Introduction

Random vibration is represented in the frequency domain by a power spectral density function. The overall root-mean-square (RMS) value is equal to the square root of the area under the curve. The purpose of this tutorial is to explain the integration procedure.

A power spectral density specification is typically represented as follows:

1. The specification is represented as a series of piecewise continuous segments.
2. Each segment is a straight line on a log-log plot.

An example is shown in Figure 1.
Note that the power spectral density amplitude is represented in units of \((G^2/\text{Hz})\). This is an abbreviated notation. The actual unit is \((G_{\text{RMS}}^2/\text{Hz})\).

**Derivation**

The equation for each segment is

\[
y(f) = \left[ \frac{y_1}{f_1^n} \right] f^n
\]

(1)

The starting coordinate is \((f_1, y_1)\).

The exponent \(n\) is a real number which represents the slope. The slope between two coordinates \((f_1, y_1)\) and \((f_2, y_2)\) is

\[
n = \frac{\log \left( \frac{y_2}{y_1} \right)}{\log \left( \frac{f_2}{f_1} \right)}
\]

(2)

The area \(a_1\) under segment 1 is

\[
a_1 = \int_{f_1}^{f_2} \left[ \frac{y_1}{f_1^n} \right] f^n df
\]

(3)

There are two cases depending on the exponent \(n\).

The first case is

\[
a_1 = \left[ \frac{y_1}{f_1^n} \right] \left[ \frac{1}{n+1} \right] f_2^{n+1} - f_1^{n+1}, \quad \text{for } n \neq -1
\]

(4)

\[
a_1 = \left[ \frac{y_1}{f_1^n} \right] \left[ \frac{1}{n+1} \right] \left[ f_2^{n+1} - f_1^{n+1} \right], \quad \text{for } n \neq -1
\]

(5)
The second case is

\[ a_1 = \int_{f_1}^{f_2} \left[ \frac{y_1}{f_1^{n+1}} \right] f^{-1} \, df, \quad \text{for } n = -1 \quad (6) \]  

\[ a_1 = \int_{f_1}^{f_2} \left[ y_1 f_1 \right] \frac{df}{f}, \quad \text{for } n = -1 \quad (7) \]  

\[ a_1 = \left[ y_1 f_1 \right] \ln(f) \bigg|_{f_1}^{f_2}, \quad \text{for } n = -1 \quad (8) \]  

\[ a_1 = \left[ y_1 f_1 \right] \left[ \ln(f_2) - \ln(f_1) \right], \quad \text{for } n = -1 \quad (9) \]  

\[ a_1 = \left[ y_1 f_1 \right] \left[ \frac{f_2}{f_1} \right], \quad \text{for } n = -1 \quad (10) \]

In summary, the area under segment \( i \) is

\[ a_i = \begin{cases} 
\left[ \frac{y_i}{f_i^n} \right] \left[ \frac{1}{n+1} \right] \left[ f_{i+1}^{n+1} - f_i^{n+1} \right], & \text{for } n \neq -1 \\
\left[ y_i f_i \right] \left[ \ln \left( \frac{f_{i+1}}{f_i} \right) \right], & \text{for } n = -1 
\end{cases} \quad (11) \]

The overall level \( L \) is

\[ L = \sqrt{\sum_{i=1}^{m} a_i} \quad (12) \]

where \( m \) is the total number of segments.
Example

Consider the power spectral density function in Figure 1. The breakpoints are given in Table 1.

<table>
<thead>
<tr>
<th>Freq (Hz)</th>
<th>Level (G^2/Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.002</td>
</tr>
<tr>
<td>100</td>
<td>0.04</td>
</tr>
<tr>
<td>1000</td>
<td>0.04</td>
</tr>
<tr>
<td>2000</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Consider the first pair of coordinates:

\[
\begin{align*}
  f_1 &= 10 \text{ Hz} & y_1 &= 0.002 \text{ G}^2/\text{Hz} \\
  f_2 &= 100 \text{ Hz} & y_2 &= 0.04 \text{ G}^2/\text{Hz}
\end{align*}
\]

Calculate the slope.

\[
\log \left( \frac{0.04}{0.002} \right) \\
\log \left( \frac{100}{10} \right)
\]  

\[
n = \frac{\log \left( \frac{0.04}{0.002} \right)}{\log \left( \frac{100}{10} \right)}
\]  

\[
n = 1.3
\]

Substitute into equation (11).

\[
a_1 = \left[ \frac{0.002}{10^{1.3}} \right] \left[ \frac{1}{1.3 + 1} \right] \left[ 100^{1.3 + 1} - 10^{1.3 + 1} \right]
\]

\[
a_1 = 1.726 \text{ G}^2
\]
Consider the second pair:

\[
\begin{array}{|c|c|}
\hline
f_2 &= 100 \text{ Hz} \\
y_2 &= 0.04 \text{ G}^2/\text{Hz} \\
\hline
f_3 &= 1000 \text{ Hz} \\
y_3 &= 0.04 \text{ G}^2/\text{Hz} \\
\hline
\end{array}
\]

Calculate the slope.

\[
n = \frac{\log \left( \frac{0.04}{0.04} \right)}{\log \left( \frac{1000}{100} \right)}
\]

(18)

\[
n = 0.
\]

(19)

Substitute into equation (11).

\[
a_2 = \left[ \frac{0.04}{100^0} \right] \left[ \frac{1}{0+1} \right] \left[ 1000^{0+1} - 100^{0+1} \right]
\]

(20)

\[
a_2 = \left[ \frac{0.04}{1} \right] \left[ \frac{1}{1} \right] \left[ 1000^{-1} - 100^{-1} \right]
\]

(21)

\[
a_2 = 36,000 \; \text{G}^2
\]

(22)

Consider the third pair:

\[
\begin{array}{|c|c|}
\hline
f_3 &= 1000 \text{ Hz} \\
y_3 &= 0.04 \text{ G}^2/\text{Hz} \\
\hline
f_4 &= 2000 \text{ Hz} \\
y_4 &= 0.02 \text{ G}^2/\text{Hz} \\
\hline
\end{array}
\]

Calculate the slope.

\[
n = \frac{\log \left( \frac{0.02}{0.04} \right)}{\log \left( \frac{2000}{1000} \right)}
\]

(23)

\[
n = -1.
\]

(24)
Substitute into equation (11).

\[ a_3 = [(0.04)(1000)] \ln \left( \frac{2000}{1000} \right) \]  
(25)

\[ a_3 = 27.726 \]  
(26)

Now substitute the individual area values into equation (12).

\[ L = \sqrt{1.726 + 36.000 + 27.726} G^2 \]  
(27)

The overall level is

\[ L = 8.09 \ G_{RMS} \]  
(28)

Additional information on slopes is given in Appendix A.
APPENDIX A

Introduction to dB/octave Slopes

NAVMAT P-9492 gives the power spectral density specification shown in Figure A-1.

![Figure A-1](image)

The task is to determine the coordinates of the endpoints.

**Derivation**

Assume that $a_1$ and $a_2$ each has an amplitude in $g^2/Hz$. The difference in dB between $a_1$ and $a_2$ is

$$\Delta dB = 10 \log \left[ \frac{a_2}{a_1} \right]$$

(A-1)
Furthermore,

$$a_2 = a_1 \left[ 10^{\frac{\Delta dB}{10}} \right]$$  \hspace{1cm} (A-2)

Additional equations are needed.

The slope $N$ between two coordinates $(f_1, a_1)$ and $(f_2, a_2)$ in a log-log plot is

$$N = \frac{\log \left( \frac{a_2}{a_1} \right)}{\log \left( \frac{f_2}{f_1} \right)}$$  \hspace{1cm} (A-3)

Solve for $a_2$.

$$N \log \left( \frac{f_2}{f_1} \right) = \log \left( \frac{a_2}{a_1} \right)$$  \hspace{1cm} (A-4)

$$\log \left( \left( \frac{f_2}{f_1} \right)^N \right) = \log \left( \frac{a_2}{a_1} \right)$$  \hspace{1cm} (A-5)

Take the anti-log.

$$\left( \frac{f_2}{f_1} \right)^N = \left[ \frac{a_2}{a_1} \right]$$  \hspace{1cm} (A-6)

$$\left[ \frac{a_2}{a_1} \right] = \left( \frac{f_2}{f_1} \right)^N$$  \hspace{1cm} (A-7)

Thus,

$$a_2 = a_1 \left[ \frac{f_2}{f_1} \right]^N$$  \hspace{1cm} (A-8)
Now consider a one-octave frequency separation.

\[ f_2 = 2f_1 \quad (A-9) \]

Substitute equation (A-9) into (A-3).

\[ N = \frac{\log \left( \frac{a_2}{a_1} \right)}{\log 2} \quad (A-10) \]

Substitute equation (A-1) into (A-10).

\[ N = \frac{\Delta dB / 10}{\log 2} \quad (A-11) \]

Note that \( \Delta dB \) represents the dB/octave slope in equation (A-11). Again, equations (A-10) and (A-11) assume a one-octave frequency separation.

Now substitute equation (A-11) into (A-8).

\[ a_2 = a_1 \left[ \frac{f_2}{f_1} \right] \left( \frac{\Delta dB / 10}{\log 2} \right) \quad (A-12) \]

Example

Calculate the amplitude at 2000 Hz for the power spectral density in Figure A-1. The slope is -3 dB/octave.

Note

\[ f_1 = 350 \text{ Hz} \]
\[ f_2 = 2000 \text{ Hz} \]
\[ a_1 = 0.04 \text{ G}^2 / \text{Hz} \]

Substitute into equation (A-12).
\[ a_2 = 0.04 \frac{G^2}{\text{Hz}} \left[ \frac{2000 \text{ Hz}}{350 \text{ Hz}} \right]^{\frac{-3 \text{ dB}/10}{\log[2]}} \]  \hspace{1cm} (A-13)

\[ a_2 = 0.007 \frac{G^2}{\text{Hz}} \text{ at 2000 Hz} \]  \hspace{1cm} (A-14)

Now calculate the amplitude at 20 Hz for the power spectral density in Figure A-1. The slope is +3dB/octave.

Note

\[ f_1 = 80 \text{ Hz} \]
\[ f_2 = 20 \text{ Hz} \]
\[ a_1 = 0.04 \frac{G^2}{\text{Hz}} \]

Substitute into equation (A-12). Note that this equation allows \( f_2 < f_1 \).

\[ a_2 = 0.04 \frac{G^2}{\text{Hz}} \left[ \frac{20 \text{ Hz}}{80 \text{ Hz}} \right]^{\frac{+3 \text{ dB}/10}{\log[2]}} \]  \hspace{1cm} (A-15)

\[ a_2 = 0.01 \frac{G^2}{\text{Hz}} \text{ at 20 Hz} \]  \hspace{1cm} (A-16)