} a_1 \tag{A-37}$$

Apply boundary condition (A-6) to equations (A-31) and (A-34).

$$\frac{2\beta^2\{\sin(\beta a)\cosh(\beta a)-\cos(\beta a)\sinh(\beta a)\}}{[\cosh(\beta a)-\cos(\beta a)]} a_1 = \frac{-2\beta^2\{\sin(\beta b)\cosh(\beta b)-\cos(\beta b)\sinh(\beta b)\}}{[\cosh(\beta b)-\cos(\beta b)]} b_1 \tag{A-38}$$

$$\frac{[\sin(\beta a)\cosh(\beta a)-\cos(\beta a)\sinh(\beta a)]}{[\cosh(\beta a)-\cos(\beta a)]} a_1 + \frac{[\sin(\beta b)\cosh(\beta b)-\cos(\beta b)\sinh(\beta b)]}{[\cosh(\beta b)-\cos(\beta b)]} b_1 = 0 \tag{A-39}$$

$$b_1 = -\frac{[\cosh(\beta b)-\cos(\beta b)][\sin(\beta a)\cosh(\beta a)-\cos(\beta a)\sinh(\beta a)]}{[\sin(\beta b)\cosh(\beta b)-\cos(\beta b)\sinh(\beta b)][\cosh(\beta a)-\cos(\beta a)]} a_1 \tag{A-40}$$

Arrange equations (A-37) and (A-39) into matrix form.

$$\begin{bmatrix} \dfrac{[1-\cosh(\beta a)\cos(\beta a)]}{[\cosh(\beta a)-\cos(\beta a)]} & \dfrac{[1-\cosh(\beta b)\cos(\beta b)]}{[\cosh(\beta b)-\cos(\beta b)]} \\[2ex] \dfrac{[\sin(\beta a)\cosh(\beta a)-\cos(\beta a)\sinh(\beta a)]}{[\cosh(\beta a)-\cos(\beta a)]} & \dfrac{[\sin(\beta b)\cosh(\beta b)-\cos(\beta b)\sinh(\beta b)]}{[\cosh(\beta b)-\cos(\beta b)]} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(A-41)

The natural frequency is calculated by setting the determinant equal to zero.

$$\frac{[1-\cosh(\beta a)\cos(\beta a)]}{[\cosh(\beta a)-\cos(\beta a)]}\frac{[\sin(\beta b)\cosh(\beta b)-\cos(\beta b)\sinh(\beta b)]}{[\cosh(\beta b)-\cos(\beta b)]}$$

$$-\frac{[1-\cosh(\beta b)\cos(\beta b)]}{[\cosh(\beta b)-\cos(\beta b)]}\frac{[\sin(\beta a)\cosh(\beta a)-\cos(\beta a)\sinh(\beta a)]}{[\cosh(\beta a)-\cos(\beta a)]} = 0$$

(A-42)

Multiply through by the common denominator.

$$[1-\cosh(\beta a)\cos(\beta a)][\sin(\beta b)\cosh(\beta b)-\cos(\beta b)\sinh(\beta b)]$$

$$-[1-\cosh(\beta b)\cos(\beta b)][\sin(\beta a)\cosh(\beta a)-\cos(\beta a)\sinh(\beta a)] = 0$$

(A-43)

Now consider the special case where a = b=L/2.

Note that either $a_1 = b_1$ or $a_1 = -b_1$ depending on the mode number for this special case. The correct polarity is found by setting the distance of the pinned support a slightly off-center with b = 1-a in equation (A-40).

The characteristic equation becomes

$$[1-\cosh(\beta a)\cos(\beta a)][\sin(\beta a)\cosh(\beta a)-\cos(\beta a)\sinh(\beta a)]$$

$$-[1-\cosh(\beta a)\cos(\beta a)][\sin(\beta a)\cosh(\beta a)-\cos(\beta a)\sinh(\beta a)] \quad = 0 \tag{A-44}$$

The left-hand-side is equal to zero, independent of the argument. This fact represents another trivial solution. Note that equation (A-44) can also be satisfied by requiring the following two equations.

$$1-\cosh(\beta a)\cos(\beta a)=0 \tag{A-45}$$

$$\sin(\beta a)\cosh(\beta a)-\cos(\beta a)\sinh(\beta a)=0 \tag{A-46}$$

The pair of characteristic equations can be written as

$$\cosh(\beta a)\cos(\beta a)=1 \tag{A-47}$$

$$\tan(\beta a)=\tanh(\beta a) \tag{A-48}$$

The first three roots are shown in Table A-1. They were obtained using the Newton Raphson method from Reference 4.

Table A-1. Clamped-Pinned-Clamped Beam, Roots for a = b=L/2 Case			
N	$\beta_n a$	$\beta_n L$	Equation
1	3.92660	7.8532	$\tan(\beta a)=\tanh(\beta a)$
2	4.73004	9.46008	$\cosh(\beta a)\cos(\beta a)=1$
3	7.06858	14.13716	$\tan(\beta a)=\tanh(\beta a)$

Recall equation (7).

$$c^2 = \beta_n{}^4 \left[\frac{EI}{\rho} \right] \qquad \text{(A-49)}$$

Note that the natural frequency ω_n is equal to the constant c.

$$\omega_n = c \qquad \text{(A-50)}$$

Thus,

$$\omega_n = \beta_n{}^2 \sqrt{\frac{EI}{\rho}} \qquad \text{(A-51)}$$

Substitute the values from Table 1 into equation (A-51) in order to obtain the first three natural frequencies.

$$\omega_n = \left[\frac{7.8532}{L} \right]^2 \sqrt{\frac{EI}{\rho}} \qquad \text{(A-52)}$$

$$\omega_n = \left[\frac{9.46008}{L} \right]^2 \sqrt{\frac{EI}{\rho}} \qquad \text{(A-53)}$$

$$\omega_n = \left[\frac{14.13716}{L} \right]^2 \sqrt{\frac{EI}{\rho}} \qquad \text{(A-54)}$$

The mode shape for mode n is given by the following pair of equations.

$$\begin{aligned} Y_{1n}(x_1) = \hat{a}_n \{ &+ [\cosh(\beta_n a) - \cos(\beta_n a)][\sinh(\beta_n x_1) - \sin(\beta_n x_1)] \\ &- [\sinh(\beta_n a) - \sin(\beta_n a)][\cosh(\beta_n x_1) - \cos(\beta_n x_1)] \; \} \end{aligned}$$

(A-55)

$$Y_{2_n}(x_2) = -\hat{a}_n \{ +[\cosh(\beta_n a) - \cos(\beta_n a)][\sinh(\beta_n x_2) - \sin(\beta_n x_2)] - [\sinh(\beta_n a) - \sin(\beta_n a)][\cosh(\beta_n x_2) - \cos(\beta_n x_2)] \} \quad \text{(A-56)}$$

Note that $\hat{a}_n$ is an arbitrary scalar for mode n. Again, this pair of equations is for the special case a = b = L/2.

The first three mode shapes are plotted in Figures A-2 through A-4, respectively.

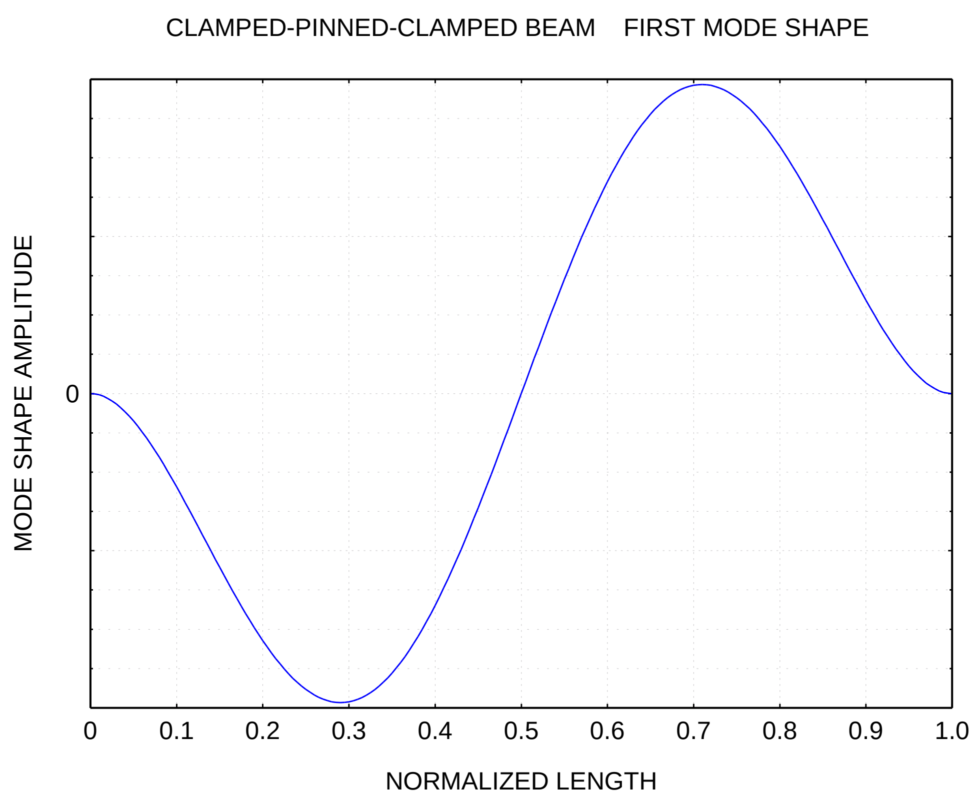

Figure A-2.

$a_1 = -b_1$ for this mode.

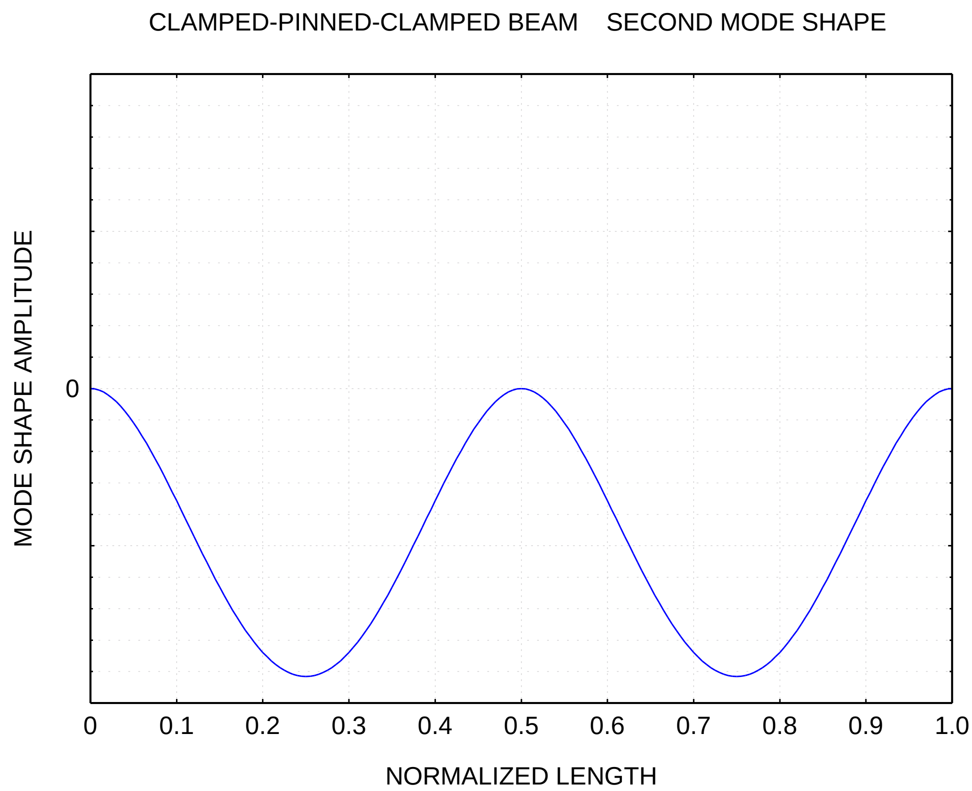

Figure A-3.

$a_1 = b_1$ for this mode

CLAMPED-PINNED-CLAMPED BEAM   THIRD MODE SHAPE

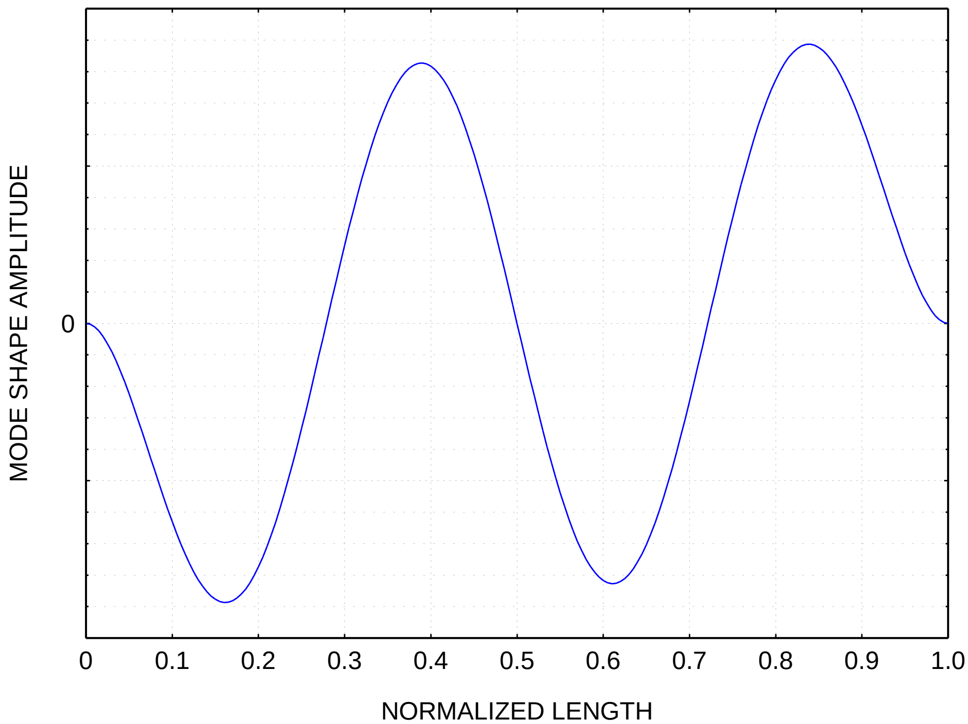

Figure A-4.

$a_1 = -b_1$ for this mode

APPENDIX B

Clamped-Pinned-Clamped Beam, Alternate Method

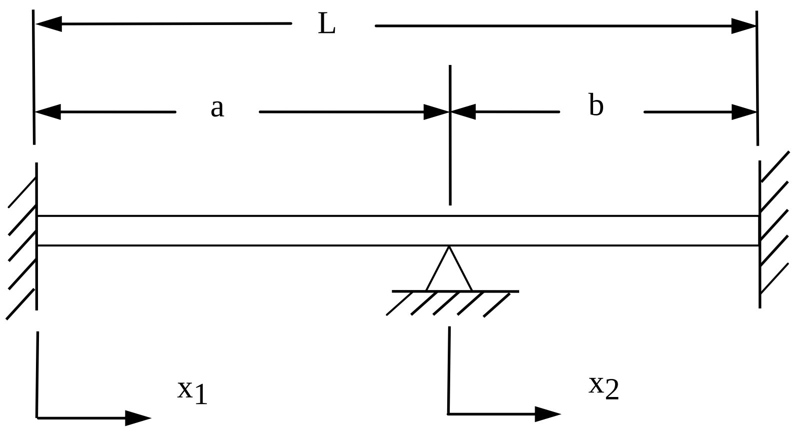

Figure B-1.

The boundary conditions at the fixed end for the first segment are

$$y_1 \big|_{x_1=0} = 0 \qquad \text{(zero displacement)} \tag{B-1}$$

$$\left.\frac{dy_1}{dx_1}\right|_{x_1=0} = 0 \qquad \text{(zero slope)} \tag{B-2}$$

The boundary conditions at the intermediate pinned location for the first segment are

$$y_1 \big|_{x_1=a} = 0 \qquad \text{(zero displacement)} \tag{B-3}$$

The boundary conditions at the intermediate pinned location for the second segment is

$$y_2 \big|_{x_2=0} = 0 \qquad \text{(zero displacement)} \tag{B-4}$$

An additional constraint at the intermediate pinned location is

$$\left.\frac{dy_1}{dx_1}\right|_{x_1=a} = \left.\frac{dy_2}{dx_2}\right|_{x_2=0} \qquad \text{(equal slope)} \tag{B-5}$$

$$\left.\frac{d^2y_1}{dx_1^{\,2}}\right|_{x_1=a} = \left.\frac{d^2y_2}{dx_2^{\,2}}\right|_{x_2=0} \qquad \text{(equal bending moment)} \tag{B-6}$$

The boundary conditions at the fixed end of the second segment are

$$\left.y_2\right|_{x_2=b} = 0 \qquad \text{(zero displacement)} \tag{B-7}$$

$$\left.\frac{dy_2}{dx_2}\right|_{x_2=b} = 0 \qquad \text{(zero slope)} \tag{B-8}$$

Eight constraints are thus defined.

Consider the displacement of first segment.

$$Y_1(x_1) = a_1 \sinh(\beta x_1) + a_2 \cosh(\beta x_1) + a_3 \sin(\beta x_1) + a_4 \cos(\beta x_1) \tag{B-9}$$

$$\frac{dY_1(x_1)}{dx} = a_1\beta \cosh(\beta x_1) + a_2\beta \sinh(\beta x_1) + a_3\beta \cos(\beta x_1) - a_4\beta \sin(\beta x_1) \tag{B-10}$$

$$\frac{d^2Y_1(x_1)}{dx^2} = a_1\beta^2 \sinh(\beta x_1) + a_2\beta^2 \cosh(\beta x_1) - a_3\beta^2 \sin(\beta x_1) - a_4\beta^2 \cos(\beta x_1) \tag{B-11}$$

$$\frac{d^3Y_1(x_1)}{dx^3} = a_1\beta^3 \cosh(\beta x_1) + a_2\beta^3 \sinh(\beta x_1) - a_3\beta^3 \cos(\beta x_1) + a_4\beta^3 \sin(\beta x_1) \tag{B-12}$$

The displacement of the second segment is

$$Y_2(x_2) = b_1 \sinh(\beta x_2) + b_2 \cosh(\beta x_2) + b_3 \sin(\beta x_2) + b_4 \cos(\beta x_2) \tag{B-13}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \sinh(\beta a) & \cosh(\beta a) & \sin(\beta a) & \cos(\beta a) & 0 & 0 & 0 & 0 \\ \cosh(\beta a) & \sinh(\beta a) & \cos(\beta a) & -\sin(\beta a) & -1 & 0 & -1 & 0 \\ \sinh(\beta a) & \cosh(\beta a) & -\sin(\beta a) & -\cos(\beta a) & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \sinh(\beta b) & \cosh(\beta b) & \sin(\beta b) & \cos(\beta b) \\ 0 & 0 & 0 & 0 & \cosh(\beta b) & \sinh(\beta b) & \cos(\beta b) & -\sin(\beta b) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(B-14)

The natural frequencies are calculated by setting the determinant of the coefficient matrix equal to zero and then solving for the roots. This can be done by a trial-and-error numerical method.

$$\omega_n = \beta_n{}^2 \sqrt{\frac{EI}{\rho}} \tag{B-15}$$

The mode shape coefficients can then be found for each mode by setting $a_1=1$ and inserting the value for the β root obtained from the determinant of the coefficient matrix from equation (B-14).

The resulting linear algebra equation is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \sinh(\beta a) & \cosh(\beta a) & \sin(\beta a) & \cos(\beta a) & 0 & 0 & 0 & 0 \\ \cosh(\beta a) & \sinh(\beta a) & \cos(\beta a) & -\sin(\beta a) & -1 & 0 & -1 & 0 \\ \sinh(\beta a) & \cosh(\beta a) & -\sin(\beta a) & -\cos(\beta a) & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \sinh(\beta b) & \cosh(\beta b) & \sin(\beta b) & \cos(\beta b) \\ 0 & 0 & 0 & 0 & \cosh(\beta b) & \sinh(\beta b) & \cos(\beta b) & -\sin(\beta b) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(B-16)

The coefficients and the corresponding β value are then applied to the pair of displacement equations.

$$Y_1(x_1) = a_1 \sinh(\beta x_1) + a_2 \cosh(\beta x_1) + a_3 \sin(\beta x_1) + a_4 \cos(\beta x_1) \tag{B-17}$$

$$Y_2(x_2) = b_1 \sinh(\beta x_2) + b_2 \cosh(\beta x_2) + b_3 \sin(\beta x_2) + b_4 \cos(\beta x_2) \tag{B-18}$$

Example

An aluminum clamped-pinned-clamped rod had a total length of 48 inches and a diameter of 1 inch. The pinned support point is located 36 inches from the left clamped end.

The natural frequencies and mode shapes are shown in Figures B-2 through B-4. The calculations were made using the equations in this appendix as implemented in Matlab script: clamped_pinned_clamped.m.

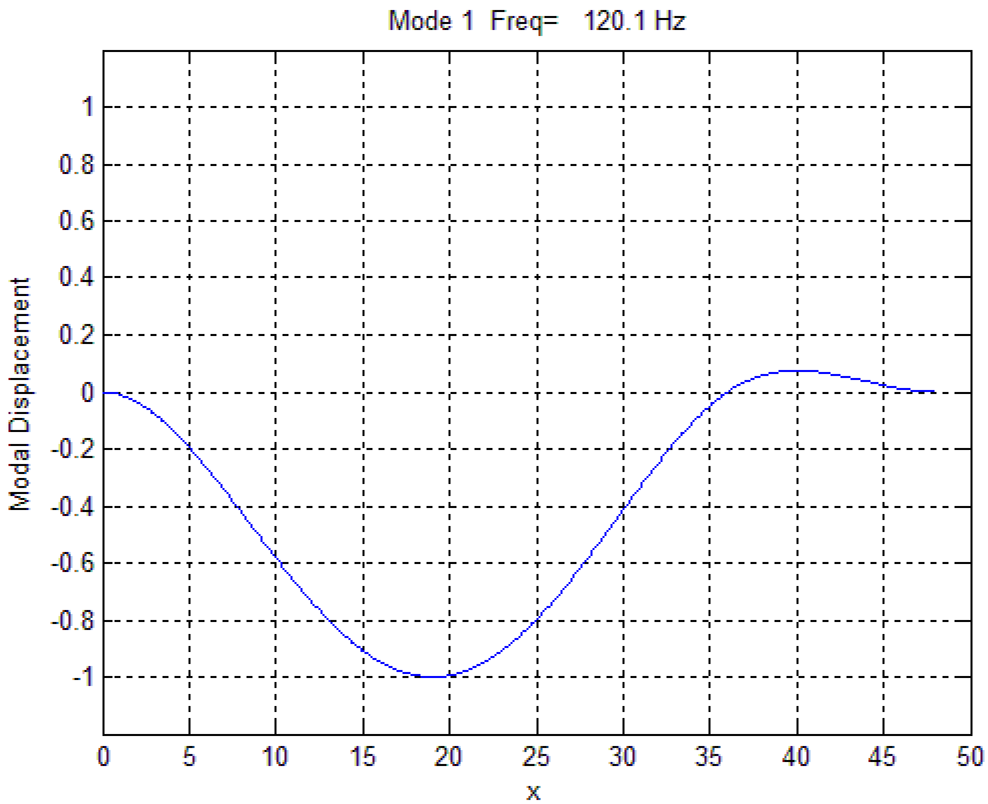
Mode 1 Freq= 120.1 Hz
Modal Displacement
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50
x

Figure B-2.

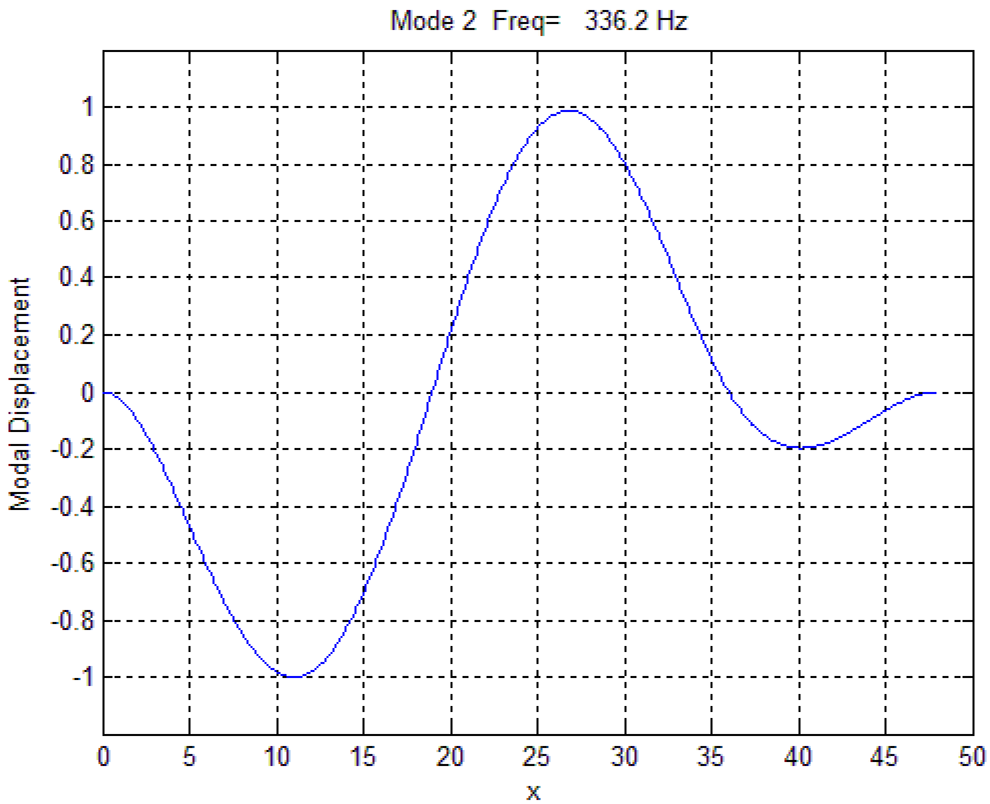
Mode 2 Freq=   336.2 Hz
Modal Displacement
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50
x

Figure B-3.

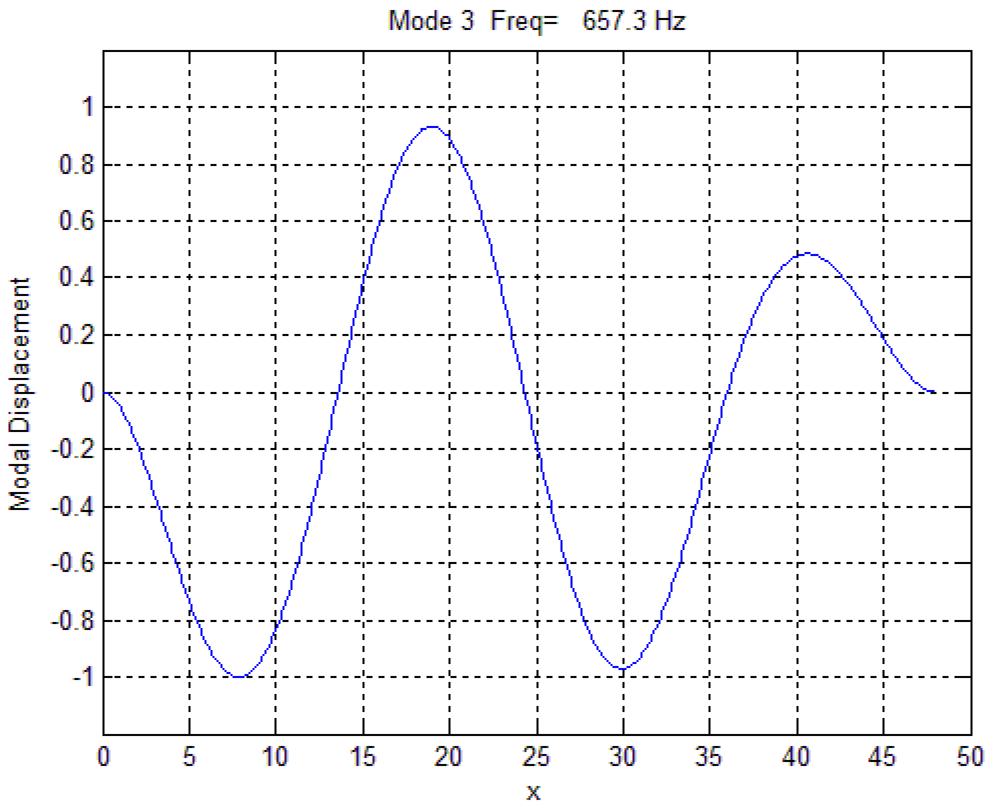
Mode 3 Freq= 657.3 Hz
Modal Displacement
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50
x

Figure B-4.

APPENDIX C

Clamped-Pinned-Free Beam

Consider the overhanging beam in Figure C-1.

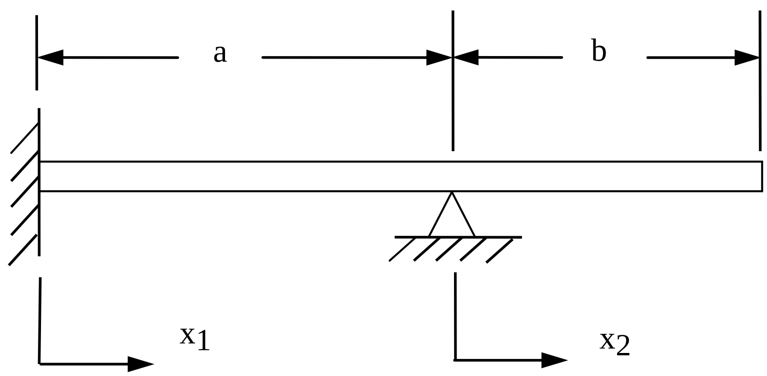

Figure C-1.

The boundary conditions at the fixed end for the first segment are

$$y_1 \big|_{x_1=0} = 0 \qquad \text{(zero displacement)} \qquad \text{(C-1)}$$

$$\left.\frac{dy_1}{dx_1}\right|_{x_1=0} = 0 \qquad \text{(slope)} \qquad \text{(C-2)}$$

The boundary conditions at the intermediate pinned location for the first segment are

$$y_1 \big|_{x_1=a} = 0 \qquad \text{(zero displacement)} \qquad \text{(C-3)}$$

The boundary conditions at the intermediate pinned location for the second segment is

$$y_2 \big|_{x_2=0} = 0 \qquad \text{(zero displacement)} \qquad \text{(C-4)}$$

An additional constraint at the intermediate pinned location is

$$\left.\frac{dy_1}{dx_1}\right|_{x_1=a} = \left.\frac{dy_2}{dx_2}\right|_{x_2=0} \qquad \text{(equal slope)} \tag{C-5}$$

$$\left.\frac{d^2y_1}{dx_1^2}\right|_{x_1=a} = \left.\frac{d^2y_2}{dx_2^2}\right|_{x_2=0} \qquad \text{(equal bending moment)} \tag{C-6}$$

The boundary conditions at the free end are

$$\left.\frac{d^2y_2}{dx_2^2}\right|_{x_2=b} = 0 \qquad \text{(zero bending moment)} \tag{C-7}$$

$$\left.\frac{d^3y_2}{dx_2^3}\right|_{x_2=b} = 0 \qquad \text{(zero shear force)} \tag{C-8}$$

Eight constraints are thus defined.

Consider the displacement of first segment.

$$Y_1(x_1) = a_1 \sinh(\beta x_1) + a_2 \cosh(\beta x_1) + a_3 \sin(\beta x_1) + a_4 \cos(\beta x_1) \tag{C-9}$$

$$\frac{dY_1(x_1)}{dx} = a_1\beta \cosh(\beta x_1) + a_2\beta \sinh(\beta x_1) + a_3\beta \cos(\beta x_1) - a_4\beta \sin(\beta x_1) \tag{C-10}$$

$$\frac{d^2Y_1(x_1)}{dx^2} = a_1\beta^2 \sinh(\beta x_1) + a_2\beta^2 \cosh(\beta x_1) - a_3\beta^2 \sin(\beta x_1) - a_4\beta^2 \cos(\beta x_1) \tag{C-11}$$

$$\frac{d^3Y_1(x_1)}{dx^3} = a_1\beta^3 \cosh(\beta x_1) + a_2\beta^3 \sinh(\beta x_1) - a_3\beta^3 \cos(\beta x_1) + a_4\beta^3 \sin(\beta x_1) \tag{C-12}$$

The displacement of the second segment is

$$Y_2(x_2) = b_1 \sinh(\beta x_2) + b_2 \cosh(\beta x_2) + b_3 \sin(\beta x_2) + b_4 \cos(\beta x_2) \tag{C-13}$$

The complete set of boundary conditions, displacement equations and their derivatives can be assembled into the following matrix.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \sinh(\beta a) & \cosh(\beta a) & \sin(\beta a) & \cos(\beta a) & 0 & 0 & 0 & 0 \\ \cosh(\beta a) & \sinh(\beta a) & \cos(\beta a) & -\sin(\beta a) & -1 & 0 & -1 & 0 \\ \sinh(\beta a) & \cosh(\beta a) & -\sin(\beta a) & -\cos(\beta a) & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \sinh(\beta b) & \cosh(\beta b) & -\sin(\beta b) & -\cos(\beta b) \\ 0 & 0 & 0 & 0 & \cosh(\beta b) & \sinh(\beta b) & -\cos(\beta b) & \sin(\beta b) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(C-14)

The natural frequencies are calculated by setting the determinant of the coefficient matrix equal to zero and then solving for the roots. This can be done by a trial-and-error numerical method.

$$\omega_n = \beta_n{}^2 \sqrt{\frac{EI}{\rho}} \qquad \text{(C-15)}$$

The mode shape coefficients can then be found for each mode by setting a_1=1 and inserting the value for the β root obtained from the determinant of the coefficient matrix from equation (C-14).

The resulting linear algebra equation is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \sinh(\beta a) & \cosh(\beta a) & \sin(\beta a) & \cos(\beta a) & 0 & 0 & 0 & 0 \\ \cosh(\beta a) & \sinh(\beta a) & \cos(\beta a) & -\sin(\beta a) & -1 & 0 & -1 & 0 \\ \sinh(\beta a) & \cosh(\beta a) & -\sin(\beta a) & -\cos(\beta a) & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \sinh(\beta b) & \cosh(\beta b) & -\sin(\beta b) & -\cos(\beta b) \\ 0 & 0 & 0 & 0 & \cosh(\beta b) & \sinh(\beta b) & -\cos(\beta b) & \sin(\beta b) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(C-16)

The coefficients and the corresponding β value are then applied to the pair of displacement equations.

$$Y_1(x_1) = a_1 \sinh(\beta x_1) + a_2 \cosh(\beta x_1) + a_3 \sin(\beta x_1) + a_4 \cos(\beta x_1) \qquad \text{(C-17)}$$

$$Y_2(x_2) = b_1 \sinh(\beta x_2) + b_2 \cosh(\beta x_2) + b_3 \sin(\beta x_2) + b_4 \cos(\beta x_2) \qquad \text{(C-18)}$$

Example

An aluminum clamped-pinned-free rod had a total length of 48 inches and a diameter of 1 inch. The pinned support point is located 36 inches from the clamped end.

The natural frequencies and mode shapes are shown in Figures C-2 through C-5. The calculations were made using the equations in this appendix as implemented in Matlab script: clamped_pinned_free.m.

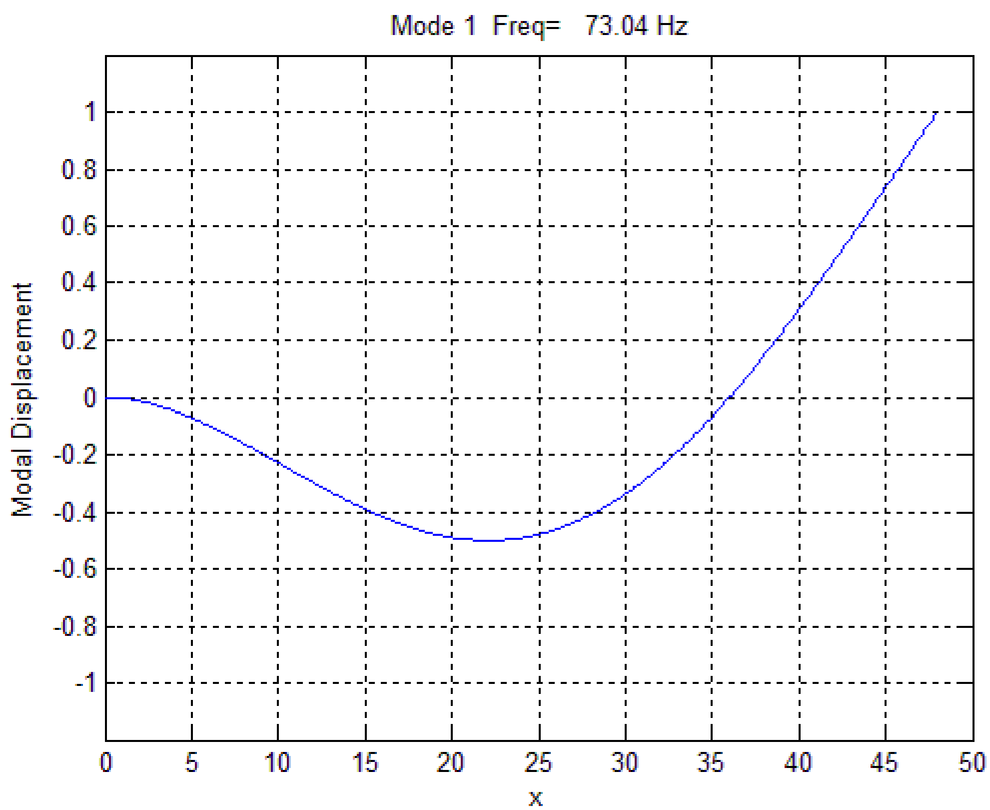

Figure C-2.

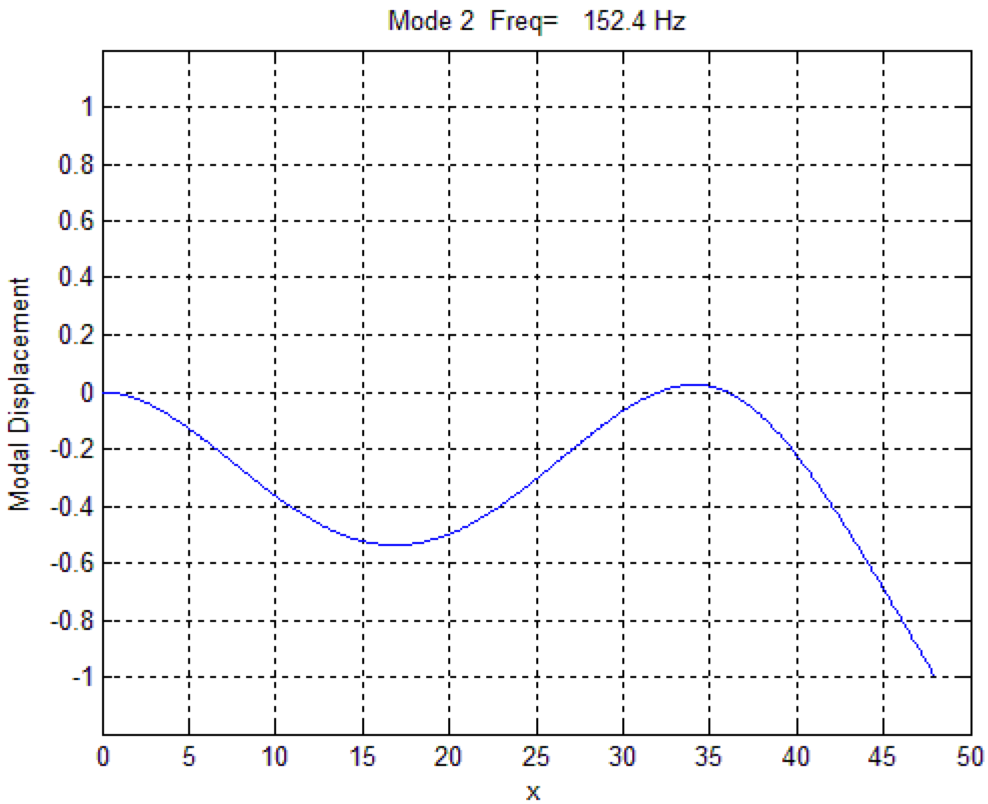
Mode 2 Freq= 152.4 Hz
Modal Displacement
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50
x

Figure C-3.

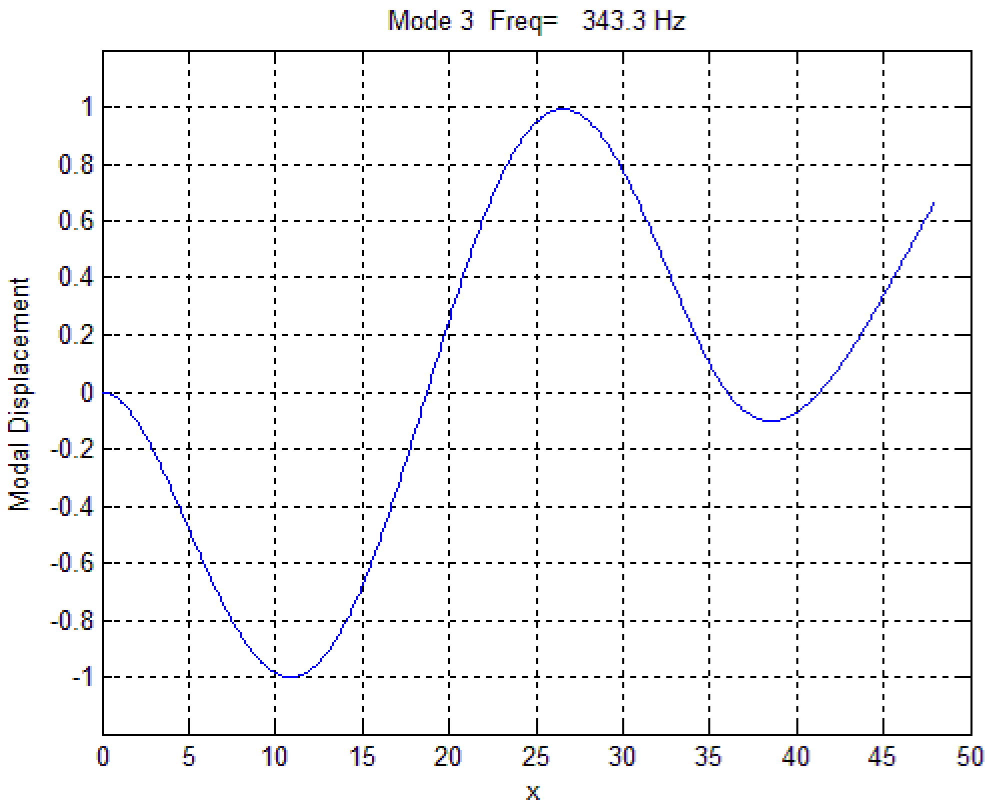
Mode 3 Freq= 343.3 Hz
Modal Displacement
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50
x

Figure C-4.

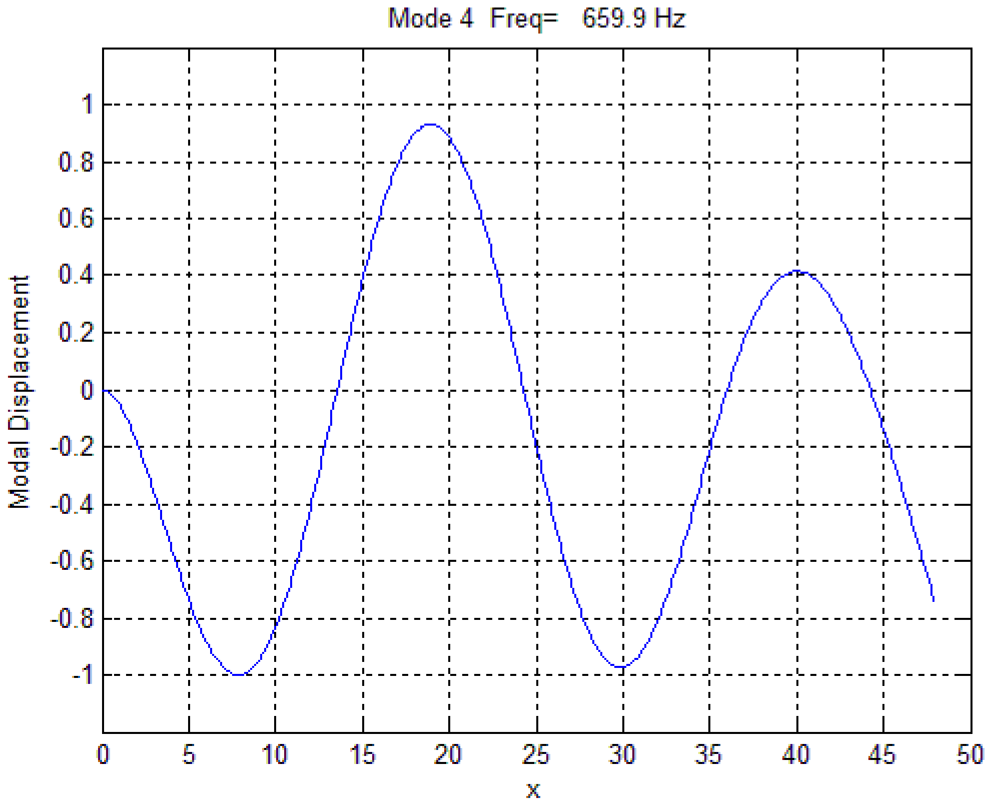
Mode 4 Freq= 659.9 Hz
Modal Displacement
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
40
45
50

Figure C-5.