Peak Response to Random Vibration
Revision F

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May 5, 2017

Readers are cautioned against seeking any perfection in the following statistical equations.

Introduction

An important parameter in random vibration analysis is the peak response, which can be the maximum relative displacement, velocity, acceleration, stress or strain. The peak response can then be compared with the threshold for yielding, ultimate failure, etc. The peak is also important for fatigue analyses, particularly for materials with higher exponents. A common approach is to consider that the peak response is 3σ, where 1σ is the standard deviation. Note that the 1σ value is equal to the overall RMS level for cases with zero mean. Further details about peak assumptions are given in Reference 1, Table 2. The peak response for Gaussian stationary random vibration is almost always greater than 3σ, however.

Basic Equations

Consider the response peak P as

$$P = \lambda\sigma \tag{1}$$

where λ is a scalar or crest factor.

A common formula for estimating the crest factor for the response of a single-degree-of-freedom (SDOF) system to stationary random vibration is

$$\lambda = \sqrt{2\ln(\mathrm{fn}\,T)} \tag{2}$$

where fn is the natural frequency and T is the total duration.

Equation (2) is given in References 1 and 2. It may be used for either base excitation or applied force.

An alternate equation from Reference 3 is

$$\lambda = \sqrt{2\ln(\mathrm{fn}\,T)} + \frac{0.5772}{\sqrt{2\ln(\mathrm{fn}\,T)}} \tag{3}$$

Note that 0.5772 is Euler's constant rounded to four decimal places.

Note that equations (2) and (3) converge as the duration becomes longer. Equation (2) thus appears to be an approximation suited for long durations.

Now consider that the crest factors in equations (2) and (3) are calculated for a family of natural frequencies. The resulting crest factor vs. natural frequency curve is referred to as the Maximax Response Spectrum (MRS) or Extreme Response Spectrum (ERS) in References 5 and 6, respectively, although these references omit the second term on the right-hand-side of equation (3).

The natural frequency term in each of the previous two equations may be replaced by the rate of positive slope zero upcrossings n_0+ for the case of a multi-degree-of-freedom system response, as discussed in Reference 4. This upcrossing rate is approximately equal to the natural frequency for the case of an SDOF response.

Let the maximax be the single highest absolute peak for a given event.

Now assume that an SDOF system is to be subjected multiple incidents of the same long-duration excitation event. The maximax peak is retained from each incident. A Rayleigh-like distribution of these peaks will form around the crest factor from equation (3). In some ideal sense, 50% of the maximax peaks would be either above or below this crest factor. This would be the case if the maximax histogram were Gaussian or otherwise symmetric.

In practice, the exceedance percent for a given numerical experiment might be 50% to 60% for a sufficiently large set of trials, based on the author's experience. Reference 6 claims that the exceedance probability is 63.2%, but this appears to be a conservative envelope. Further research and experimentation is needed.

Advanced Equation

A more conservative estimate of the crest factor is needed for some design and analysis cases. A goal might be to find the crest factor above which only 10% of the maximax peaks will occur.

This alternate crest factor λ_α derived from References 5 and 6 is

$$\lambda_\alpha = \left\{ \sqrt{2\ln(n_0^+ T)} + \frac{0.5772}{\sqrt{2\ln(n_0^+ T)}} \right\} \left\{ \sqrt{\frac{-\ln\left(1-(1-\alpha)^{1/(n_0^+ T)}\right)}{\ln(n_0^+ T)}} \right\} \tag{4}$$

where α is the probability of exceedance for the maximax peak for a given event.

Equation (4) is a more conservative approach than the previous ones, as it requires a stated probability of exceedance.

As an aside, Reference 6 omitted the $0.5772/\sqrt{2\ln(n_0^+ T)}$ term, which is needed for small $n_0^+ T$ values.

Now consider that the crest factor in equation (4) is calculated for a family of natural frequencies. The resulting crest factor vs. natural frequency curve is referred to as the X Response Spectrum (XRS) or Up-Crossing Risk Spectrum (URS) in References 5 and 6, respectively. A still more conservative equation is given in Appendix A, as taken from Reference 7.

Numerical Experiment I

An incident of an event is represented by a trial for the experiment. The event could be a test specification or an actual field environment that the system must withstand. For experimental purposes, the event has a stated specification, but each trial has a unique time history.

The equation of motion for an SDOF system subjected to base excitation is given in Appendix A.

An SDOF system with fn = 800 Hz and 5% damping was subjected to base acceleration for an event duration of 1200 seconds. The base input was white noise, 10,000 samples/sec, and lowpass filtered at 1600 Hz. The number of response peaks was approximately 960,000 per trial. The positive slope zero-crossing rate was 783 Hz, just below the natural frequency.

The response was calculated in the time domain using the ramp invariant, digital recursive filtering relationship in Reference 8. All maximax peaks were represented in terms of their respective crest factors. The peak acceleration response was determined for each of 12,000 trials. The time history synthesis and response calculations were performed using a Matlab script.

A sample base input and response pair are shown in Figures 1 and 2, respectively. The maximax histogram for the collection of trials is shown in Figure 3. The theoretical response probability from equation (4) and the experimental results are shown together in Figure 4 as a function of crest factor.

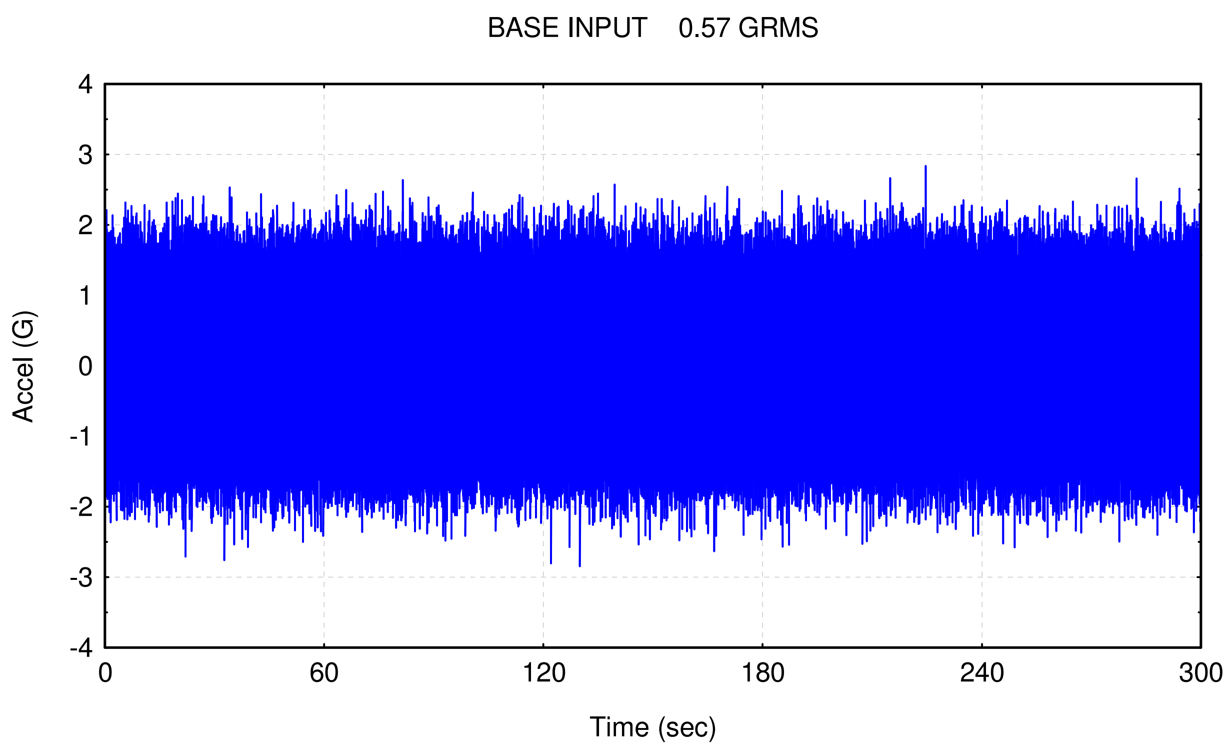

Figure 1. Band-limited White Noise Input, Experiment I

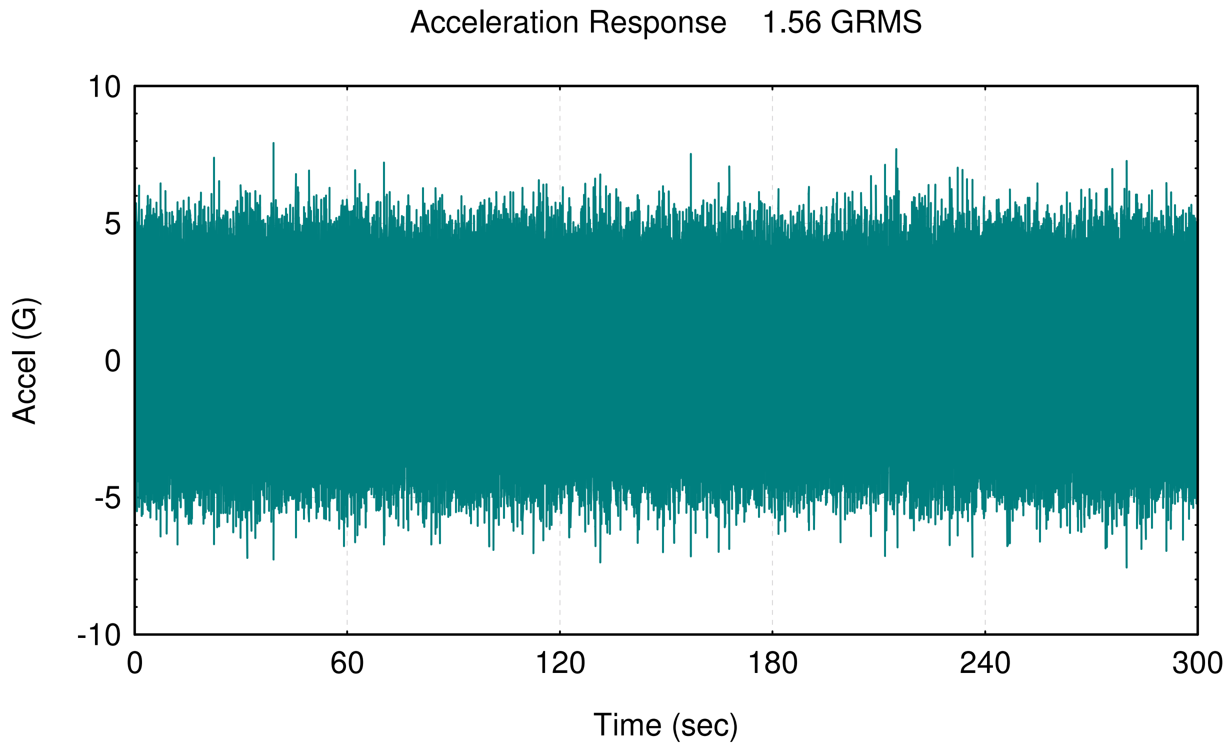

Figure 2. SDOF System Response, Experiment I

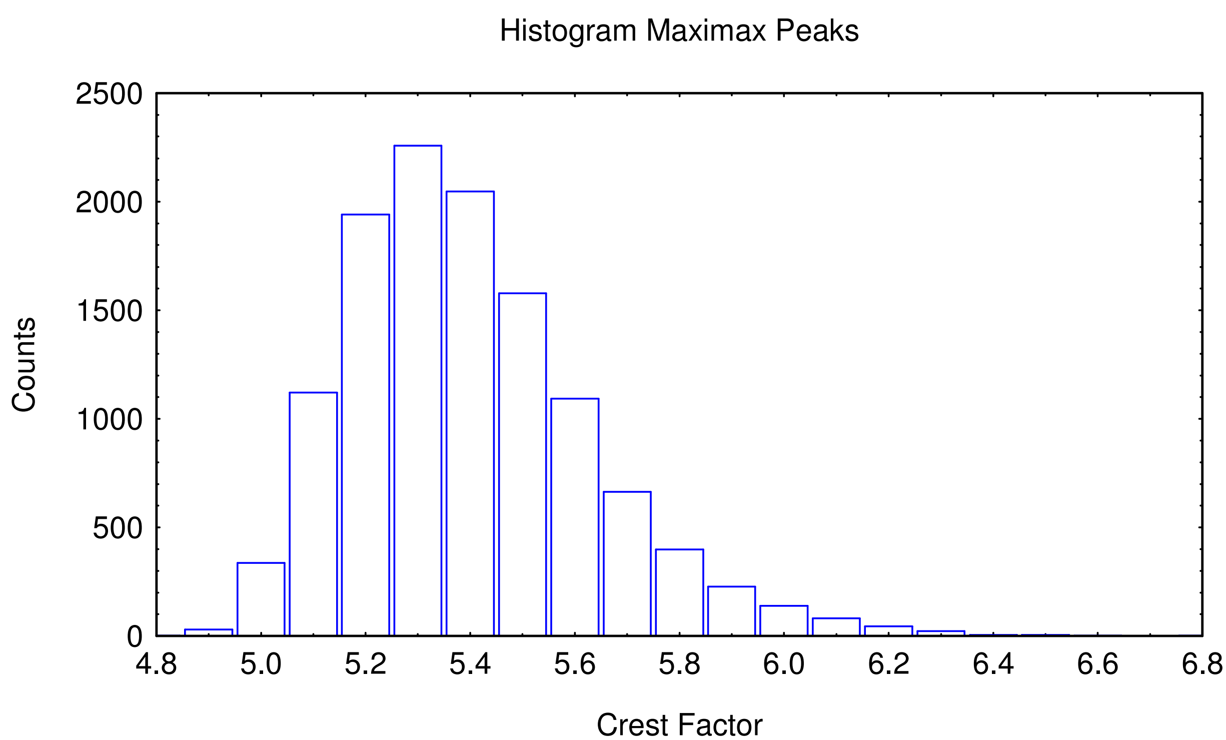

Figure 3. Maximax Histogram for 12,000 Trials, Experiment I

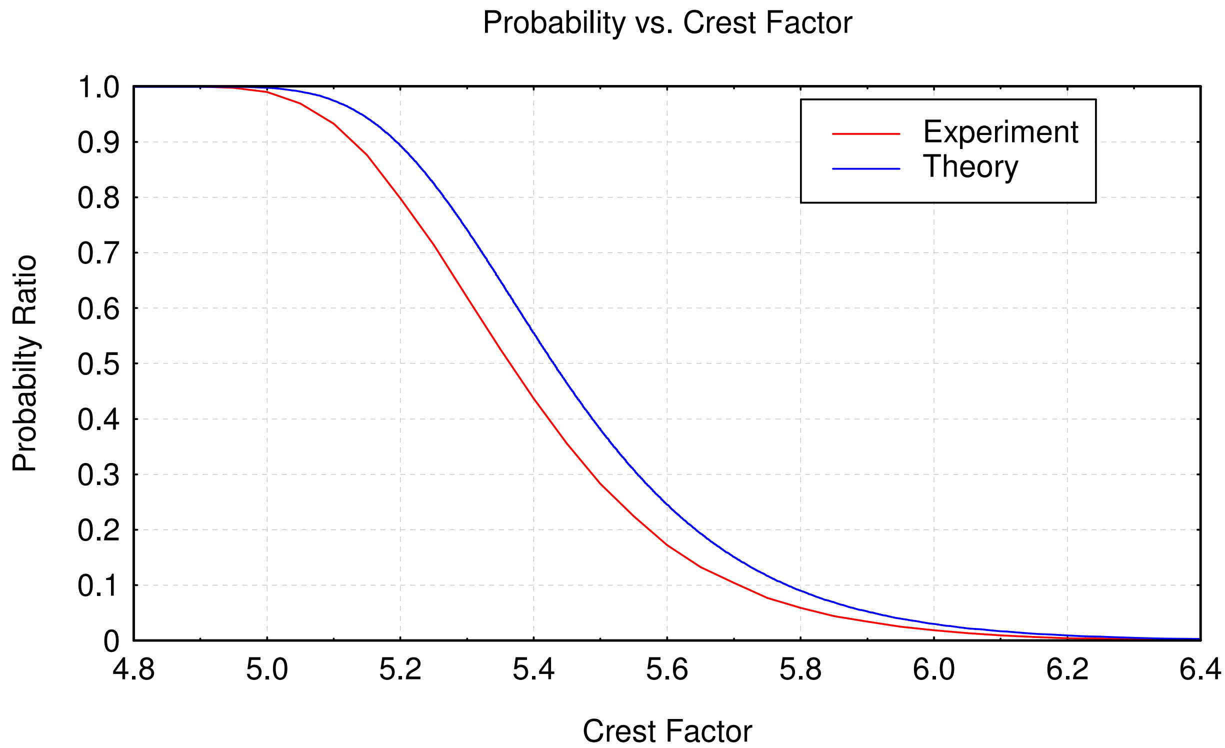

Figure 4. Probability Comparison, Experiment I

The theoretical crest factor for experiment I is

$$\lambda = \sqrt{2\ln\left(\text{fn}\left(800\,\text{Hz}\cdot 1200\,\text{sec}\right)\right)} + \frac{0.5772}{\sqrt{2\ln\left(800\,\text{Hz}\cdot 1200\,\text{sec}\right)}} = 5.35 \tag{5}$$

The theoretical crest factor has a 63.2% exceedance probability.

The crest factor for 10% exceedance probability is 5.8.

<u>Numerical Experiment II</u>

An SDOF system with fn = 250 Hz and 5% damping was subjected to base acceleration for an event duration of 1000 seconds. The base input was white noise, 10,000 samples/sec, and lowpass filtered at 1000 Hz. The number of response peaks was approximately 250,000 per trial, with 4000 total trials.

The maximax histogram is shown in Figure 3. The theoretical response probability from equation (4) and the experimental results are shown together in Figure 4 as a function of crest factor.

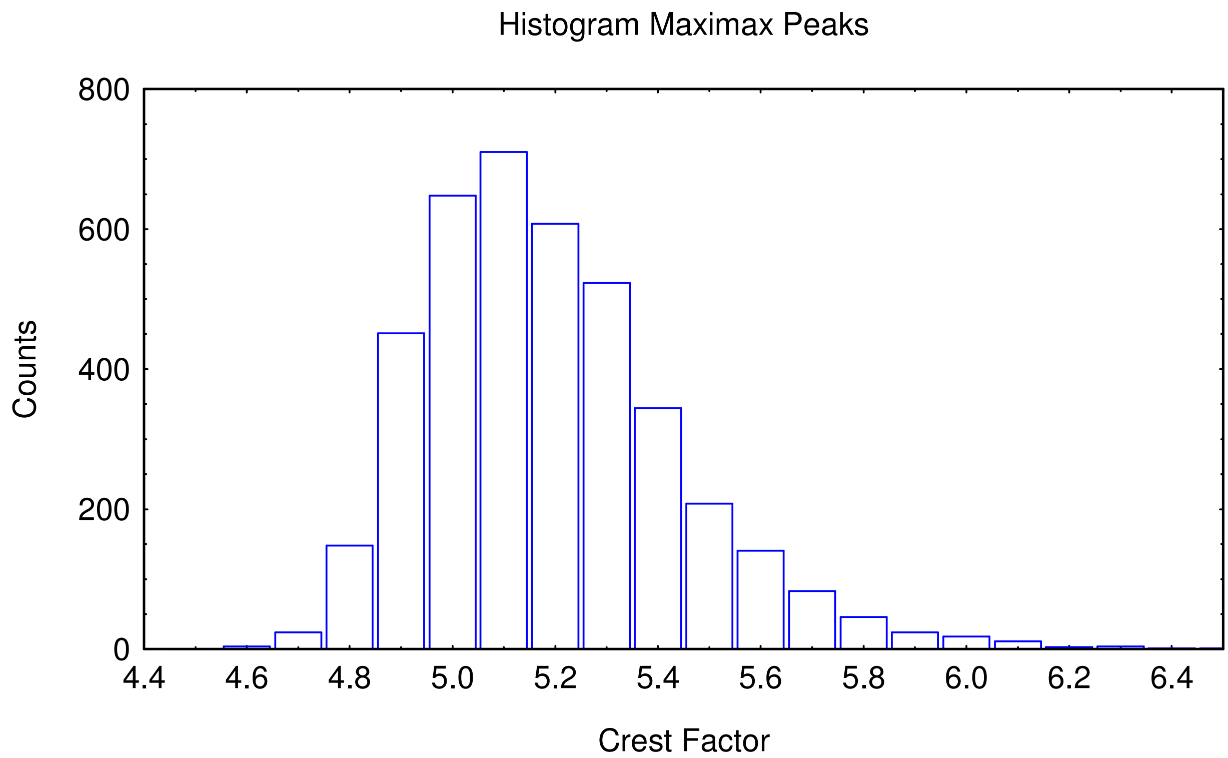

Figure 5. Maximax Histogram for 4000 Trials, Experiment II

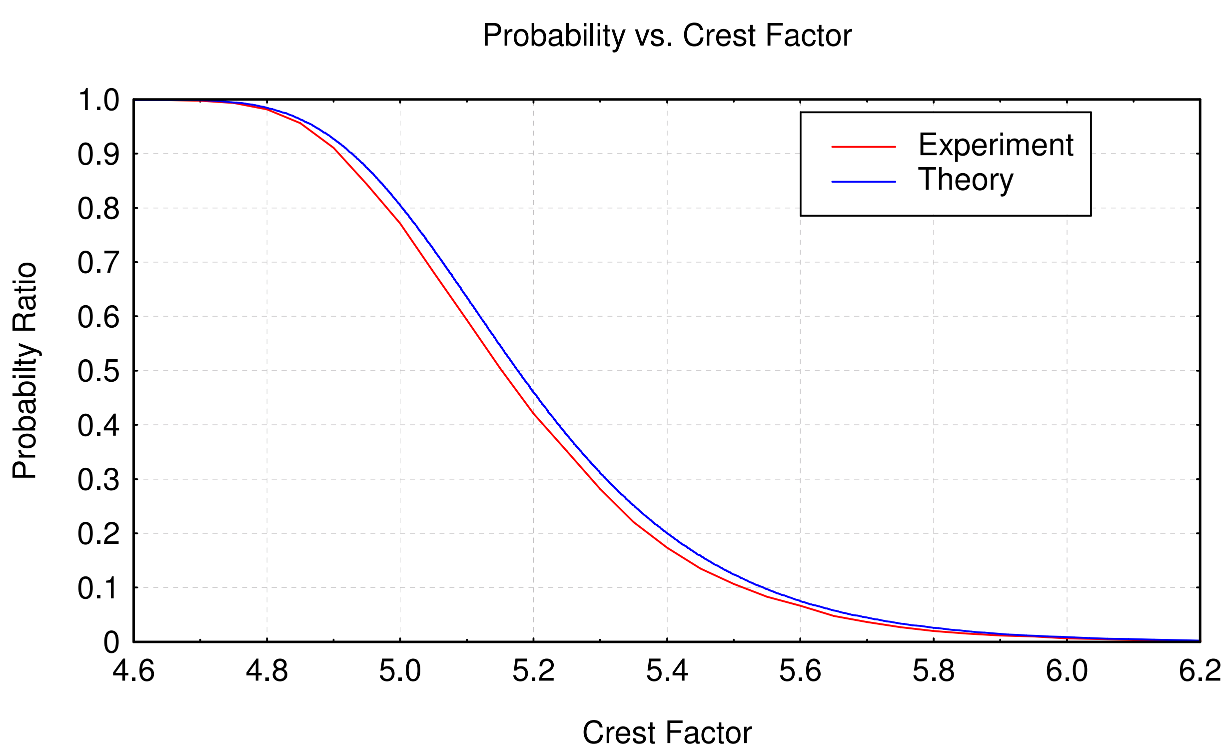

Figure 6. Probability Comparison, Experiment II

The theoretical curve envelops that of the experiment in Figure 6.

The theoretical crest factor for experiment II is.

$$\lambda = \sqrt{2\ln\left(\mathrm{fn}(250\,\mathrm{Hz}\cdot 1000\,\mathrm{sec})\right)} + \frac{0.5772}{\sqrt{2\ln(250\,\mathrm{Hz}\cdot 1000\,\mathrm{sec})}} = 5.10 \tag{6}$$

The theoretical crest factor has a 63.2% exceedance probability.

The crest factor for 10% exceedance probability is 5.54.

Conclusions

The crest factors in Figures 4 and 6 were calculated as a function of probability via equation (4). The data was then represented in terms of probability vs. crest factor. The theoretical curve in each case enveloped that of the respective experiment. The margin for experiment I in Figure 4 was up to 10%. The margin for experiment II in Figure 6 was within 5%.

Clearly more test cases are needed, including multi-degree-of-freedom system response.

Select the appropriate crest factor for a given design or analysis problem is a matter of engineering judgment.

References

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4. T. Irvine, Peak Response of a Multi-degree-of-freedom System, Vibrationdata, 2015.
5. C. Lalanne, Mechanical Vibration and Shock Analysis, second edition – volume 5, Specification Development, Wiley, Hoboken, NJ, 2009.
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9. Dave S. Steinberg, Vibration Analysis for Electronic Equipment, Second Edition, Wiley-Interscience, New York, 1988.

APPENDIX A

Alternate Equation

The XRS equation from Reference 7 is

$$\lambda_\alpha = \sqrt{2\ln(n_0^+ T)}\sqrt{1 - \frac{\ln(\alpha)}{\ln(n_0^+ T)}} \qquad \text{(A-1)}$$

This equation is more conservative than that in equation (4). It is given in this paper for historical purposes only.

APPENDIX B

SDOF System Equation of Motion Derivation

Consider a single degree-of-freedom system subjected to base excitation.

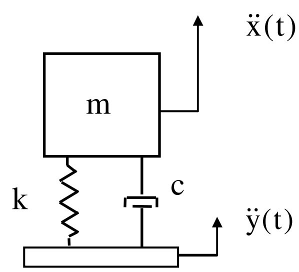


Figure B-1.

where

m	is the mass
c	is the viscous damping coefficient
k	is the stiffness
x	is the absolute displacement of the mass
y	is the base input displacement

The double-dot denotes acceleration.

The free-body diagram is

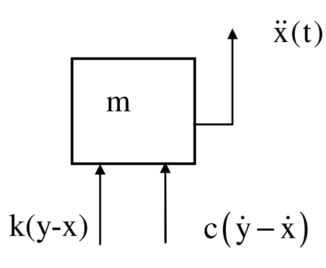


Figure B-2.

Summation of forces in the vertical direction,

$$\sum F = m\ddot{x} \tag{B-1}$$

$$m\ddot{x} = f(t) + c(\dot{y} - \dot{x}) + k(y - x) \tag{B-2}$$

Define a relative displacement

$$z = x - y \tag{B-3}$$

Substituting the relative displacement terms into equation (B-2) yields

$$m(\ddot{z} + \ddot{y}) = -c\dot{z} - kz \tag{B-4}$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{B-5}$$

Dividing through by mass yields,

$$\ddot{z} + (c/m)\dot{z} + (k/m)z = -\ddot{y} \tag{B-6}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{B-7}$$

$$(k/m) = \omega_n{}^2 \tag{B-8}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substituting the convention terms into equation (B-6) yields

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z = -\ddot{y} \tag{B-9}$$

This equation can be solved for arbitrary base excitation using Reference (7).

APPENDIX C

Continuation of Interrupted Random Vibration Test

Consider a component which must be subjected to a 60-second vibration test on a shaker table. Assume that the base input acceleration is stationary, broadband random vibration with a Gaussian distribution. Also assume that the component can be modeled as a linear, single-degree-of-freedom (SDOF) system.

Can two separate 30-second tests be substituted for one 60-second test? This question could arise, for example, if the test shut down halfway through due to a power outage, etc. Consideration of both peak response and fatigue damage are needed to answer this question.

The quick answer to the question is "Reasonably Yes" for engineering purposes but with caveats.

Ideally, a shock test would be used to cover peak response, and random vibration used for fatigue. This assumes that the shock response level is much higher than the peak random response level. But there may be a component test plan which specify random vibration only.

Dave Steinberg wrote:

> The most obvious characteristic of random vibration is that it is nonperiodic. A knowledge of the past history of random motion is adequate to predict the probability of occurrence of various acceleration and displacement magnitudes, but it is not sufficient to predict the precise magnitude at a specific instant.

Thus, there is uncertainty as to the peak response value of a system subjected to random vibration.

Probability Density Functions

An SDOF system will have a narrowband, Gaussian response to a broadband Gaussian input. This tends to also be true for a non-Gaussian input due to the Central Limit Theorem. There are three probability density function or equivalent histograms associated with the SDOF response as shown in Figure C-1.

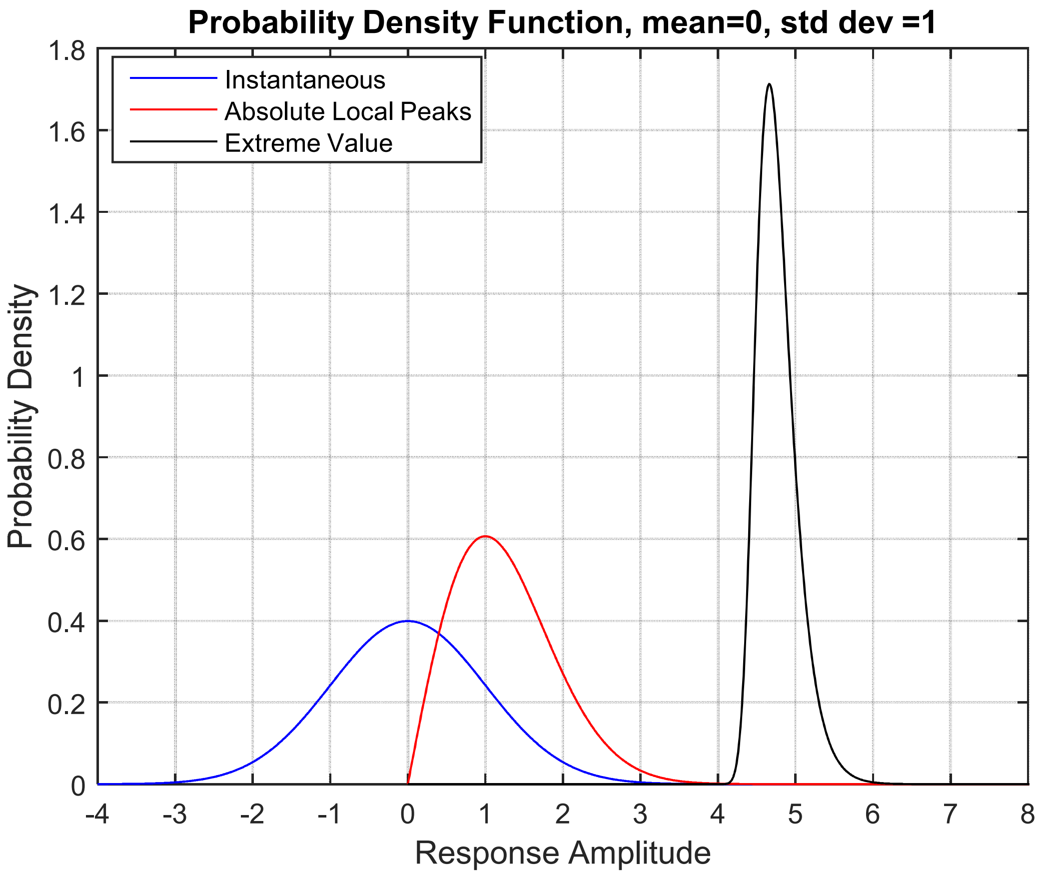


Figure C-1. Sample Response Probability Density Functions

The area under each curve is one.

The blue curve for the instantaneous amplitude has a Gaussian or normal distribution.

The red curve for the local peaks has a Rayleigh distribution.

The black curve is for the extreme value, which is the maximum absolute peak.

Now consider the blue and red curves to be those for a single record of some test duration. Assume there are numerous repetitions of the test, where the extreme value is retained from each. The black curve is the distribution of extreme values for the whole set of tests.

The black curve shows the potential variation of extreme value from one test to another. The crest factor could vary from 4 to 6. Recall that the crest factor is the (extreme peak / standard deviation).

Also note that the black curve will shift according to the total number of response cycles, whereas the blue and red curves are independent of this number.

Numerical Experiment

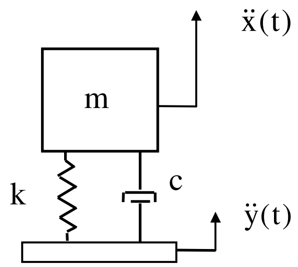
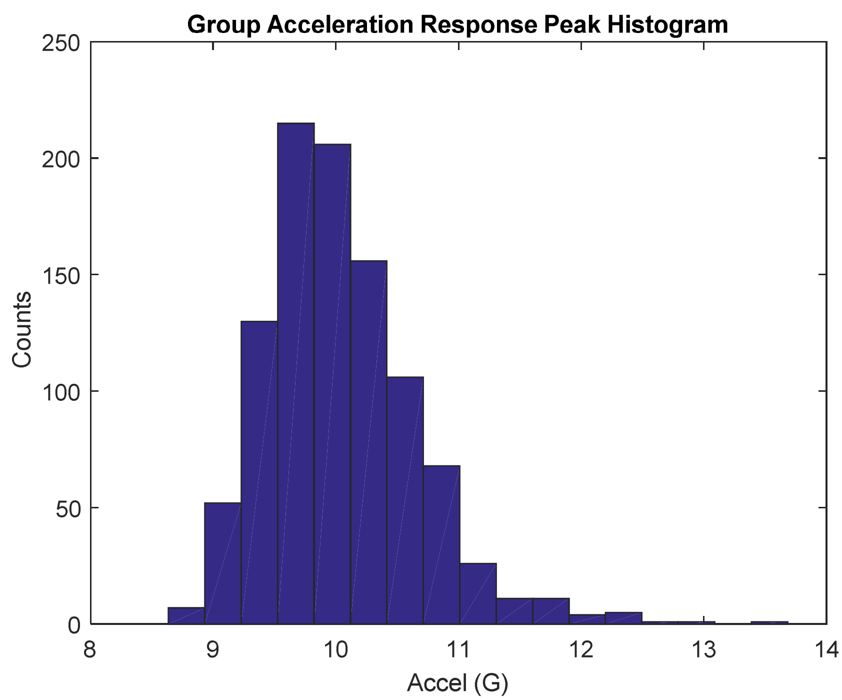


Figure C-2. SDOF System

The SDOF system has a natural frequency of 300 Hz with an amplification factor Q=10. It is subjected to band-limited white noise base input with a standard deviation of 1 G for 60-seconds. The white noise is lowpass filtered at 1000 Hz.

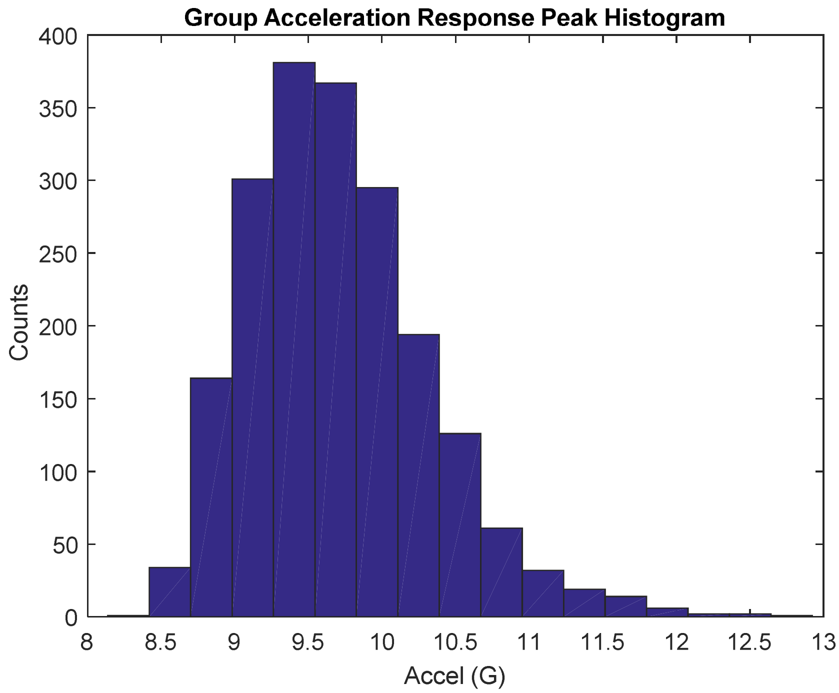
A set of 1000 such time histories are generated using these parameters. The response acceleration is calculated for each case. The extreme value is retained for each case. The histogram of these values is shown in Figure C-3. The average response standard deviation of the instantaneous response was approximately 2.17 G.



Range:	8.77 to 12.83 G
Mean:	10.04 G
Std Dev:	0.61 G

Figure C-3. Extreme Value Histogram for 1000 Responses with 60-second Duration Each

The range has a 3.3 dB spread.



Range: 8.28 to 12.78 G
Mean: 9.70 G
Std Dev: 0.62 G

Figure C-4. Extreme Value Histogram for 2000 Responses with 30-second Duration Each

The calculation is repeated for 2000 inputs of 30-seconds each. The extreme value histogram is shown in Figure C-4. The mean value for this set is about 3% lower than that for the group of 1000 inputs of 60-seconds each. This difference is expected per equation (4).

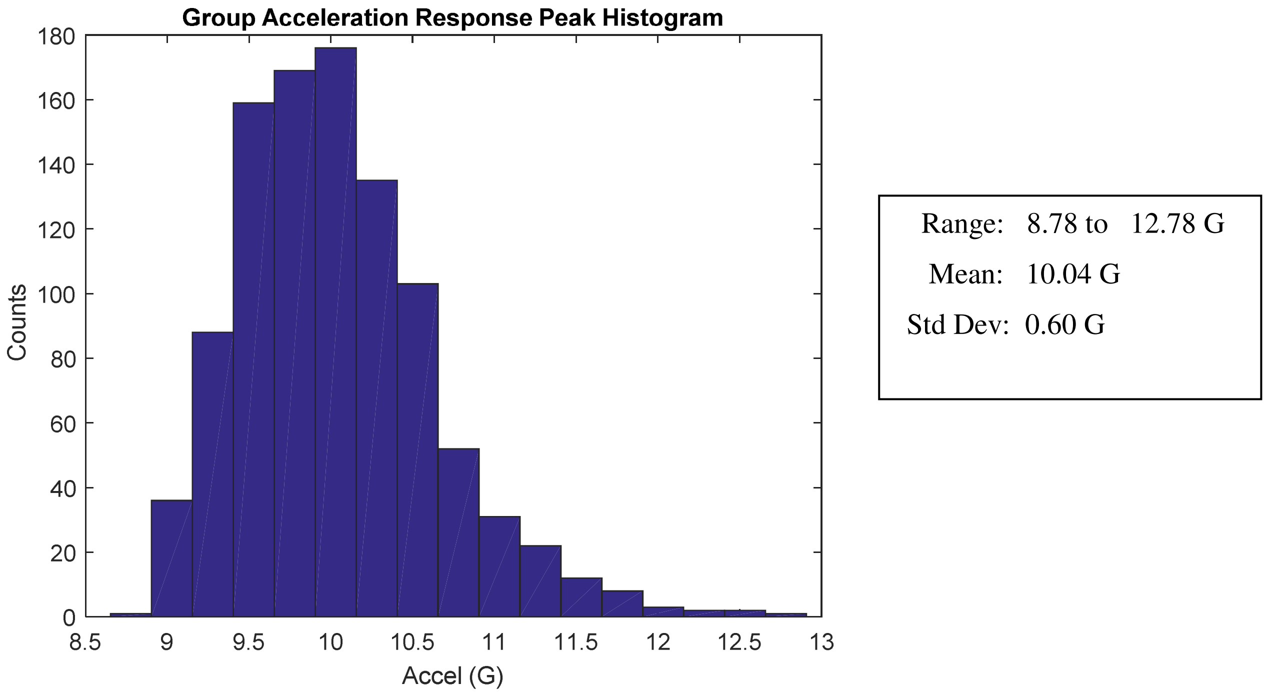


Figure C-5. Extreme Value Histogram for 1000 Response Pairs

Now create extreme value pairs from the 30-second records, such that the pair represents a total of 60 seconds. The number of records is thus reduced from 2000 to 1000. Each pair represents two discontinuous 30-second tests.

Reduce the two values within each pair to the highest value for each of the 1000 pairs. The resulting histogram is shown in Figure C-5. The descriptive statistics for this case closely match those for the 1000, 60-second records, as shown in Table C-1.

Table C-1. Extreme Value Comparison for Groups of 1000 Tests

Metric	60-sec Test	Paired 30-sec Tests
Range (G)	8.77 to 12.83	8.78 to 12.78
Mean (G)	10.04	10.04
Standard Deviation (G)	0.61	0.60

Note that the 10.04 G mean corresponds to a crest factor of 4.63. Approximately 50% of the extreme values were above this crest factor per equation (4).

Conclusion

The extreme response probability for one 60-second test is essentially the same as that for a pair of separate 30-second tests, assuming stationary input level. But the precise extreme value is subject to test-to-test variation similar to the histograms shown in Figures C-3 through C-5 for the sample case.

A shock test would be better than random vibration for covering a peak response requirement. A sine test could also be applied if precision is needed.

Further consideration is needed for fatigue. Fatigue damage accumulates in a linear manner per the Palmgren-Miner rule. So, a 60-second test should have the same fatigue damage as a pair of 30-seconds tests, but again some statistical variation is expected for the case of random vibration. This will be explored in the next revision of this paper.

APPENDIX D

Crest Factor Equation with Probability

Recall the equation for the crest factor λ_α.

$$\lambda_\alpha = \left\{\sqrt{2\ln(n_0^+ T)} + \frac{0.5772}{\sqrt{2\ln(n_0^+ T)}}\right\}\left\{\sqrt{\frac{-\ln\left(1-(1-\alpha)^{1/(n_0^+ T)}\right)}{\ln(n_0^+ T)}}\right\} \tag{D-1}$$

The equation can be expressed in terms of the probability of exceedance α.

$$\alpha = 1 - \left\{1 - \exp\left\{\frac{-\lambda_\alpha^{2}\ln(n_0^+ T)}{\left[\sqrt{2\ln(n_0^+ T)} + \frac{0.5772}{\sqrt{2\ln(n_0^+ T)}}\right]^2}\right\}\right\}^{(n_0^+ T)} \tag{D-2}$$

The probability density function is

$$-\frac{d\alpha}{d\lambda_\alpha}=\frac{2\lambda_\alpha(n_0^+T)\ln(n_0^+T)}{\left[\sqrt{2\ln(n_0^+T)}+\frac{0.5772}{\sqrt{2\ln(n_0^+T)}}\right]^2}\left\{1-\exp\left\{\frac{-\lambda_\alpha{}^2\ln(n_0^+T)}{\left[\sqrt{2\ln(n_0^+T)}+\frac{0.5772}{\sqrt{2\ln(n_0^+T)}}\right]^2}\right\}\right\}^{(n_0^+T-1)}\exp\left\{\frac{-\lambda_\alpha{}^2\ln(n_0^+T)}{\left[\sqrt{2\ln(n_0^+T)}+\frac{0.5772}{\sqrt{2\ln(n_0^+T)}}\right]^2}\right\}$$

(D-3)