VIBRATION ANALYSIS OF AN ISOLATED MASS
WITH SIX DEGREES OF FREEDOM
Revision G

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Introduction

An avionics component may be mounted with isolator grommets, which act as soft springs. The goal of the isolator design is to provide attenuation of shock and vibration energy. This is achieved by lowering the natural frequency of the component system.

Consider a component with a complex geometry that is to be mounted via four isolators, as shown in Figures 1 and 2. Assume that the component's hardmounted natural frequency is at least one octave greater than any of its isolation frequencies.

The objective is to derive the equations of motion for this system, accounting for six degrees-of-freedom.

Background

The motivation for this report came from a case history where the component's center of gravity was offset such that the a2 dimension in Figure 2 was negative. Thus, the component's C.G. was overhanging the footprint.

The detailed finite element model of the isolated component had some numerical stability problems[1] due to the combination of the C.G. offset and the soft springs. As a result, the model incorrectly gave some of the natural frequencies as zero, or approximately zero.[2]

Hand calculations should be used to check finite element results regardless of numerically stability, but this verification is particularly critical when the model is potentially unstable.

The "hand calculation" approach in this report represents the component by a point mass with inertia. The isolators are represented by springs. The natural frequencies for a real-world example are calculated using a software program that implements the hand calculation model. The program is given in Appendix A.

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[1] To some extent, the numerical instability may represent mechanical instability.

[2] The finite element model was eventually rendered numerically stable by adding rotational dof springs for each isolator. This was rather artificial because the true rotational stiffness of each isolator was unknown.

Derivation

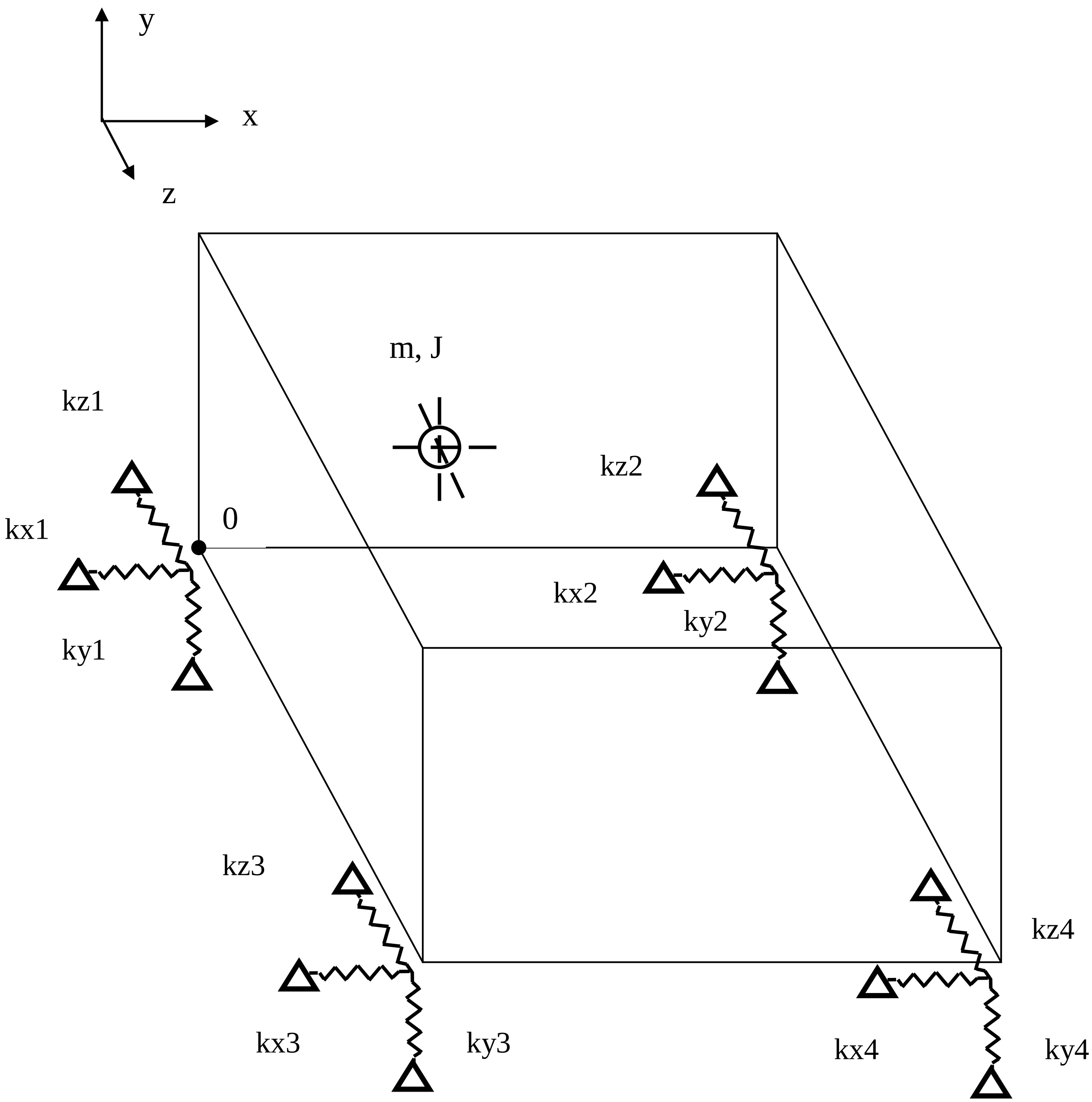


Figure 1. Isolated Avionics Component Model

The mass and inertia are represented at a point with the circle symbol. Each isolator is modeled by three orthogonal DOF springs. The springs are mounted at each corner. The springs are shown with an offset from the corners for clarity. The triangles indicate fixed constraints. “0” indicates the origin.

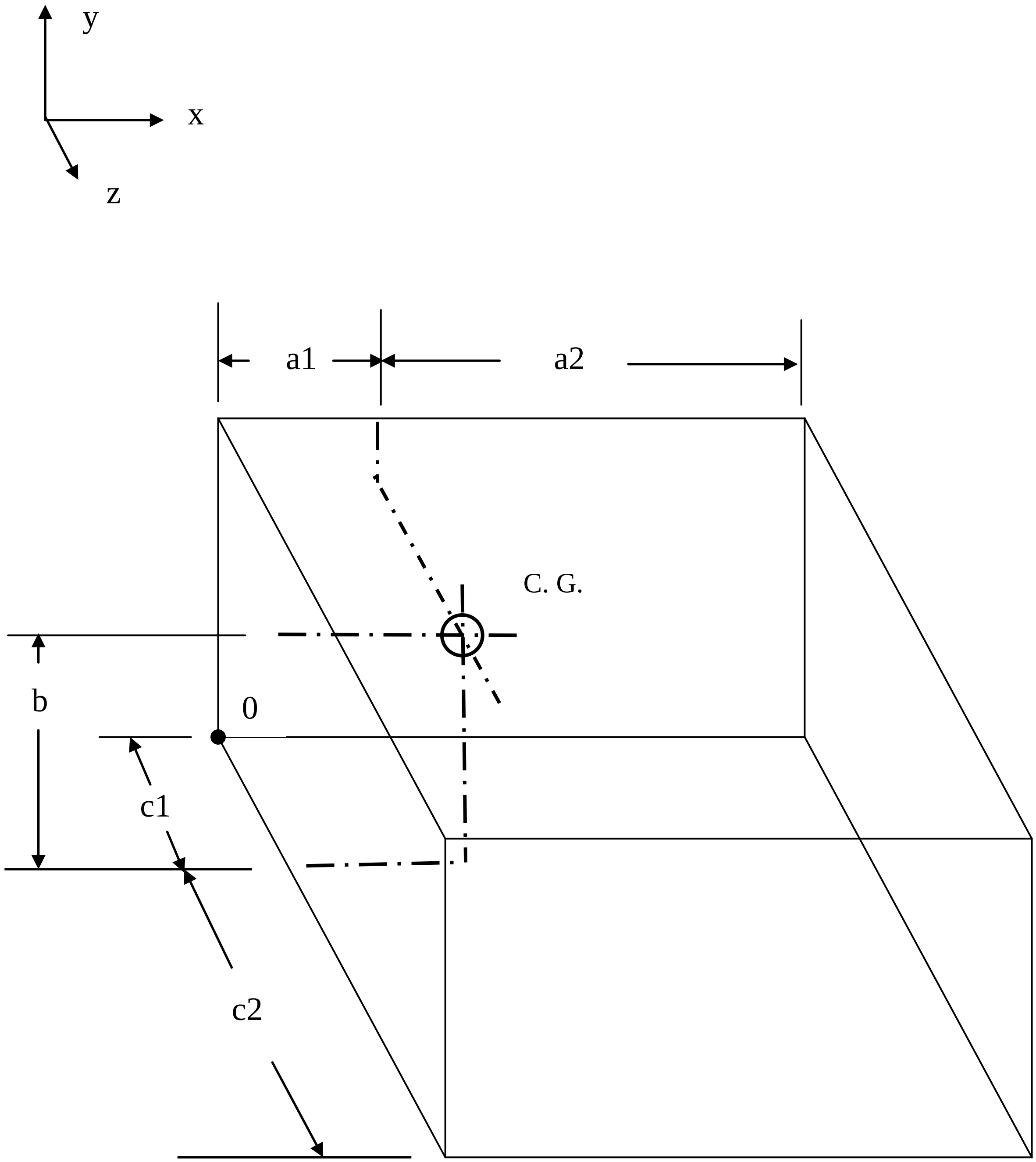


Figure 2. Isolated Avionics Component Model with Dimensions

All dimensions are positive as long as the C.G. is "inside the box." At least one dimension will be negative otherwise.

The variables $\alpha, \beta,$ and θ represent rotations about the X, Y, and Z axes, respectively, using the right-hand rule convention.

The total kinetic energy is

$$T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{1}{2} J_x \dot{\alpha}^2 + \frac{1}{2} J_y \dot{\beta}^2 + \frac{1}{2} J_z \dot{\theta}^2 \tag{1}$$

The total potential energy is

$$\begin{aligned}
V = {} & \\
& + \frac{1}{2} k_{x1}(x - c_1 \beta + b\theta)^2 \\
& + \frac{1}{2} k_{x2}(x - c_1 \beta + b\theta)^2 \\
& + \frac{1}{2} k_{x3}(x + c_2 \beta + b\theta)^2 \\
& + \frac{1}{2} k_{x4}(x + c_2 \beta + b\theta)^2 \\
& + \frac{1}{2} k_{y1}(y + c_1 \alpha - a_1\theta)^2 \\
& + \frac{1}{2} k_{y2}(y + c_1 \alpha + a_2\theta)^2 \\
& + \frac{1}{2} k_{y3}(y - c_2 \alpha - a_1\theta)^2 \\
& + \frac{1}{2} k_{y4}(y - c_2 \alpha + a_2\theta)^2 \\
& + \frac{1}{2} k_{z1}(z + a_1 \beta - b\alpha)^2 \\
& + \frac{1}{2} k_{z2}(z - a_2 \beta - b\alpha)^2 \\
& + \frac{1}{2} k_{z3}(z + a_1 \beta - b\alpha)^2 \\
& + \frac{1}{2} k_{z4}(z - a_2 \beta - b\alpha)^2
\end{aligned} \tag{2}$$

The energy is

$$
\begin{aligned}
E = \\
&+\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{1}{2}J_x\,\dot{\alpha}^2+\frac{1}{2}J_y\,\dot{\beta}^2+\frac{1}{2}J_z\,\dot{\theta}^2\\
&+\frac{1}{2}k_{x1}(x-c_1\beta+b\theta)^2\\
&+\frac{1}{2}k_{x2}(x-c_1\beta+b\theta)^2\\
&+\frac{1}{2}k_{x3}(x+c_2\beta+b\theta)^2\\
&+\frac{1}{2}k_{x4}(x+c_2\beta+b\theta)^2\\
&+\frac{1}{2}k_{y1}(y+c_1\alpha-a_1\theta)^2\\
&+\frac{1}{2}k_{y2}(y+c_1\alpha+a_2\theta)^2\\
&+\frac{1}{2}k_{y3}(y-c_2\alpha-a_1\theta)^2\\
&+\frac{1}{2}k_{y4}(y-c_2\alpha+a_2\theta)^2\\
&+\frac{1}{2}k_{z1}(z+a_1\beta-b\alpha)^2\\
&+\frac{1}{2}k_{z2}(z-a_2\beta-b\alpha)^2\\
&+\frac{1}{2}k_{z3}(z+a_1\beta-b\alpha)^2\\
&+\frac{1}{2}k_{z4}(z-a_2\beta-b\alpha)^2
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
E = \\
&+\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{1}{2}J_x\,\dot{\alpha}^2+\frac{1}{2}J_y\,\dot{\beta}^2+\frac{1}{2}J_z\,\dot{\theta}^2 \\
&+\frac{1}{2}k_{x1}(x-c_1\beta+b\theta)(x-c_1\beta+b\theta) \\
&+\frac{1}{2}k_{x2}(x-c_1\beta+b\theta)(x-c_1\beta+b\theta) \\
&+\frac{1}{2}k_{x3}(x+c_2\beta+b\theta)(x+c_2\beta+b\theta) \\
&+\frac{1}{2}k_{x4}(x+c_2\beta+b\theta)(x+c_2\beta+b\theta) \\
&+\frac{1}{2}k_{y1}(y+c_1\alpha-a_1\theta)(y+c_1\alpha-a_1\theta) \\
&+\frac{1}{2}k_{y2}(y+c_1\alpha+a_2\theta)(y+c_1\alpha+a_2\theta) \\
&+\frac{1}{2}k_{y3}(y-c_2\alpha-a_1\theta)(y-c_2\alpha-a_1\theta) \\
&+\frac{1}{2}k_{y4}(y-c_2\alpha+a_2\theta)(y-c_2\alpha+a_2\theta) \\
&+\frac{1}{2}k_{z1}(z+a_1\beta-b\alpha)(z+a_1\beta-b\alpha) \\
&+\frac{1}{2}k_{z2}(z-a_2\beta-b\alpha)(z-a_2\beta-b\alpha) \\
&+\frac{1}{2}k_{z3}(z+a_1\beta-b\alpha)(z+a_1\beta-b\alpha) \\
&+\frac{1}{2}k_{z4}(z-a_2\beta-b\alpha)(z-a_2\beta-b\alpha)
\end{aligned}
\tag{4}
$$

$$E =$$

$$+\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{1}{2}J_x\,\dot{\alpha}^2+\frac{1}{2}J_y\,\dot{\beta}^2+\frac{1}{2}J_z\,\dot{\theta}^2$$

$$+\frac{1}{2}k_{x1}\left(x\left(x-c_1\beta+b\theta\right)-c_1\beta\left(x-c_1\beta+b\theta\right)+b\theta\left(x-c_1\beta+b\theta\right)\right)$$
$$+\frac{1}{2}k_{x2}\left(x\left(x-c_1\beta+b\theta\right)-c_1\beta\left(x-c_1\beta+b\theta\right)+b\theta\left(x-c_1\beta+b\theta\right)\right)$$
$$+\frac{1}{2}k_{x3}\left(x\left(x+c_2\beta+b\theta\right)+c_2\beta\left(x+c_2\beta+b\theta\right)+b\theta\left(x+c_2\beta+b\theta\right)\right)$$
$$+\frac{1}{2}k_{x4}\left(x\left(x+c_2\beta+b\theta\right)+c_2\beta\left(x+c_2\beta+b\theta\right)+b\theta\left(x+c_2\beta+b\theta\right)\right)$$

$$+\frac{1}{2}k_{y1}\left(y\left(y+c_1\alpha-a_1\theta\right)+c_1\alpha\left(y+c_1\alpha-a_1\theta\right)-a_1\theta\left(y+c_1\alpha-a_1\theta\right)\right)$$
$$+\frac{1}{2}k_{y2}\left(y\left(y+c_1\alpha+a_2\theta\right)+c_1\alpha\left(y+c_1\alpha+a_2\theta\right)+a_2\theta\left(y+c_1\alpha+a_2\theta\right)\right)$$
$$+\frac{1}{2}k_{y3}\left(y\left(y-c_2\alpha-a_1\theta\right)-c_2\alpha\left(y-c_2\alpha-a_1\theta\right)-a_1\theta\left(y-c_2\alpha-a_1\theta\right)\right)$$
$$+\frac{1}{2}k_{y4}\left(y\left(y-c_2\alpha+a_2\theta\right)-c_2\alpha\left(y-c_2\alpha+a_2\theta\right)+a_2\theta\left(y-c_2\alpha+a_2\theta\right)\right)$$

$$+\frac{1}{2}k_{z1}\left(z\left(z+a_1\beta-b\alpha\right)+a_1\beta\left(z+a_1\beta-b\alpha\right)-b\alpha\left(z+a_1\beta-b\alpha\right)\right)$$
$$+\frac{1}{2}k_{z2}\left(z\left(z-a_2\beta-b\alpha\right)-a_2\beta\left(z-a_2\beta-b\alpha\right)-b\alpha\left(z-a_2\beta-b\alpha\right)\right)$$
$$+\frac{1}{2}k_{z3}\left(z\left(z+a_1\beta-b\alpha\right)+a_1\beta\left(z+a_1\beta-b\alpha\right)-b\alpha\left(z+a_1\beta-b\alpha\right)\right)$$
$$+\frac{1}{2}k_{z4}\left(z\left(z-a_2\beta-b\alpha\right)-a_2\beta\left(z-a_2\beta-b\alpha\right)-b\alpha\left(z-a_2\beta-b\alpha\right)\right) \tag{5}$$

$$E =$$

$$+\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{1}{2}J_x\,\dot{\alpha}^2+\frac{1}{2}J_y\,\dot{\beta}^2+\frac{1}{2}J_z\,\dot{\theta}^2$$

$$
\begin{aligned}
&+\frac{1}{2}k_{x1}\left(\left(x^2-c_1\beta x+b\theta x\right)+\left(-c_1x\beta+c_1^2\beta^2-c_1b\theta\beta\right)+\left(bx\theta-bc_1\beta\theta+b^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{x2}\left(\left(x^2-c_1\beta x+b\theta x\right)+\left(-c_1x\beta+c_1^2\beta^2-c_1b\theta\beta\right)+\left(bx\theta-bc_1\beta\theta+b^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{x3}\left(\left(x^2+c_2\beta x+b\theta x\right)+\left(c_2x\beta+c_2{}^2\beta^2+c_2b\theta\beta\right)+\left(bx\theta+bc_2\beta\theta+b^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{x4}\left(\left(x^2+c_2\beta x+b\theta x\right)+\left(c_2x\beta+cc_2{}^2\beta^2+c_2b\theta\beta\right)+\left(bx\theta+bc_2\beta\theta+b^2\theta^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}k_{y1}\left(\left(y^2+c_1\alpha y-a_1\theta y\right)+\left(c_1y\alpha+c_1^2\alpha^2-c_1a_1\theta\alpha\right)+\left(-a_1y\theta-a_1c_1\alpha\theta+a_1^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{y2}\left(\left(y^2+c_1\alpha y+a_2\theta y\right)+\left(c_1y\alpha+c_1^2\alpha^2+a_2c_1\theta\alpha\right)+\left(a_2y\theta+a_2c_1\alpha\theta+a_2^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{y3}\left(\left(y^2-c_2\alpha y-a_1\theta y\right)+\left(-c_2y\alpha+c_2{}^2\alpha^2+a_1c_2\theta\alpha\right)+\left(-a_1y\theta+a_1c_2\alpha\theta+a_1^2\theta^2\right)\right)\\
&+\frac{1}{2}k_{y4}\left(\left(y^2-c_2\alpha y+a_2\theta y\right)+\left(-c_2y\alpha+c_2{}^2\alpha^2-a_2c_2\theta\alpha\right)+\left(a_2y\theta-a_2c_2\alpha\theta+a_2^2\theta^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}k_{z1}\left(\left(z^2+a_1\beta z-b\alpha z\right)+\left(a_1z\beta+a_1^2\beta^2-a_1b\alpha\beta\right)+\left(-bz\alpha-ba_1\beta\alpha+b^2\alpha^2\right)\right)\\
&+\frac{1}{2}k_{z2}\left(\left(z^2-a_2\beta z-b\alpha z\right)+\left(-a_2z\beta+a_2^2\beta^2+a_2b\alpha\beta\right)+\left(-bz\alpha+ba_2\beta\alpha+b^2\alpha^2\right)\right)\\
&+\frac{1}{2}k_{z3}\left(\left(z^2+a_1\beta z-b\alpha z\right)+\left(a_1z\beta+a_1^2\beta^2-a_1b\alpha\beta\right)+\left(-bz\alpha-ba_1\beta\alpha+b^2\alpha^2\right)\right)\\
&+\frac{1}{2}k_{z4}\left(\left(z^2-a_2\beta z-b\alpha z\right)+\left(-a_2z\beta+a_2^2\beta^2+a_2b\alpha\beta\right)+\left(-bz\alpha+ba_2\beta\alpha+b^2\alpha^2\right)\right)
\end{aligned}
\tag{6}
$$

$$E =$$

$$+\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{1}{2}J_x\,\dot{\alpha}^2+\frac{1}{2}J_y\,\dot{\beta}^2+\frac{1}{2}J_z\,\dot{\theta}^2$$

$$+\frac{1}{2}\left(k_{x1}+k_{x2}\right)\left(x^2-2c_1\beta x+2b\theta x+c_1^2\beta^2-2c_1b\theta\beta+b^2\theta^2\right)$$
$$+\frac{1}{2}\left(k_{x3}+k_{x4}\right)\left(x^2+2c_2\beta x+2b\theta x+c_2^{\ 2}\beta^2+2c_2b\theta\beta+b^2\theta^2\right)$$

$$+\frac{1}{2}k_{y1}\left(y^2-2c_1\alpha y-2a_1\theta y+c_1^2\alpha^2+2a_1c_1\theta\alpha+a_1^2\theta^2\right)$$
$$+\frac{1}{2}k_{y2}\left(y^2-2c_1\alpha y+2a_2\theta y+c_1^2\alpha^2-2a_2c_1\theta\alpha+a_2^2\theta^2\right)$$
$$+\frac{1}{2}k_{y3}\left(y^2+2c_2y\alpha-2a_1\theta y+c_2^{\ 2}\alpha^2-2a_1c_2\theta\alpha+a_1^2\theta^2\right)$$
$$+\frac{1}{2}k_{y4}\left(y^2+2c_2\alpha y+2a_2\theta y+c_2^{\ 2}\alpha^2+2a_2c_2\theta\alpha+a_2^2\theta^2\right)$$

$$+\frac{1}{2}\left(k_{z1}+k_{z3}\right)\left(z^2+2a_1\beta z-2b\alpha z+a_1^2\beta^2-2a_1b\alpha\beta+b^2\alpha^2\right)$$
$$+\frac{1}{2}\left(k_{z2}+k_{z4}\right)\left(z^2-2a_2\beta z-2b\alpha z+a_2^2\beta^2+2a_2b\alpha\beta+b^2\alpha^2\right)$$

(7)

$$
\begin{aligned}
E \quad = \quad & \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{1}{2}J_x\,\dot{\alpha}^2+\frac{1}{2}J_y\,\dot{\beta}^2+\frac{1}{2}J_z\,\dot{\theta}^2 \\
& +\frac{1}{2}\left(k_{x1}+k_{x2}+k_{x3}+k_{x4}\right)x^2 \\
& +\left(-\left(k_{x1}+k_{x2}\right)c_1+\left(k_{x3}+k_{x4}\right)c_2\right)\beta x \\
& +\left(k_{x1}+k_{x2}+k_{x3}+k_{x4}\right)b\theta x \\
& +\frac{1}{2}\left(\left(k_{x1}+k_{x2}\right)c_1^2+\left(k_{x3}+k_{x4}\right)c_2^2\right)\beta^2 \\
& +\left(-\left(k_{x1}+k_{x2}\right)c_1+\left(k_{x3}+k_{x4}\right)c_2\right)b\theta\beta \\
& +\frac{1}{2}\left(k_{x1}+k_{x2}+k_{x3}+k_{x4}\right)b^2\theta^2 \\
& +\frac{1}{2}\left(k_{y1}+k_{y2}+k_{y3}+k_{y4}\right)y^2 \\
& +\frac{1}{2}\left(\left(k_{y1}+k_{y2}\right)c_1^2+\left(k_{y3}+k_{y4}\right)c_2^{\ 2}\right)\alpha^2 \\
& +\frac{1}{2}\left(\left(k_{y1}+k_{y3}\right)a_1^2+\left(k_{y2}+k_{y4}\right)a_2^2\right)\theta^2 \\
& +\left(\left(k_{y1}+k_{y2}\right)c_1-\left(k_{y3}+k_{y4}\right)c_2\right)\alpha y \\
& +\left(\left(-k_{y1}-k_{y3}\right)a_1+\left(k_{y2}+k_{y4}\right)a_2\right)\theta y \\
& +\left(-k_{y1}a_1c_1+k_{y2}a_2c_1+k_{y3}a_1c_2-k_{y4}a_2c_2\right)\theta\alpha \\
& +\frac{1}{2}\left(k_{z1}+k_{z2}+k_{z3}+k_{z4}\right)z^2 \\
& +\frac{1}{2}\left(\left(k_{z1}+k_{z3}\right)a_1^2+\left(k_{z2}+k_{z4}\right)a_2^2\right)\beta^2 \\
& +\frac{1}{2}\left(k_{z1}+k_{z2}+k_{z3}+k_{z4}\right)b^2\alpha^2 \\
& +\left(\left(k_{z1}+k_{z3}\right)a_1-\left(k_{z2}+k_{z4}\right)a_2\right)\beta z \\
& +\left(-k_{z1}-k_{z2}-k_{z3}-k_{z4}\right)b\alpha z \\
& +\left(-\left(k_{z1}+k_{z3}\right)a_1+\left(k_{z2}+k_{z4}\right)a_2\right)b\alpha\beta
\end{aligned}
\tag{8}
$$

The energy method is based on conservation of energy.

$$\frac{d}{dt} E = 0 \tag{9}$$

Apply the method.

$$
\begin{aligned}
&+ m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) + J_x \dot{\alpha}\ddot{\alpha} + J_y \dot{\beta}\ddot{\beta} + J_z \dot{\theta}\ddot{\theta} \\
\\
&+ (k_{x1} + k_{x2} + k_{x3} + k_{x4}) x\dot{x} \\
&+ \left((k_{x1} + k_{x2}) c_1^2 + (k_{x3} + k_{x4}) c_2^{\ 2}\right) \beta\dot{\beta} \\
&+ (k_{x1} + k_{x2} + k_{x3} + k_{x4}) b^2 \theta\dot{\theta} \\
&+ (k_{x1} + k_{x2} + k_{x3} + k_{x4})(b)(\theta\dot{x} + \dot{\theta}x) \\
&+ (-(k_{x1} + k_{x2}) c_1 + (k_{x3} + k_{x4}) c_2)(\dot{\beta}x + \beta\dot{x}) \\
&+ (-(k_{x1} + k_{x2}) c_1 + (k_{x3} + k_{x4}) c_2)(b)(\theta\dot{\beta} + \dot{\theta}\beta) \\
\\
&+ (k_{y1} + k_{y2} + k_{y3} + k_{y4}) y\dot{y} \\
&+ \left((k_{y1} + k_{y2}) c_1^2 + (k_{y3} + k_{y4}) c_2^{\ 2}\right) \alpha\dot{\alpha} \\
&+ \left((k_{y1} + k_{y3}) a_1^2 + (k_{y2} + k_{y4}) a_2^2\right) \theta\dot{\theta} \\
&+ ((k_{y1} + k_{y2}) c_1 - (k_{y3} + k_{y4}) c_2)(\dot{\alpha}y + \alpha\dot{y}) \\
&+ ((-k_{y1} - k_{y3}) a_1 + (k_{y2} + k_{y4}) a_2)(\dot{\theta}y + \theta\dot{y}) \\
&+ (-k_{y1} a_1 c_1 + k_{y2} a_2 c_1 + k_{y3} a_1 c_2 - k_{y4} a_2 c_2)(\dot{\theta}\alpha + \theta\dot{\alpha}) \\
\\
&+ (k_{z1} + k_{z2} + k_{z3} + k_{z4}) z\dot{z} \\
&+ \left((k_{z1} + k_{z3}) a_1^2 + (k_{z2} + k_{z4}) a_2^2\right) \beta\dot{\beta} \\
&+ (k_{z1} + k_{z2} + k_{z3} + k_{z4}) b^2 \alpha\dot{\alpha} \\
&+ ((k_{z1} + k_{z3}) a_1 - (k_{z2} + k_{z4}) a_2)(\dot{\beta}z + \beta\dot{z}) \\
&+ (-k_{z1} - k_{z2} - k_{z3} - k_{z4})(b)(\dot{\alpha}z + \alpha\dot{z}) \\
&+ (-(k_{z1} + k_{z3}) a_1 + (k_{z2} + k_{z4}) a_2)(b)(\dot{\alpha}\beta + \alpha\dot{\beta}) = 0
\end{aligned}
\tag{10}
$$

Equation (10) can be separated into six individual equations.

$$
\begin{aligned}
m\dot{x}\ddot{x} &+ \left(k_{x1}+k_{x2}+k_{x3}+k_{x4}\right)x\dot{x} + \left(k_{x1}+k_{x2}+k_{x3}+k_{x4}\right)b\theta\dot{x} \\
&+ \left(-\left(k_{x1}+k_{x2}\right)c_1 + \left(k_{x3}+k_{x4}\right)c_2\right)\beta\dot{x} = 0
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
m\dot{y}\ddot{y} &+ \left(k_{y1}+k_{y2}+k_{y3}+k_{y4}\right)y\dot{y} + \left(\left(k_{y1}+k_{y2}\right)c_1 - \left(k_{y3}+k_{y4}\right)c_2\right)\alpha\dot{y} \\
&+ \left(\left(-k_{y1}-k_{y3}\right)a_1 + \left(k_{y2}+k_{y4}\right)a_2\right)\theta\dot{y} = 0
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
m\dot{z}\ddot{z} &+ \left(k_{z1}+k_{z2}+k_{z3}+k_{z4}\right)z\dot{z} + \left(\left(k_{z1}+k_{z3}\right)a_1 - \left(k_{z2}+k_{z4}\right)a_2\right)\beta\dot{z} \\
&+ \left(-k_{z1}-k_{z2}-k_{z3}-k_{z4}\right)b\alpha\dot{z} = 0
\end{aligned}
$$

(13)

$$
\begin{aligned}
&J_x\,\dot{\alpha}\ddot{\alpha} + \left(\left(k_{y1}+k_{y2}\right)c_1^{2} + \left(k_{y3}+k_{y4}\right)c_2^{\ 2}\right)\alpha\dot{\alpha} \\
&+ \left(\left(k_{y1}+k_{y2}\right)c_1 - \left(k_{y3}+k_{y4}\right)c_2\right)\dot{\alpha}y \\
&+ \left(-k_{y1}a_1c_1 + k_{y2}a_2c_1 + k_{y3}a_1c_2 - k_{y4}a_2c_2\right)\theta\dot{\alpha} \\
&+ \left(k_{z1}+k_{z2}+k_{z3}+k_{z4}\right)b^2\alpha\dot{\alpha} \\
&+ \left(-k_{z1}-k_{z2}-k_{z3}-k_{z4}\right)b\dot{\alpha}z \\
&+ \left(\left(k_{z1}+k_{z3}\right)a_1 - \left(k_{z2}+k_{z4}\right)a_2\right)b\dot{\alpha}\beta = 0
\end{aligned}
$$

(14)

$$J_y\,\dot{\beta}\ddot{\beta}$$

$$
\begin{aligned}
&+ \left(\left(k_{x1}+k_{x2}\right)c_1^{2} + \left(k_{x3}+k_{x4}\right)c_2^{\ 2}\right)\beta\dot{\beta} \\
&+ \left(-\left(k_{x1}+k_{x2}\right)c_1 + \left(k_{x3}+k_{x4}\right)c_2\right)\dot{\beta}x \\
&+ \left(-\left(k_{x1}+k_{x2}\right)c_1 + \left(k_{x3}+k_{x4}\right)c_2\right)b\theta\dot{\beta}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\left(k_{z1}+k_{z3}\right)a_1^{2} + \left(k_{z2}+k_{z4}\right)a_2^{2}\right)\beta\dot{\beta} \\
&+ \left(\left(k_{z1}+k_{z3}\right)a_1 - \left(k_{z2}+k_{z4}\right)a_2\right)\dot{\beta}z \\
&+ \left(-\left(k_{z1}+k_{z3}\right)a_1 + \left(k_{z2}+k_{z4}\right)a_2\right)b\alpha\dot{\beta} = 0
\end{aligned}
$$

(15)

$$
\begin{aligned}
& J_z \dot{\theta}\ddot{\theta} \\
& + (k_{x1} + k_{x2} + k_{x3} + k_{x4}) b^2 \theta\dot{\theta} \\
& + (k_{x1} + k_{x2} + k_{x3} + k_{x4}) b\dot{\theta} x \\
& + (-(k_{x1} + k_{x2}) c_1 + (k_{x3} + k_{x4}) c_2) b\dot{\theta}\beta \\
& + \left((k_{y1} + k_{y3}) a_1^2 + (k_{y2} + k_{y4}) a_2^2 \right) \theta\dot{\theta} \\
& + ((- k_{y1} - k_{y3}) a_1 + (k_{y2} + k_{y4}) a_2) \dot{\theta} y \\
& + (- k_{y1} a_1 c_1 + k_{y2} a_2 c_1 + k_{y3} a_1 c_2 - k_{y4} a_2 c_2) \dot{\theta}\alpha = 0
\end{aligned}
$$

(16)

Typically,

$$k_{x1} = k_{x2} = k_{x3} = k_{x4} = k_x \tag{17}$$

$$k_{y1} = k_{y2} = k_{y3} = k_{y4} = k_y \tag{18}$$

$$k_{z1} = k_{z2} = k_{z3} = k_{z4} = k_z \tag{19}$$

Thus

$$m\dot{x}\ddot{x}+4k_x\ x\dot{x}+4k_x\ b\theta\dot{x}+2k_x(-c_1+c_2)\beta\dot{x}=0 \tag{20}$$

$$m\dot{y}\ddot{y}+4k_y y\dot{y}+2k_y(c_1-c_2)\alpha\dot{y}+2k_y(-a_1+a_2)\theta\dot{y}=0 \tag{21}$$

$$m\dot{z}\ddot{z}+4k_z z\dot{z}-4k_z b\alpha\dot{z}+2k_z(a_1-a_2)\beta\dot{z}=0 \tag{22}$$

$$\begin{aligned}&J_x\,\dot{\alpha}\ddot{\alpha}+2k_y\left(c_1^2+c_2^{\ 2}\right)\alpha\dot{\alpha}+2k_y(c_1-c_2)\dot{\alpha}y\quad+k_y(-a_1+a_2)(c_1-c_2)\theta\dot{\alpha}\\&+4k_z b^2\alpha\dot{\alpha}-4k_z b\dot{\alpha}z+2k_z(a_1-a_2)b\dot{\alpha}\beta=0\end{aligned} \tag{23}$$

$$\begin{aligned}&J_y\,\dot{\beta}\ddot{\beta}+2k_x\left(c_1^2+c_2^2\right)\beta\dot{\beta}+2k_x(-c_1+c_2)b\theta\dot{\beta}+2k_x(-c_1+c_2)\dot{\beta}x\\&+2k_z\left(a_1^2+a_2^2\right)\beta\dot{\beta}+2k_z(a_1-a_2)\dot{\beta}z+2k_z(-a_1+a_2)b\alpha\dot{\beta}=0\end{aligned} \tag{24}$$

$$\begin{aligned}&J_z\,\dot{\theta}\ddot{\theta}+4k_x b^2\theta\dot{\theta}+4k_x b\dot{\theta}x+2k_x(-c_1+c_2)b\dot{\theta}\beta\\&+2k_y\left(a_1^2+a_2^2\right)\theta\dot{\theta}+2k_y(-a_1+a_2)\dot{\theta}y+k_y(-a_1+a_2)(c_1-c_2)\dot{\theta}\alpha=0\end{aligned} \tag{25}$$

The equations can be simplified as

$$m\ddot{x} + 4k_x x + 4k_x b\theta + 2k_x(-c_1 + c_2)\beta = 0 \tag{26}$$

$$m\ddot{y} + 4k_y y + 2k_y(c_1 - c_2)\alpha + 2k_y(-a_1 + a_2)\theta = 0 \tag{27}$$

$$m\ddot{z} + 4k_z z - 4k_z b\alpha + 2k_z(a_1 - a_2)\beta = 0 \tag{28}$$

$$\begin{aligned} &J_x \ddot{\alpha} + 2k_y\left(c_1^2 + c_2^2\right)\alpha + 2k_y(c_1 - c_2)y + k_y(-a_1 + a_2)(c_1 - c_2)\theta \\ &+ 4k_z b^2\alpha - 4k_z bz + 2k_z(a_1 - a_2)b\beta = 0 \end{aligned} \tag{29}$$

$$\begin{aligned} &J_y \ddot{\beta} + 2k_x\left(c_1^2 + c_2^2\right)\beta + 2k_x(-c_1 + c_2)b\theta + 2k_x(-c_1 + c_2)x \\ &+ 2k_z\left(a_1^2 + a_2^2\right)\beta + 2k_z(a_1 - a_2)z + 2k_z(-a_1 + a_2)b\alpha = 0 \end{aligned} \tag{30}$$

$$\begin{aligned} &J_z \ddot{\theta} + 4k_x b^2\theta + 4k_x bx + 2k_x(-c_1 + c_2)b\beta \\ &+ 2k_y\left(a_1^2 + a_2^2\right)\theta + 2k_y(-a_1 + a_2)y + k_y(-a_1 + a_2)(c_1 - c_2)\alpha = 0 \end{aligned} \tag{31}$$

The equations can be arranged in matrix format.

$$\underline{M}\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \ddot{\alpha} \\ \ddot{\beta} \\ \ddot{\theta} \end{bmatrix} + \underline{K}\begin{bmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{32}$$

The mass and stiffness matrices are shown in upper triangular form due to symmetry.

$$\underline{M} = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ & m & 0 & 0 & 0 & 0 \\ & & m & 0 & 0 & 0 \\ & & & J_x & 0 & 0 \\ & & & & J_y & 0 \\ & & & & & J_z \end{bmatrix} \tag{33}$$

$$\underline{K} = \begin{bmatrix} 4k_x & 0 & 0 & 0 & 2k_x(-c_1 + c_2) & 4k_x b \\ & 4k_y & 0 & 2k_y(c_1 - c_2) & 0 & 2k_y(-a_1 + a_2) \\ & & 4k_z & -4k_z b & 2k_z(a_1 - a_2) & 0 \\ & & & 4k_z b^2 + 2k_y\left(c_1^2 + c_2^2\right) & 2k_z(-a_1 + a_2)b & k_y(-a_1 + a_2)(c_1 - c_2) \\ & & & & 2k_x\left(c_1^2 + c_2^2\right) + 2k_z\left(a_1^2 + a_2^2\right) & 2k_x(-c_1 + c_2)b \\ & & & & & 4k_x b^2 + 2k_y\left(a_1^2 + a_2^2\right) \end{bmatrix} \tag{34}$$

Acceleration Base Excitation in the X-axis

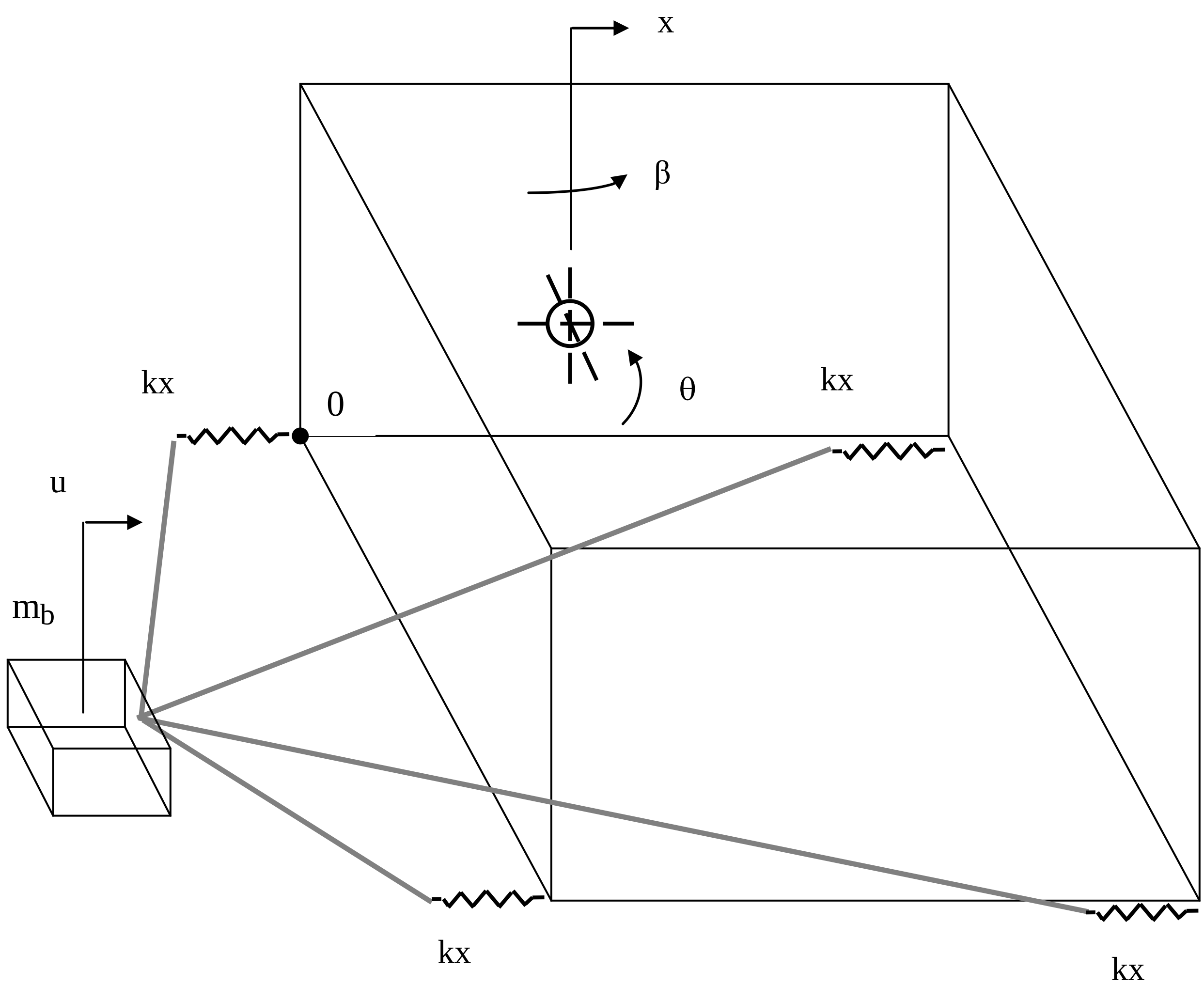


Figure 3. Isolated Avionics Component Model, Base Excitation in X-axis

The base mass is connected to the springs via rigid links.

The total kinetic energy is

$$T = \frac{1}{2}m_b\,\dot{u}^2 + \frac{1}{2}m\,\dot{x}^2 + \frac{1}{2}m\,\dot{y}^2 + \frac{1}{2}m\,\dot{z}^2 + \frac{1}{2}J_y\,\dot{\beta}^2 + \frac{1}{2}J_z\,\dot{\theta}^2 \tag{35}$$

Let

$$r_1 = x - u \tag{36}$$

$$T = \frac{1}{2}m_b\,\dot{u}^2 + \frac{1}{2}m\left(u + r_1\right)^2 + \frac{1}{2}m\,\dot{y}^2 + \frac{1}{2}m\,\dot{z}^2 + \frac{1}{2}J_y\,\dot{\beta}^2 + \frac{1}{2}J_z\,\dot{\theta}^2 \tag{37}$$

$$T = \frac{1}{2}m_b\,\dot{u}^2 + \frac{1}{2}m\,\dot{u}^2 + m\,\dot{u}\,\dot{r}_1 + \frac{1}{2}m\,\dot{r}_1^2 + \frac{1}{2}m\,\dot{y}^2 + \frac{1}{2}m\,\dot{z}^2 + \frac{1}{2}J_y\,\dot{\beta}^2 + \frac{1}{2}J_z\,\dot{\theta}^2 \tag{38}$$

$$T = \frac{1}{2}\left(m_b + m\right)\dot{u}^2 + m\,\dot{u}\,\dot{r}_1 + \frac{1}{2}m\,\dot{r}_1^2 + \frac{1}{2}m\,\dot{y}^2 + \frac{1}{2}m\,\dot{z}^2 + \frac{1}{2}J_y\,\dot{\beta}^2 + \frac{1}{2}J_z\,\dot{\theta}^2 \tag{39}$$

The total potential energy is

$$
\begin{aligned}
V = \\
&+ \frac{1}{2} k_{x1} (r_1 - c_1 \beta + b\theta)^2 \\
&+ \frac{1}{2} k_{x2} (r_1 - c_1 \beta + b\theta)^2 \\
&+ \frac{1}{2} k_{x3} (r_1 + c_2 \beta + b\theta)^2 \\
&+ \frac{1}{2} k_{x4} (r_1 + c_2 \beta + b\theta)^2 \\
\\
&+ \frac{1}{2} k_{y1} (y + c_1 \alpha - a_1 \theta)^2 \\
&+ \frac{1}{2} k_{y2} (y + c_1 \alpha + a_2 \theta)^2 \\
&+ \frac{1}{2} k_{y3} (y - c_2 \alpha - a_1 \theta)^2 \\
&+ \frac{1}{2} k_{y4} (y - c_2 \alpha + a_2 \theta)^2 \\
\\
&+ \frac{1}{2} k_{z1} (z + a_1 \beta - b\alpha)^2 \\
&+ \frac{1}{2} k_{z2} (z - a_2 \beta - b\alpha)^2 \\
&+ \frac{1}{2} k_{z3} (z + a_1 \beta - b\alpha)^2 \\
&+ \frac{1}{2} k_{z4} (z - a_2 \beta - b\alpha)^2
\end{aligned} \tag{40}
$$

The energy is

$$E =$$

$$+\frac{1}{2}(m_b + m)\dot{u}^2 + m\dot{u}\dot{r}_1 + \frac{1}{2}m\dot{r}_1^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 + \frac{1}{2}J_y\ \dot{\beta}^2 + \frac{1}{2}J_z\ \dot{\theta}^2$$

$$+\frac{1}{2}k_{x1}(r_1 - c_1\beta + b\theta)^2$$
$$+\frac{1}{2}k_{x2}(r_1 - c_1\beta + b\theta)^2$$
$$+\frac{1}{2}k_{x3}(r_1 + c_2\beta + b\theta)^2$$
$$+\frac{1}{2}k_{x4}(r_1 + c_2\beta + b\theta)^2$$

$$+\frac{1}{2}k_{y1}(y + c_1\alpha - a_1\theta)^2$$
$$+\frac{1}{2}k_{y2}(y + c_1\alpha + a_2\theta)^2$$
$$+\frac{1}{2}k_{y3}(y - c_2\alpha - a_1\theta)^2$$
$$+\frac{1}{2}k_{y4}(y - c_2\alpha + a_2\theta)^2$$

$$+\frac{1}{2}k_{z1}(z + a_1\beta - b\alpha)^2$$
$$+\frac{1}{2}k_{z2}(z - a_2\beta - b\alpha)^2$$
$$+\frac{1}{2}k_{z3}(z + a_1\beta - b\alpha)^2$$
$$+\frac{1}{2}k_{z4}(z - a_2\beta - b\alpha)^2$$

(41)

$$
\begin{aligned}
E \;=\; & +\frac{1}{2}(m_b + m)\dot{u}^2 + m\dot{u}\dot{r}_1 + \frac{1}{2}m\dot{r}_1^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 + \frac{1}{2}J_y\,\dot{\beta}^2 + \frac{1}{2}J_z\,\dot{\theta}^2 \\
& +\frac{1}{2}(k_{x1}+k_{x2}+k_{x3}+k_{x4})r_1^2 \\
& +(-(k_{x1}+k_{x2})c_1+(k_{x3}+k_{x4})c_2)\beta r_1 \\
& +(k_{x1}+k_{x2}+k_{x3}+k_{x4})b\theta r_1 \\
& +\frac{1}{2}\left((k_{x1}+k_{x2})c_1^2+(k_{x3}+k_{x4})c_2^2\right)\beta^2 \\
& +(-(k_{x1}+k_{x2})c_1+(k_{x3}+k_{x4})c_2)b\theta\beta \\
& +\frac{1}{2}(k_{x1}+k_{x2}+k_{x3}+k_{x4})b^2\theta^2 \\
& +\frac{1}{2}(k_{y1}+k_{y2}+k_{y3}+k_{y4})y^2 \\
& +\frac{1}{2}\left((k_{y1}+k_{y2})c_1^2+(k_{y3}+k_{y4})c_2^{\;2}\right)\alpha^2 \\
& +\frac{1}{2}\left((k_{y1}+k_{y3})a_1^2+(k_{y2}+k_{y4})a_2^2\right)\theta^2 \\
& +((k_{y1}+k_{y2})c_1-(k_{y3}+k_{y4})c_2)\alpha y \\
& +((-k_{y1}-k_{y3})a_1+(k_{y2}+k_{y4})a_2)\theta y \\
& +(-k_{y1}a_1c_1+k_{y2}a_2c_1+k_{y3}a_1c_2-k_{y4}a_2c_2)\theta\alpha \\
& +\frac{1}{2}(k_{z1}+k_{z2}+k_{z3}+k_{z4})z^2 \\
& +\frac{1}{2}\left((k_{z1}+k_{z3})a_1^2+(k_{z2}+k_{z4})a_2^2\right)\beta^2 \\
& +\frac{1}{2}(k_{z1}+k_{z2}+k_{z3}+k_{z4})b^2\alpha^2 \\
& +((k_{z1}+k_{z3})a_1-(k_{z2}+k_{z4})a_2)\beta z \\
& +(-k_{z1}-k_{z2}-k_{z3}-k_{z4})b\alpha z \\
& +(-(k_{z1}+k_{z3})a_1+(k_{z2}+k_{z4})a_2)b\alpha\beta
\end{aligned}
\tag{42}
$$

The energy method is based on conservation of energy.

$$\frac{d}{dt}E = 0 \tag{43}$$

Apply the method.

$$
\begin{aligned}
&+(m_b + m)\dot{u}\ddot{u} + m\dot{u}\ddot{r}_1 + m\ddot{u}\dot{r}_1 + m\dot{r}_1\ddot{r}_1 + m\dot{y}\ddot{y} + m\dot{z}\ddot{z} \\
&+ J_x\dot{\alpha}\ddot{\alpha} + J_y\dot{\beta}\ddot{\beta} + J_z\dot{\theta}\ddot{\theta} \\
&+ (k_{x1} + k_{x2} + k_{x3} + k_{x4})r_1\dot{r}_1 \\
&+ \left((k_{x1} + k_{x2})c_1^2 + (k_{x3} + k_{x4})c_2^2\right)\beta\dot{\beta} \\
&+ (k_{x1} + k_{x2} + k_{x3} + k_{x4})b^2\theta\dot{\theta} \\
&+ (k_{x1} + k_{x2} + k_{x3} + k_{x4})(b)\left(\theta\dot{r}_1 + \dot{\theta}r_1\right) \\
&+ \left(-(k_{x1} + k_{x2})c_1 + (k_{x3} + k_{x4})c_2\right)\left(\dot{\beta}r_1 + \beta\dot{r}_1\right) \\
&+ \left(-(k_{x1} + k_{x2})c_1 + (k_{x3} + k_{x4})c_2\right)(b)\left(\theta\dot{\beta} + \dot{\theta}\beta\right) \\
&+ (k_{y1} + k_{y2} + k_{y3} + k_{y4})y\dot{y} \\
&+ \left((k_{y1} + k_{y2})c_1^2 + (k_{y3} + k_{y4})c_2^2\right)\alpha\dot{\alpha} \\
&+ \left((k_{y1} + k_{y3})a_1^2 + (k_{y2} + k_{y4})a_2^2\right)\theta\dot{\theta} \\
&+ \left((k_{y1} + k_{y2})c_1 - (k_{y3} + k_{y4})c_2\right)(\dot{\alpha}y + \alpha\dot{y}) \\
&+ \left((-k_{y1} - k_{y3})a_1 + (k_{y2} + k_{y4})a_2\right)(\dot{\theta}y + \theta\dot{y}) \\
&+ \left(-k_{y1}a_1c_1 + k_{y2}a_2c_1 + k_{y3}a_1c_2 - k_{y4}a_2c_2\right)(\dot{\theta}\alpha + \theta\dot{\alpha}) \\
&+ (k_{z1} + k_{z2} + k_{z3} + k_{z4})z\dot{z} \\
&+ \left((k_{z1} + k_{z3})a_1^2 + (k_{z2} + k_{z4})a_2^2\right)\beta\dot{\beta} \\
&+ (k_{z1} + k_{z2} + k_{z3} + k_{z4})b^2\alpha\dot{\alpha} \\
&+ \left((k_{z1} + k_{z3})a_1 - (k_{z2} + k_{z4})a_2\right)(\dot{\beta}z + \beta\dot{z}) \\
&+ (-k_{z1} - k_{z2} - k_{z3} - k_{z4})(b)(\dot{\alpha}z + \alpha\dot{z}) \\
&+ \left(-(k_{z1} + k_{z3})a_1 + (k_{z2} + k_{z4})a_2\right)(b)\left(\dot{\alpha}\beta + \alpha\dot{\beta}\right) = 0
\end{aligned}
\tag{44}
$$

Equation (10) can be separated into seven individual equations.

$$(m_b + m)\dot{u}\ddot{u} + m\dot{u}\ddot{r}_1 = 0 \tag{45}$$

$$\begin{aligned} &m\ddot{u}\dot{r}_1 + m\dot{r}_1\ddot{r}_1 + (k_{x1} + k_{x2} + k_{x3} + k_{x4})r_1\dot{r}_1 + (k_{x1} + k_{x2} + k_{x3} + k_{x4})b\theta\dot{r}_1 \\ &\qquad + (-(k_{x1} + k_{x2})c_1 + (k_{x3} + k_{x4})c_2)\beta\dot{r}_1 = 0 \end{aligned} \tag{46}$$

$$\begin{aligned} &m\dot{y}\ddot{y} + (k_{y1} + k_{y2} + k_{y3} + k_{y4})y\dot{y} + ((k_{y1} + k_{y2})c_1 - (k_{y3} + k_{y4})c_2)\alpha\dot{y} \\ &\qquad + ((-k_{y1} - k_{y3})a_1 + (k_{y2} + k_{y4})a_2)\theta\dot{y} = 0 \end{aligned} \tag{47}$$

$$\begin{aligned} &m\dot{z}\ddot{z} + (k_{z1} + k_{z2} + k_{z3} + k_{z4})z\dot{z} + ((k_{z1} + k_{z3})a_1 - (k_{z2} + k_{z4})a_2)\beta\dot{z} \\ &\qquad + (-k_{z1} - k_{z2} - k_{z3} - k_{z4})b\alpha\dot{z} = 0 \end{aligned} \tag{48}$$

$$\begin{aligned} &J_x\dot{\alpha}\ddot{\alpha} + ((k_{y1} + k_{y2})c_1^2 + (k_{y3} + k_{y4})c_2^2)\alpha\dot{\alpha} \\ &+ ((k_{y1} + k_{y2})c_1 - (k_{y3} + k_{y4})c_2)\dot{\alpha}y \\ &+ (-k_{y1}a_1c_1 + k_{y2}a_2c_1 + k_{y3}a_1c_2 - k_{y4}a_2c_2)\theta\dot{\alpha} \\ &+ (k_{z1} + k_{z2} + k_{z3} + k_{z4})b^2\alpha\dot{\alpha} \\ &+ (-k_{z1} - k_{z2} - k_{z3} - k_{z4})b\dot{\alpha}z \\ &+ ((k_{z1} + k_{z3})a_1 - (k_{z2} + k_{z4})a_2)b\dot{\alpha}\beta = 0 \end{aligned} \tag{49}$$

$$J_y\,\dot{\beta}\ddot{\beta}$$

$$+\left(\left(k_{x1}+k_{x2}\right)c_1{}^2+\left(k_{x3}+k_{x4}\right)c_2{}^2\right)\beta\dot{\beta}$$
$$+\left(-\left(k_{x1}+k_{x2}\right)c_1+\left(k_{x3}+k_{x4}\right)c_2\right)\dot{\beta}r_1$$
$$+\left(-\left(k_{x1}+k_{x2}\right)c_1+\left(k_{x3}+k_{x4}\right)c_2\right)b\theta\dot{\beta}$$

$$+\left(\left(k_{z1}+k_{z3}\right)a_1{}^2+\left(k_{z2}+k_{z4}\right)a_2{}^2\right)\beta\dot{\beta}$$
$$+\left(\left(k_{z1}+k_{z3}\right)a_1-\left(k_{z2}+k_{z4}\right)a_2\right)\dot{\beta}z$$
$$+\left(-\left(k_{z1}+k_{z3}\right)a_1+\left(k_{z2}+k_{z4}\right)a_2\right)b\alpha\dot{\beta}=0$$

(50)

$$J_z\,\dot{\theta}\ddot{\theta}$$
$$+\left(k_{x1}+k_{x2}+k_{x3}+k_{x4}\right)b^2\theta\dot{\theta}$$
$$+\left(k_{x1}+k_{x2}+k_{x3}+k_{x4}\right)b\dot{\theta}r_1$$
$$+\left(-\left(k_{x1}+k_{x2}\right)c_1+\left(k_{x3}+k_{x4}\right)c_2\right)b\dot{\theta}\beta$$

$$+\left(\left(k_{y1}+k_{y3}\right)a_1{}^2+\left(k_{y2}+k_{y4}\right)a_2{}^2\right)\theta\dot{\theta}$$
$$+\left(\left(-k_{y1}-k_{y3}\right)a_1+\left(k_{y2}+k_{y4}\right)a_2\right)\dot{\theta}y$$
$$+\left(-k_{y1}a_1c_1+k_{y2}a_2c_1+k_{y3}a_1c_2-k_{y4}a_2c_2\right)\dot{\theta}\alpha=0$$

(51)

Typically,

$$k_{x1}=k_{x2}=k_{x3}=k_{x4}=k_x \tag{52}$$

$$k_{y1}=k_{y2}=k_{y3}=k_{y4}=k_y \tag{53}$$

$$k_{z1}=k_{z2}=k_{z3}=k_{z4}=k_z \tag{54}$$

Thus

$$\left(m_b + m\right)\dot{u}\ddot{u} + m\dot{u}\ddot{r}_1 = 0 \tag{55}$$

$$m\ddot{u}\dot{r}_1 + m\dot{r}_1\ddot{r}_1 + 4k_x r_1\dot{r}_1 + 4k_x b\theta\dot{r}_1 + \left(-2k_x + 2k_x c_2\right)\beta\dot{r}_1 = 0 \tag{56}$$

$$m\dot{y}\ddot{y} + 4k_y y\dot{y} + 2k_y\left(c_1 - c_2\right)\alpha\dot{y} + 2k_y\left(-a_1 + a_2\right)\theta\dot{y} = 0 \tag{57}$$

$$m\dot{z}\ddot{z} + 4k_z z\dot{z} - 4k_z b\alpha\dot{z} + 2k_z\left(a_1 - a_2\right)\beta\dot{z} = 0 \tag{58}$$

$$\begin{aligned}&J_x\dot{\alpha}\ddot{\alpha} + 2k_y\left(c_1^2 + c_2^{\,2}\right)\alpha\dot{\alpha} + 2k_y\left(c_1 - c_2\right)\dot{\alpha}y \quad + k_y\left(-a_1 + a_2\right)\left(c_1 - c_2\right)\theta\dot{\alpha}\\&+ 4k_z b^2\alpha\dot{\alpha} - 4k_z b\dot{\alpha}z + 2k_z\left(a_1 - a_2\right)b\dot{\alpha}\beta = 0\end{aligned} \tag{59}$$

$$\begin{aligned}&J_y\dot{\beta}\ddot{\beta} + 2k_x\left(c_1^2 + c_2^2\right)\beta\dot{\beta} + 2k_x\left(-c_1 + c_2\right)b\theta\dot{\beta} + 2k_x\left(-c_1 + c_2\right)\dot{\beta}r_1\\&+ 2k_z\left(a_1^2 + a_2^2\right)\beta\dot{\beta} + 2k_z\left(a_1 - a_2\right)\dot{\beta}z + 2k_z\left(-a_1 + a_2\right)b\alpha\dot{\beta} = 0\end{aligned} \tag{60}$$

$$\begin{aligned}&J_z\dot{\theta}\ddot{\theta} + 4k_x b^2\theta\dot{\theta} + 4k_x b\dot{\theta}r_1 + 2k_x\left(-c_1 + c_2\right)b\dot{\theta}\beta\\&+ 2k_y\left(a_1^2 + a_2^2\right)\theta\dot{\theta} + 2k_y\left(-a_1 + a_2\right)\dot{\theta}y + k_y\left(-a_1 + a_2\right)\left(c_1 - c_2\right)\dot{\theta}\alpha = 0\end{aligned} \tag{61}$$

The equations can be simplified as

$$(m_b + m)\ddot{u} + m\,\ddot{r}_1 = 0 \tag{62}$$

$$m\,\ddot{r}_1 + 4k_x\,\dot{r}_1 + 4k_x\,b\theta + 2k_x(-c_1 + c_2)\beta = -m\ddot{u} \tag{63}$$

$$m\ddot{y} + 4k_y y + 2k_y(c_1 - c_2)\alpha + 2k_y(-a_1 + a_2)\theta = 0 \tag{64}$$

$$m\ddot{z} + 4k_z z - 4k_z b\alpha + 2k_z(a_1 - a_2)\beta = 0 \tag{65}$$

$$J_x\,\ddot{\alpha} + 2k_y\left(c_1^2 + c_2^2\right)\alpha + 2k_y(c_1 - c_2)y \quad + k_y(-a_1 + a_2)(c_1 - c_2)\theta$$
$$+ 4k_z b^2\alpha - 4k_z bz + 2k_z(a_1 - a_2)b\beta = 0 \tag{66}$$

$$J_y\,\ddot{\beta} + 2k_x\left(c_1^2 + c_2^2\right)\beta + 2k_x(-c_1 + c_2)b\theta + 2k_x(-c_1 + c_2)r_1$$
$$+ 2k_z\left(a_1^2 + a_2^2\right)\beta + 2k_z(a_1 - a_2)z + 2k_z(-a_1 + a_2)b\alpha = 0 \tag{67}$$

$$J_z\,\ddot{\theta} + 4k_x b^2\theta + 4k_x b\,r_1 + 2k_x(-c_1 + c_2)b\beta$$
$$+ 2k_y\left(a_1^2 + a_2^2\right)\theta + 2k_y(-a_1 + a_2)y + k_y(-a_1 + a_2)(c_1 - c_2)\alpha = 0 \tag{68}$$

The equations can be arranged in matrix format.

$$\underline{M}\begin{bmatrix}\ddot{r}_1\\ \ddot{y}\\ \ddot{z}\\ \ddot{\alpha}\\ \ddot{\beta}\\ \ddot{\theta}\end{bmatrix} + \underline{K}\begin{bmatrix}r_1\\ y\\ z\\ \alpha\\ \beta\\ \theta\end{bmatrix} = \begin{bmatrix}-m\ddot{u}\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \tag{69}$$

The mass and stiffness matrices are the same as those given in equations (33) and (34) respectively.

Acceleration Base Excitation in the Y-axis

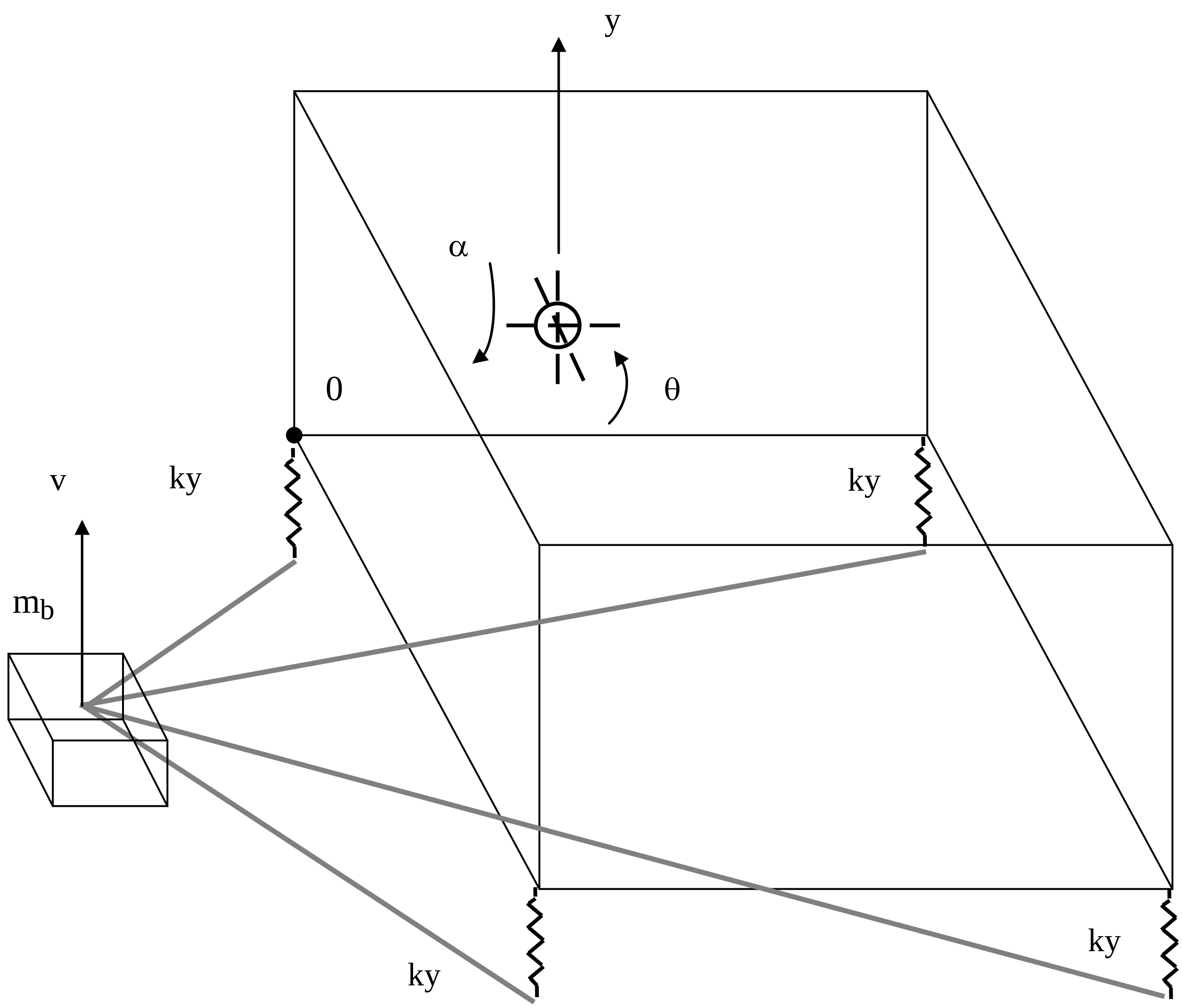


Figure 4. Isolated Avionics Component Model, Base Excitation in Y-axis

The base mass is connected to the springs via rigid links.

Let

$$r_2 = y - v \tag{70}$$

The equations can be arranged in matrix format.

$$\underline{M}\begin{bmatrix} \ddot{x} \\ \ddot{r}_2 \\ \ddot{z} \\ \ddot{\alpha} \\ \ddot{\beta} \\ \ddot{\theta} \end{bmatrix} + \underline{K}\begin{bmatrix} x \\ r_2 \\ z \\ \alpha \\ \beta \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ -m\,\ddot{v} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{71}$$

The mass and stiffness matrices are the same as those given in equations (33) and (34)

Acceleration Base Excitation in the Z-axis

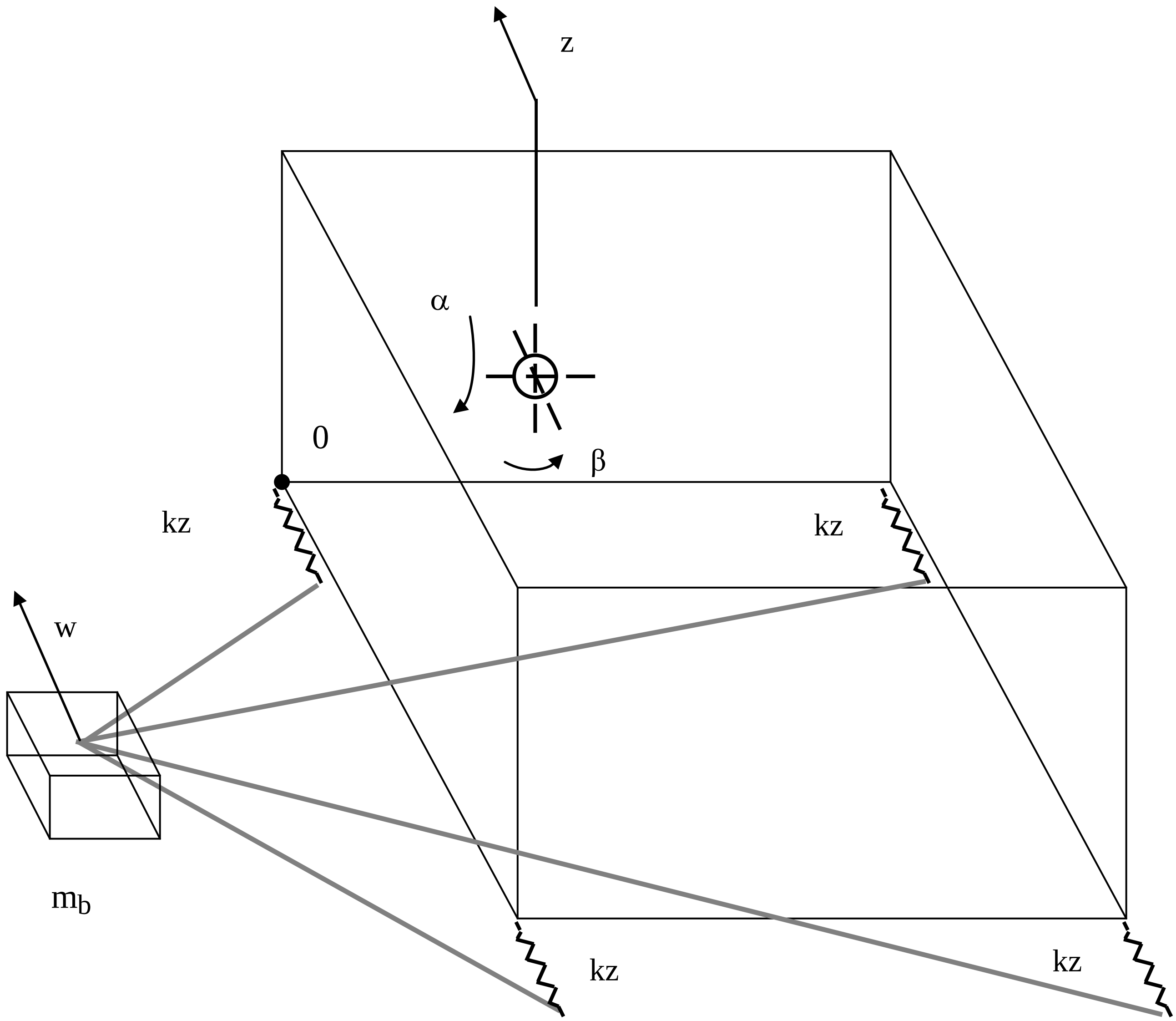


Figure 5. Isolated Avionics Component Model, Base Excitation in Z-axis

The base mass is connected to the springs via rigid links.

Let

$$r_3 = z - w \tag{72}$$

The equations can be arranged in matrix format.

$$\underline{M}\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{r}_3 \\ \ddot{\alpha} \\ \ddot{\beta} \\ \ddot{\theta} \end{bmatrix} + \underline{K}\begin{bmatrix} x \\ y \\ r_3 \\ \alpha \\ \beta \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -m\ddot{w} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{73}$$

The acceleration transmissibility functions of equations (69), (71) and (73) can then be determined via Reference 1.

The modal transient response to a base input time history can be calculated via References 2 and 3.

References

1. T. Irvine, Frequency Response Function Analysis of a Multi-degree-of-freedom System with Enforced Motion, Vibrationdata, 2011.

2. T. Irvine, The Generalized Coordinate Method for Discrete Systems, Subjected to Base Excitation, Revision B, Vibrationdata, 2004.

3. T. Irvine, Shock Response of Multi-degree-of-freedom Systems, Revision F, Vibrationdata, 2010.

APPENDIX A

Example

A mass is mounted to a surface with four isolators. The system has the following properties.

M	=	4.28 lbm
Jx	=	44.9 lbm in^2
Jy	=	39.9 lbm in^2
Jz	=	18.8 lbm in^2
Kx	=	80 lbf/in
Ky	=	80 lbf/in
Kz	=	80 lbf/in
a1	=	6.18 in
a2	=	-2.68 in
b	=	3.85 in
c1	=	3. in
c2	=	3. in

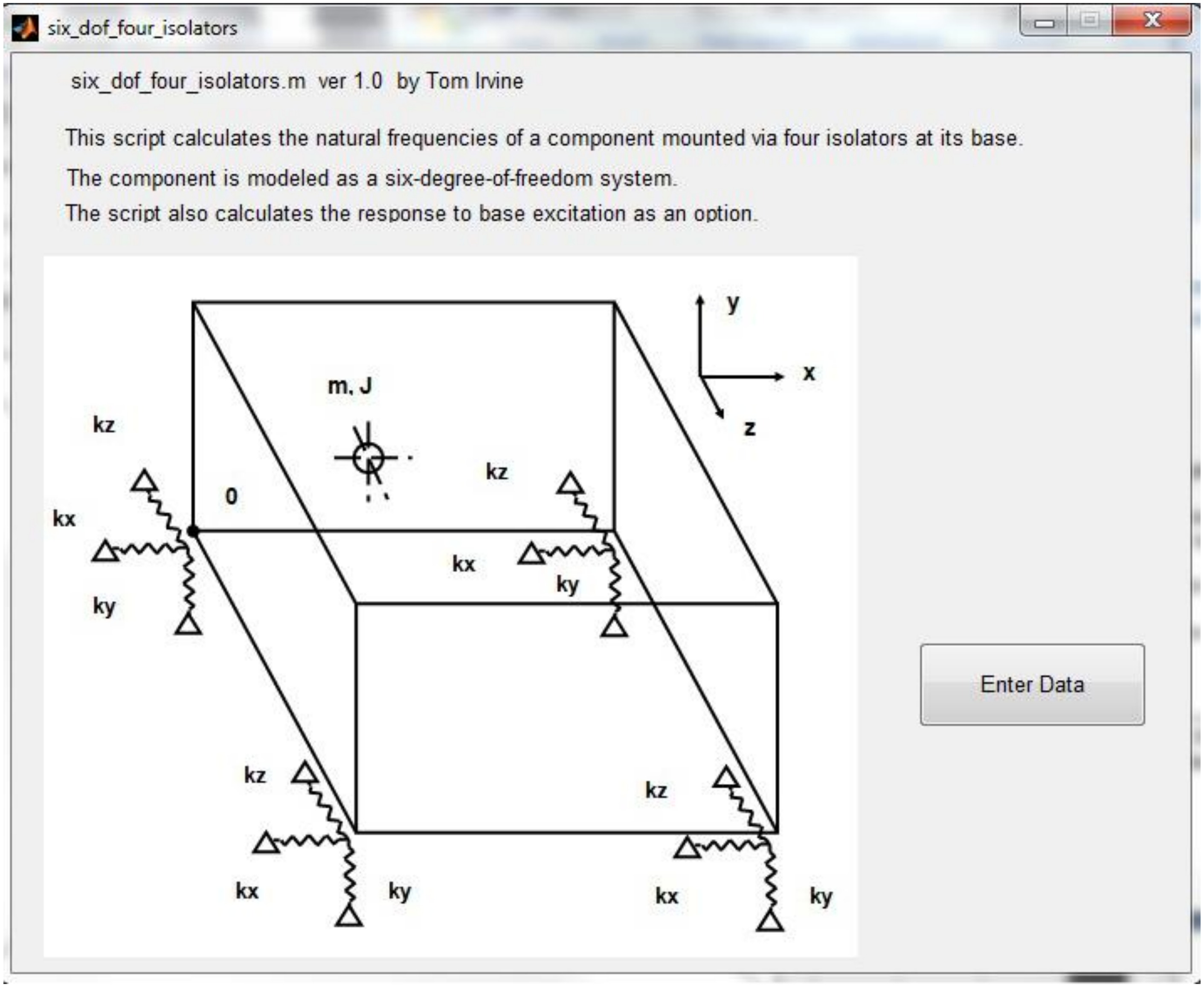
Again, a2 is negative due to overhang.

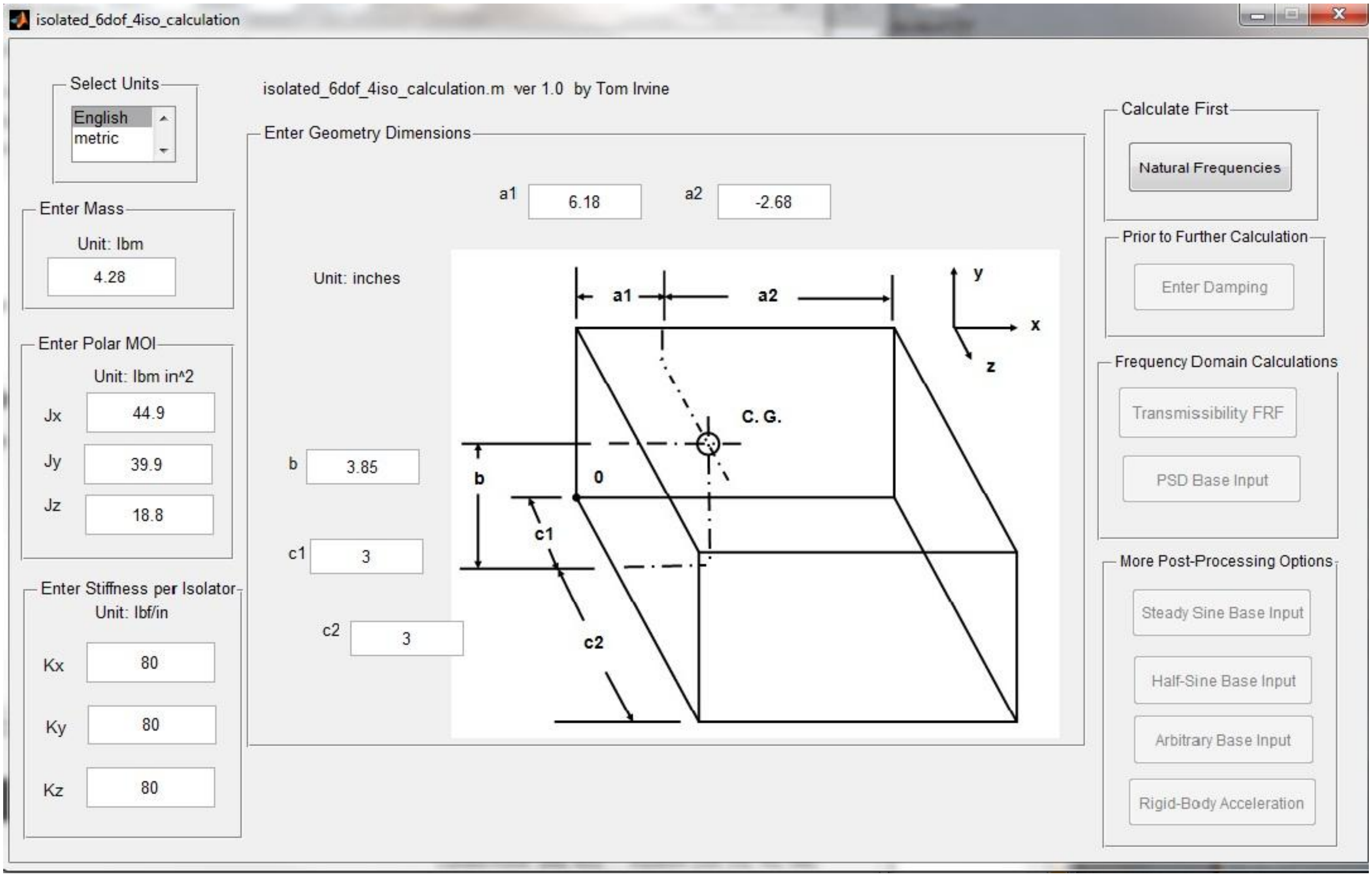
Assume that the damping is 8% for all modes.[3] Note that the isolators provide damping by converting kinetic energy to heat energy.

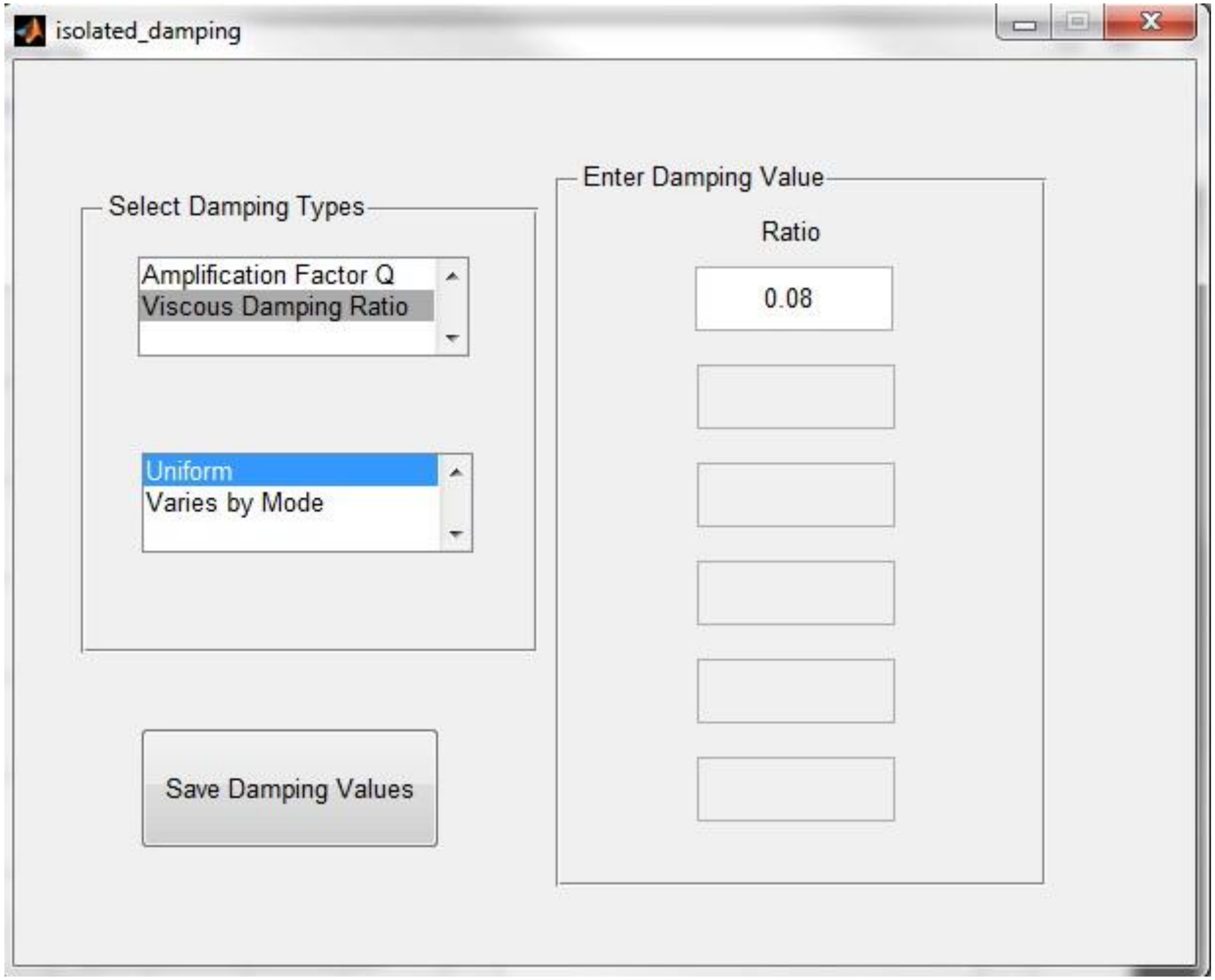
The natural frequency results are calculated using the Matlab script: six_dof_four_iso.m

The output is given on the next pages.

[3] Other damping values could be used to perform a trade study if the actual values were unknown. Damping cannot be predicted analytically. It must be measured.


six_dof_four_isolators
six_dof_four_isolators.m ver 1.0 by Tom Irvine
This script calculates the natural frequencies of a component mounted via four isolators at its base.
The component is modeled as a six-degree-of-freedom system.
The script also calculates the response to base excitation as an option.
y
x
z
m, J
0
kz
kx
ky
kz
kx
ky
kz
kx
ky
kz
kx
ky
Enter Data


isolated_6dof_4iso_calculation
Select Units
English
metric
isolated_6dof_4iso_calculation.m ver 1.0 by Tom Irvine
Enter Geometry Dimensions
Calculate First
Natural Frequencies
a1
6.18
a2
-2.68
Enter Mass
Unit: lbm
4.28
Prior to Further Calculation
Enter Damping
Unit: inches
a1
a2
y
x
z
Enter Polar MOI
Unit: lbm in^2
Jx
44.9
Jy
39.9
Jz
18.8
Frequency Domain Calculations
Transmissibility FRF
PSD Base Input
C. G.
b
3.85
b
0
c1
c1
3
c2
3
c2
Enter Stiffness per Isolator
Unit: lbf/in
Kx
80
Ky
80
Kz
80
More Post-Processing Options
Steady Sine Base Input
Half-Sine Base Input
Arbitrary Base Input
Rigid-Body Acceleration


isolated_damping
Select Damping Types
Amplification Factor Q
Viscous Damping Ratio
Uniform
Varies by Mode
Save Damping Values
Enter Damping Value
Ratio
0.08

```
Geometry Dimensions (inch)
 a1=   6.18  a2=  -2.68  b=   3.85  c1=      3  c2=      3

 Mass (lbm)
 Mass=   4.28

 Polar MOI (lbm in^2)
 Jx=   44.9  Jy=   39.9  Jz=   18.8

 Stiffness per Isolator (lbf/in)
 Kx=     80  Ky=     80  Kz=     80


 The mass matrix is

m =

    0.0111         0         0         0         0         0
         0    0.0111         0         0         0         0
         0         0    0.0111         0         0         0
         0         0         0    0.1163         0         0
         0         0         0         0    0.1034         0
         0         0         0         0         0    0.0487


 The stiffness matrix is

k =

   1.0e+04 *

    0.0320         0         0         0         0    0.1232
         0    0.0320         0         0         0   -0.1418
         0         0    0.0320   -0.1232    0.1418         0
         0         0   -0.1232    0.7623   -0.5458         0
         0         0    0.1418   -0.5458    1.0140         0
    0.1232   -0.1418         0         0         0    1.2003


 Eigenvalues

lambda =

   1.0e+05 *

    0.0213    0.0570    0.2886    0.2980    1.5699    2.7318
```

Natural Frequencies =

1.	7.338 Hz
2.	12.02 Hz
3.	27.04 Hz
4.	27.47 Hz
5.	63.06 Hz
6.	83.19 Hz

Modes Shapes (rows represent modes)

	x	y	z	alpha	beta	theta
1.	5.914	-6.805	0.000	0.000	0.000	-1.423
2.	0.000	0.000	8.687	0.954	-0.744	0.000
3.	7.168	6.230	0.000	0.000	0.000	0.000
4.	0.000	0.000	1.041	-2.258	-1.955	0.000
5.	0.000	0.000	-3.692	1.609	-2.302	0.000
6.	1.956	-2.251	0.000	0.000	0.000	4.302

Participation Factors (rows represent modes)

	x	y	z	alpha	beta	theta
1.	25.313	-29.127	0.000	0.000	0.000	-26.753
2.	0.000	0.000	37.182	42.846	-29.693	0.000
3.	30.679	26.662	0.000	0.000	0.000	0.000
4.	0.000	0.000	4.455	-101.382	-77.991	0.000
5.	0.000	0.000	-15.802	72.231	-91.854	0.000
6.	8.373	-9.635	0.000	0.000	0.000	80.877

Effective Modal Mass (rows represent modes)

	x	y	z	alpha	beta	theta
1.	1.660	2.198	0.000	0.000	0.000	1.854
2.	0.000	0.000	3.582	4.756	2.284	0.000
3.	2.438	1.842	0.000	0.000	0.000	0.000
4.	0.000	0.000	0.051	26.628	15.758	0.000
5.	0.000	0.000	0.647	13.516	21.858	0.000
6.	0.182	0.240	0.000	0.000	0.000	16.946

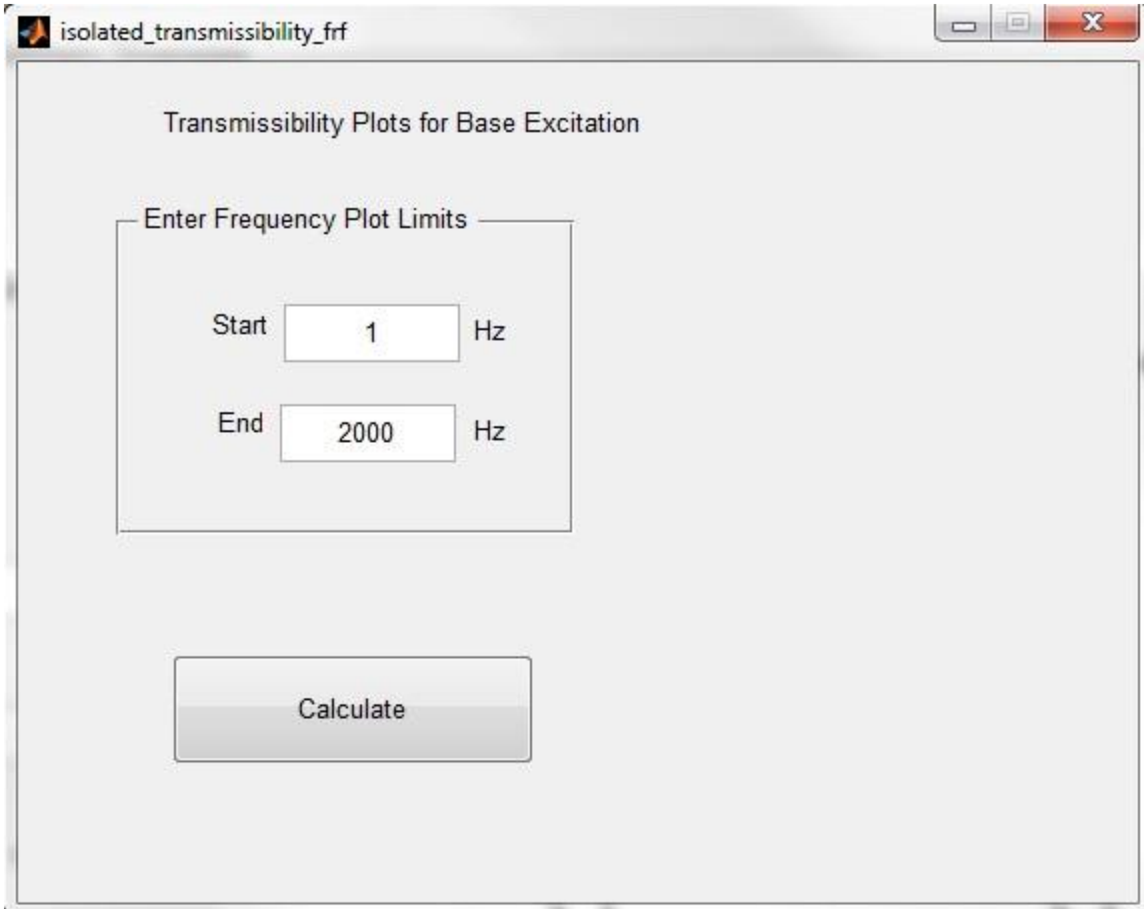
Total Modal Mass

4.280	4.280	4.280	44.900	39.900	18.800

Viscous Damping Ratios for Six Modes

Mode 1	2	3	4	5	6
0.08	0.08	0.08	0.08	0.08	0.08

Frequency Response Functions



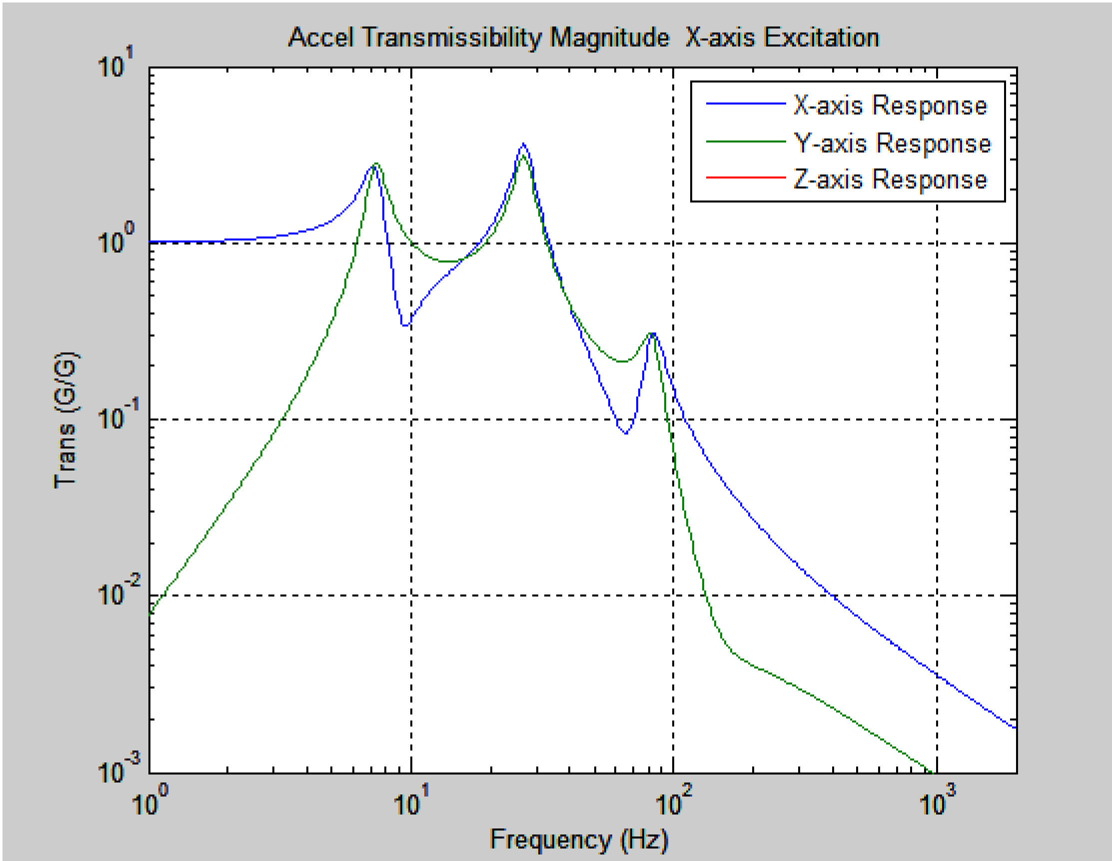


Figure A-1.

The Z-axis response was below the lower amplitude plot limit.

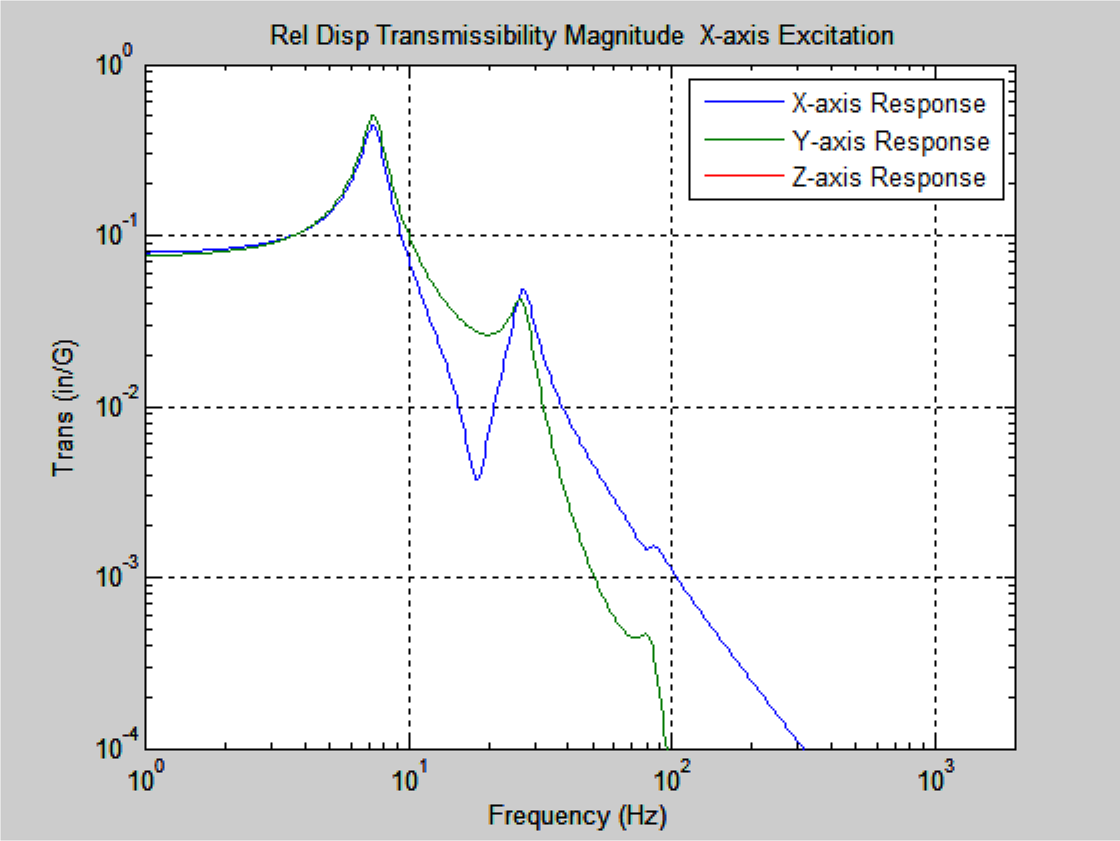


Figure A-2.

The Z-axis response was below the lower amplitude plot limit.

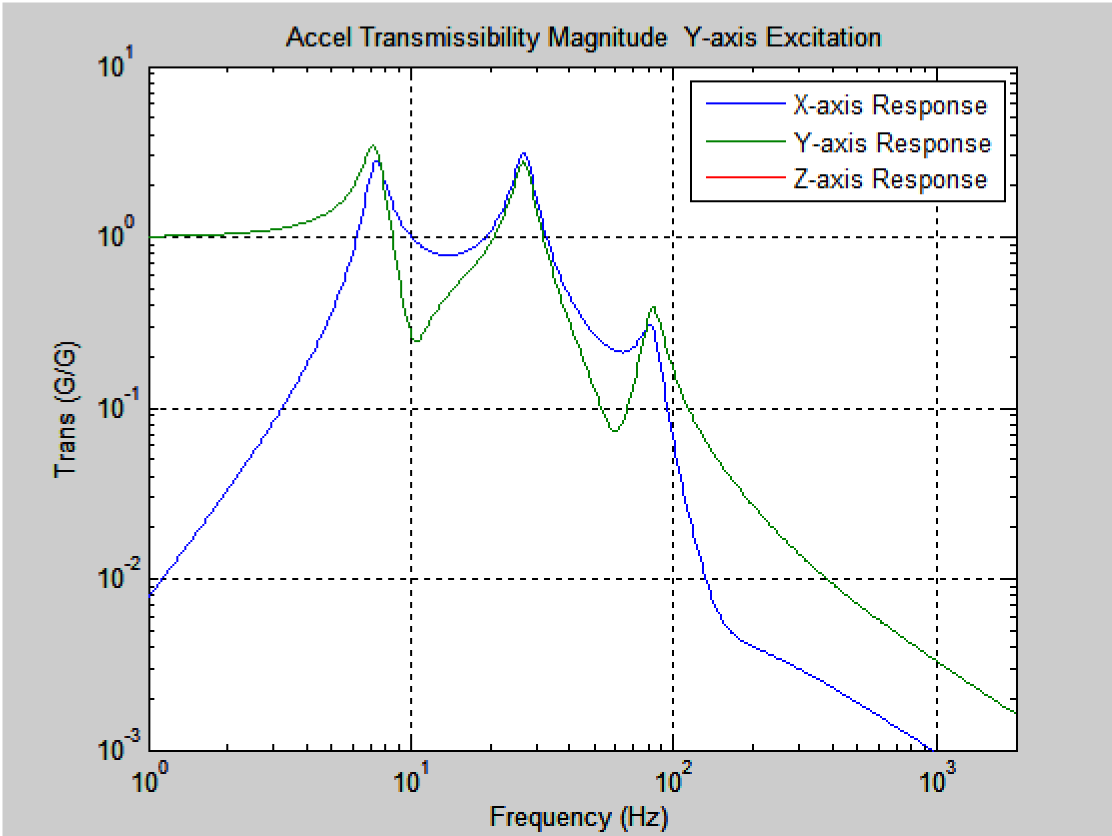


Figure A-3.

The Z-axis response was below the lower amplitude plot limit.

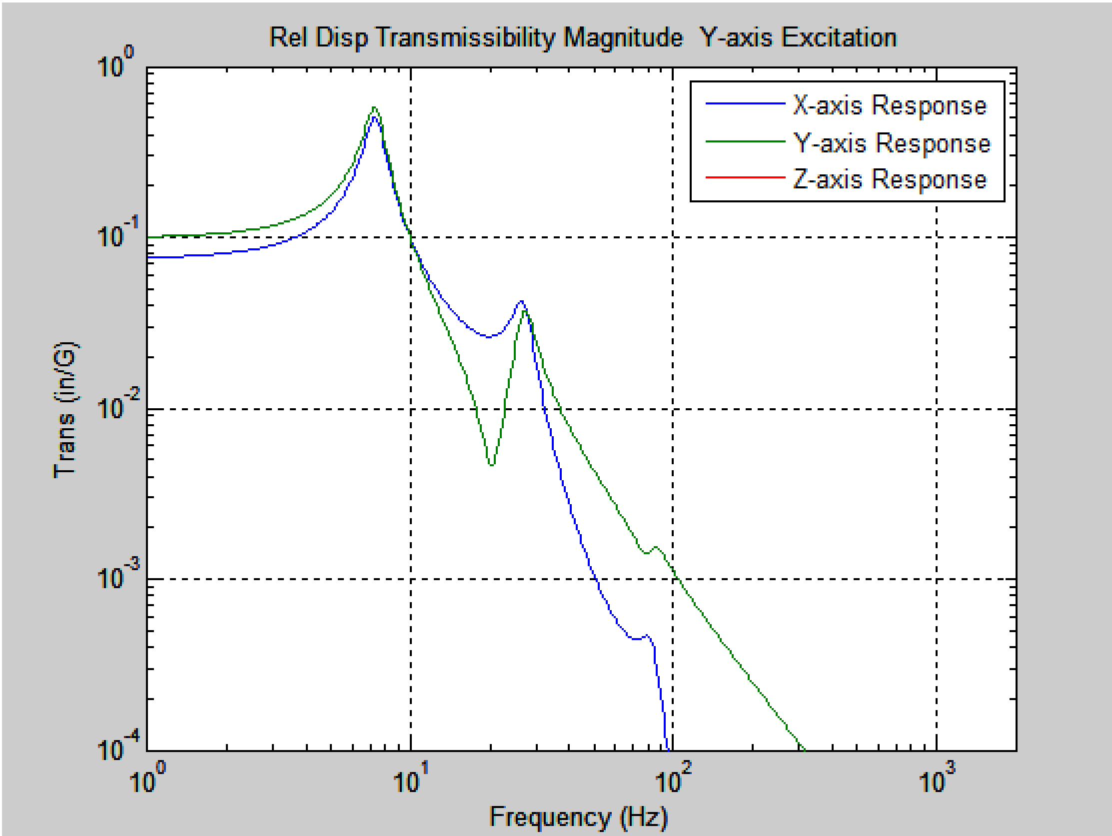


Figure A-4.

The Z-axis response was below the lower amplitude plot limit.

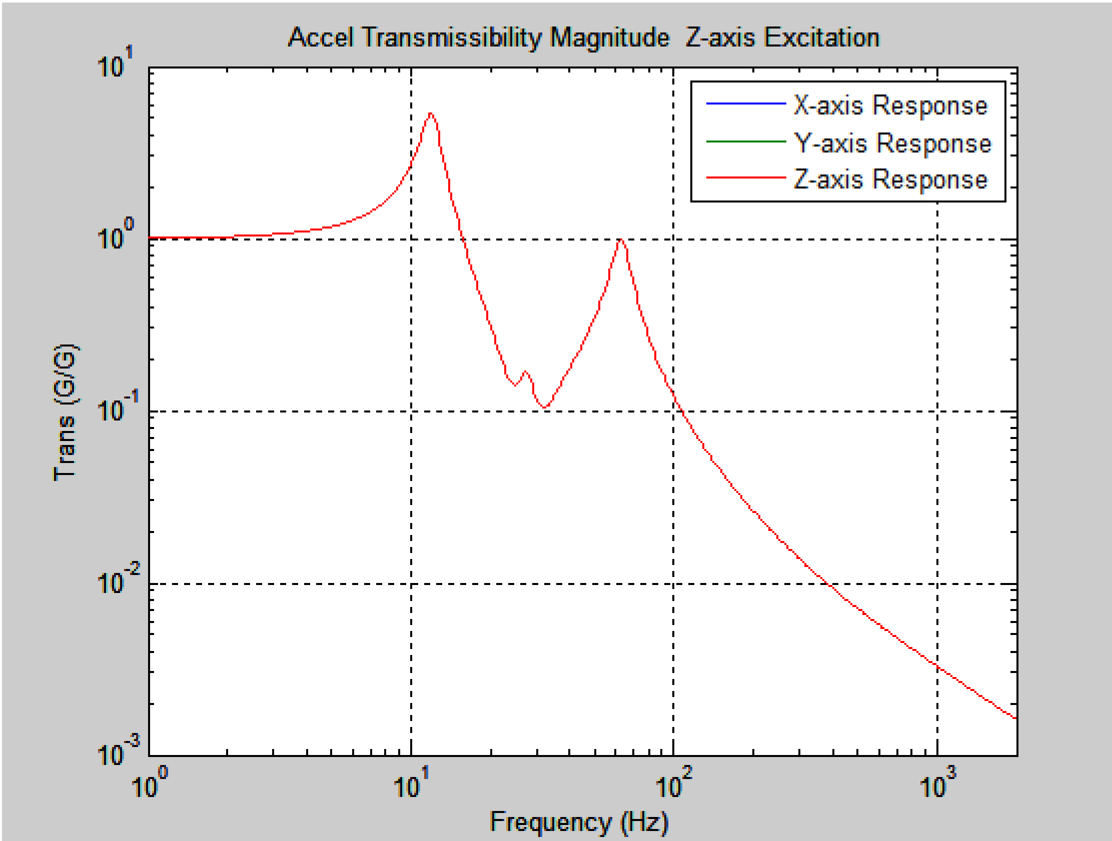


Figure A-5.

Both the X and Y-axes responses were below the lower amplitude plot limit.

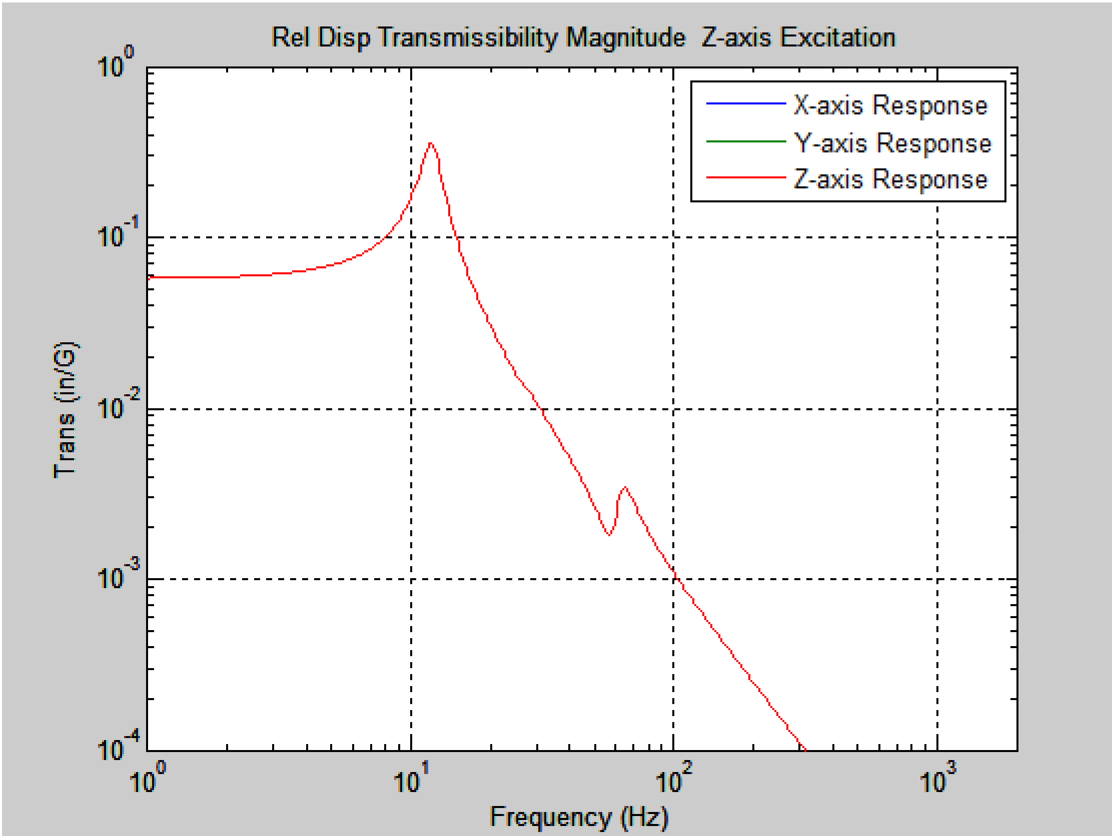
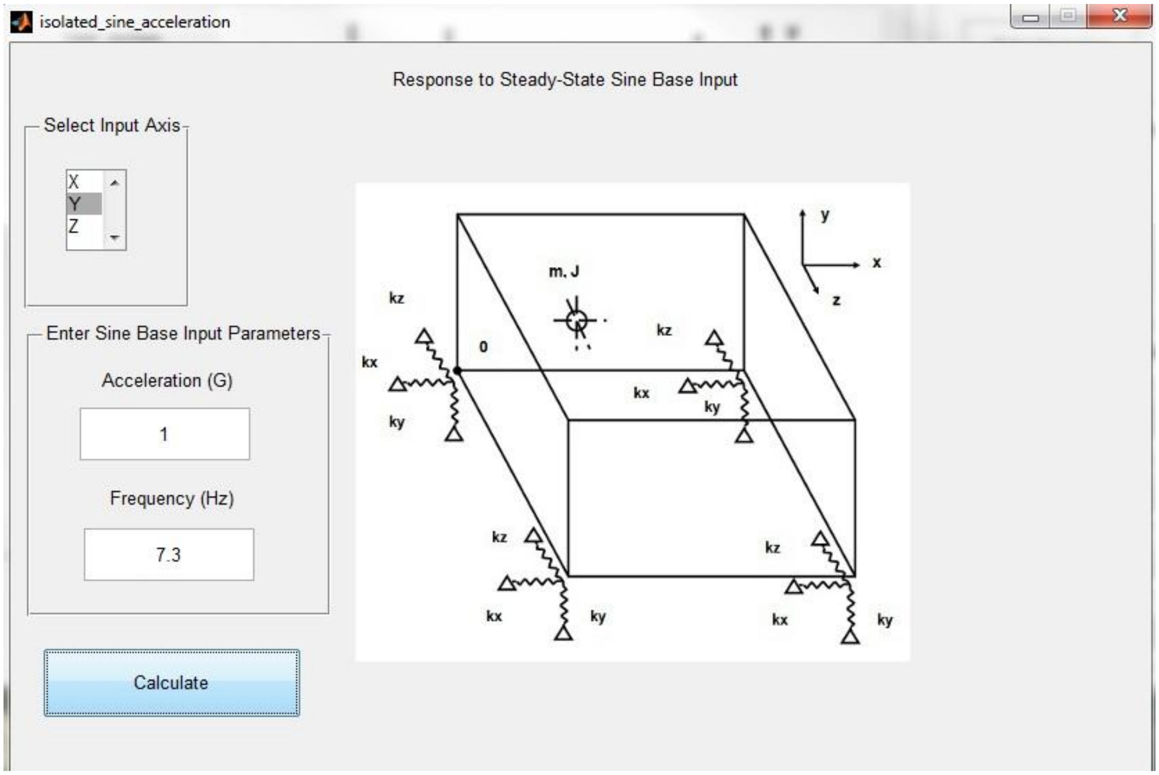


Figure A-6.

Both the X and Y-axes responses were below the lower amplitude plot limit.

<u>Steady-state Sine Base Input</u>

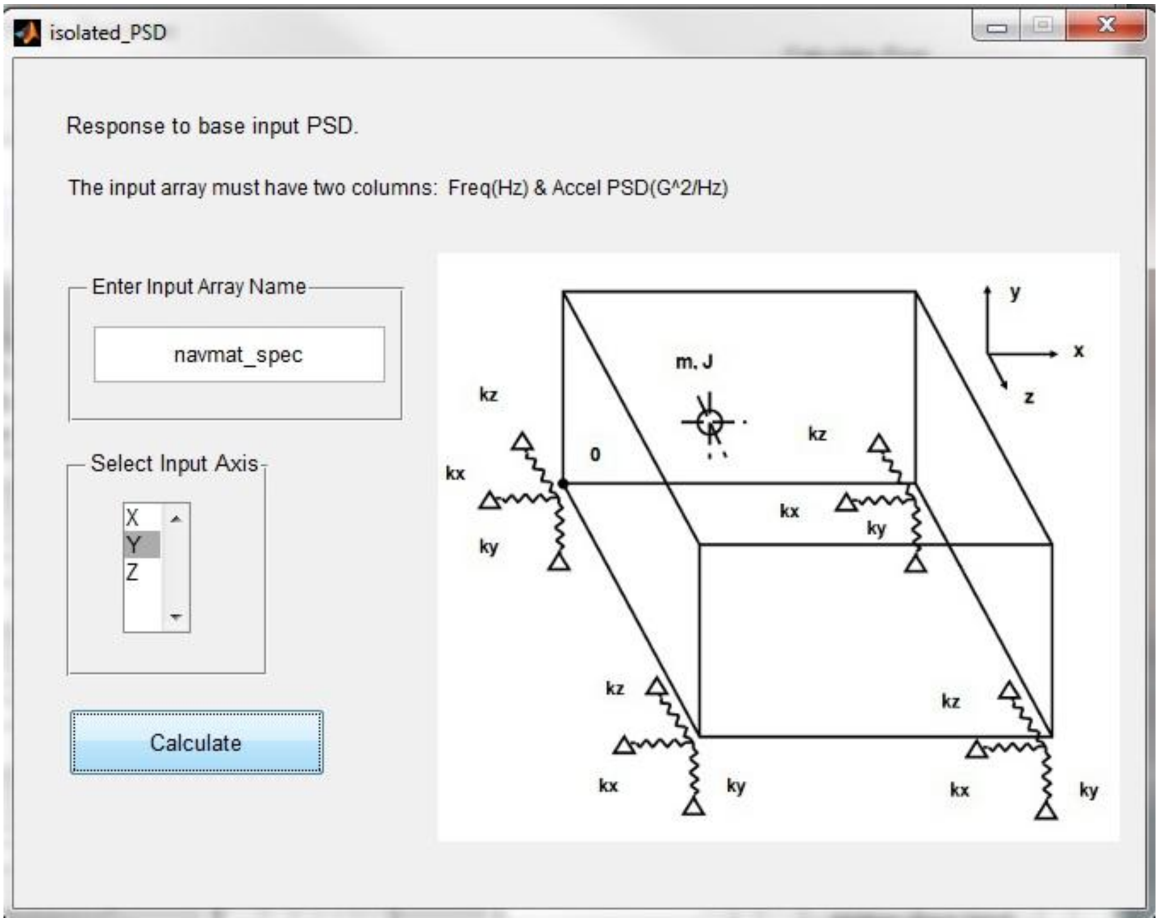
Apply a sine input as follows:




```
Y-axis input

        1 G        7.3 Hz



 C.G. Acceleration Response
  X-axis:      2.765 G
  Y-axis:      3.414 G
  Z-axis:          0 G

 C.G. Relative Displacement Response
  X-axis:     0.5073 in
  Y-axis:     0.5853 in
  Z-axis:          0 in
```

Random Vibration Base Input

Apply a random base input using the NAVMAT P-9492 level as follows:



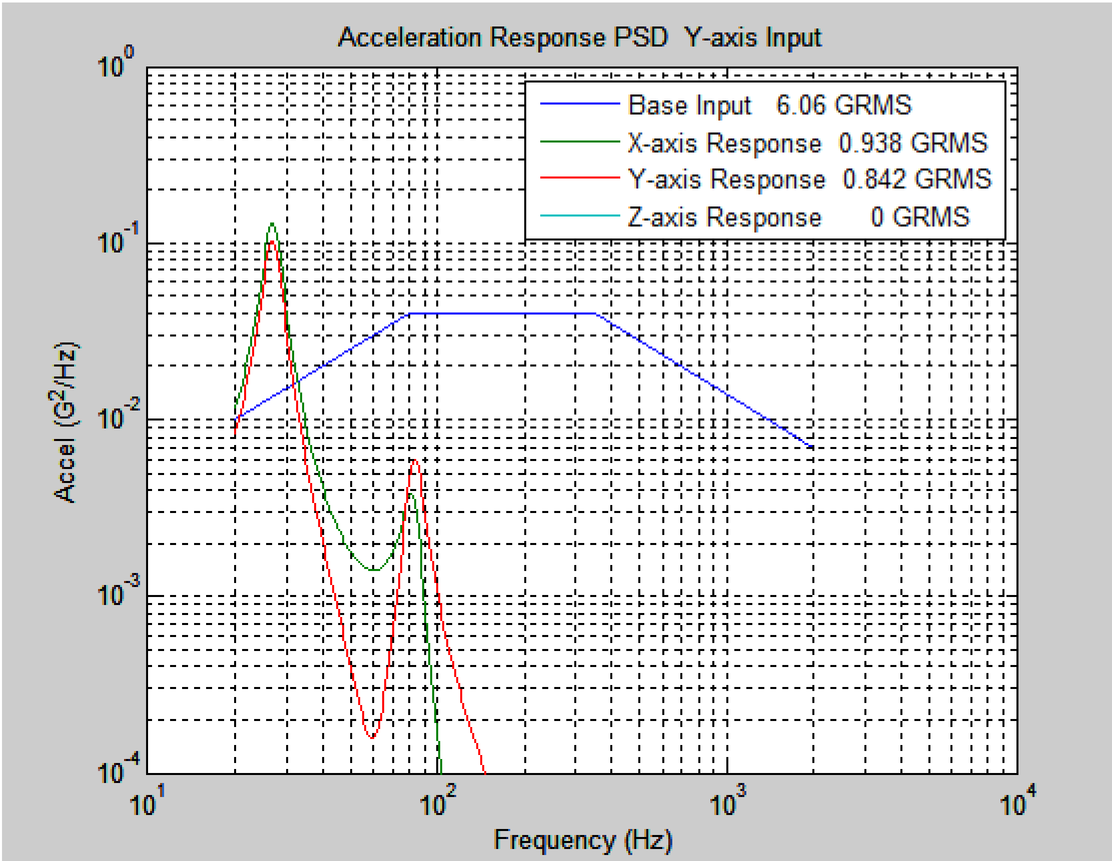


Figure A-7.

The Z-axis response is below the lower plot limit.

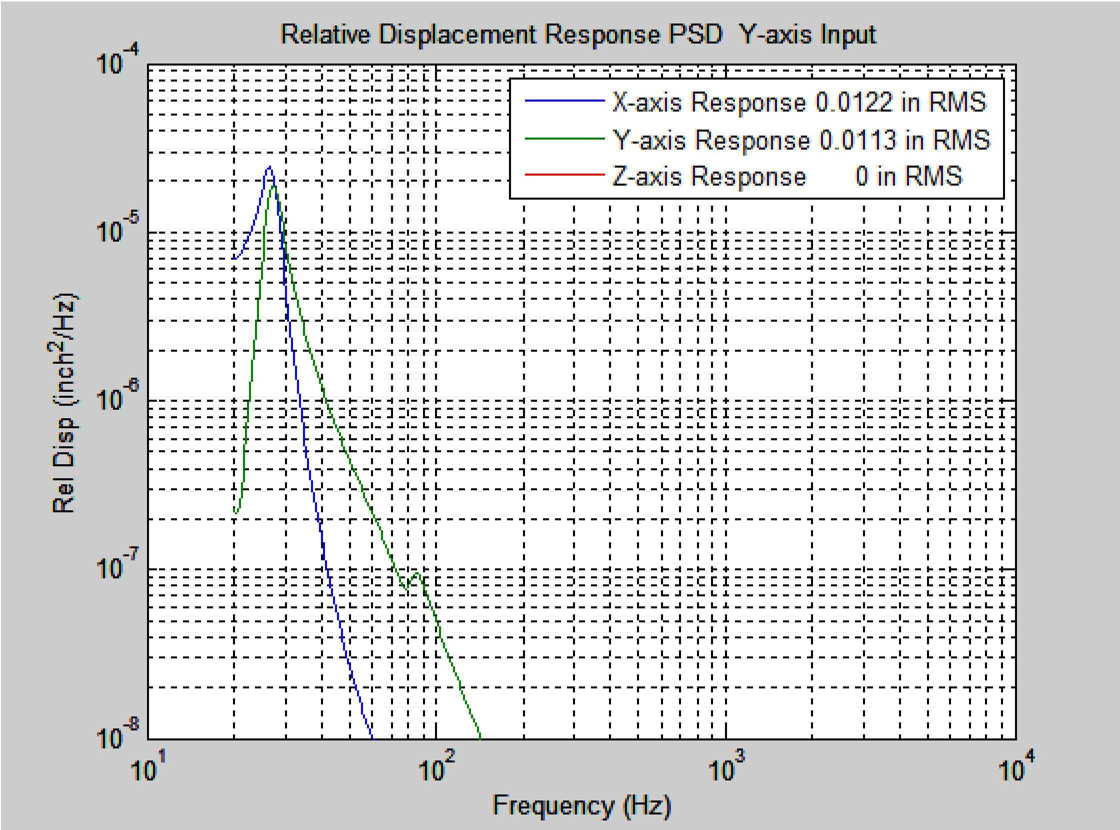


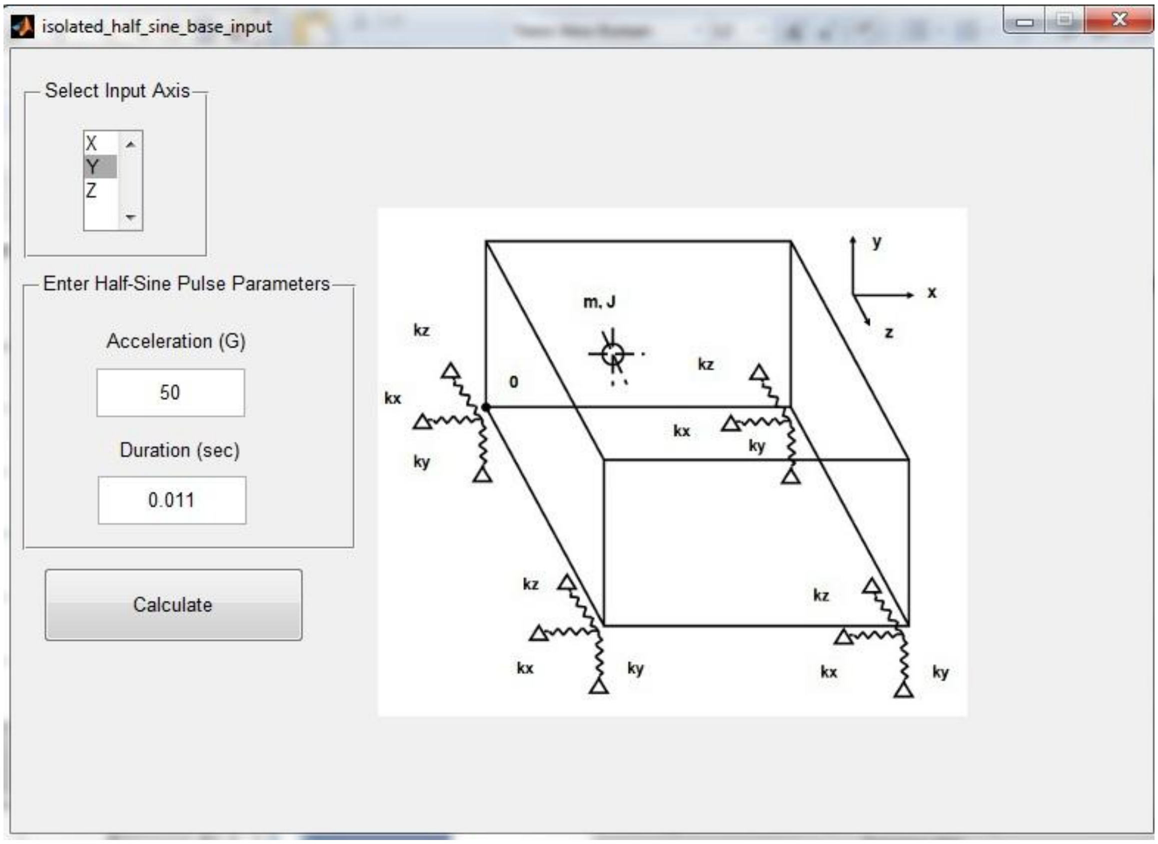
Figure A-8.

The Z-axis response is below the lower plot limit.

```
Y-axis input

 C.G. Acceleration Response
  X-axis:     0.9383 GRMS
  Y-axis:     0.8416 GRMS
  Z-axis:          0 GRMS

 C.G. Relative Displacement Response
  X-axis:    0.01215 in RMS
  Y-axis:    0.01132 in RMS
  Z-axis:          0 in RMS
```

Half-sine Pulse Base Input

Apply a 50 G, 11 msec half-sine pulse as follows:




```
Y-axis input

       50 G     0.011 sec Half-Sine Pulse


  Acceleration Response (G)
               max       min
  X-axis:     23.18    -26.55
  Y-axis:     23.52    -11.71
  Z-axis:         0         0

  Relative Displacement Response (inch)
               max      min
  X-axis:     1.382   -0.8369
  Y-axis:      1.11    -1.338
  Z-axis:         0         0
```

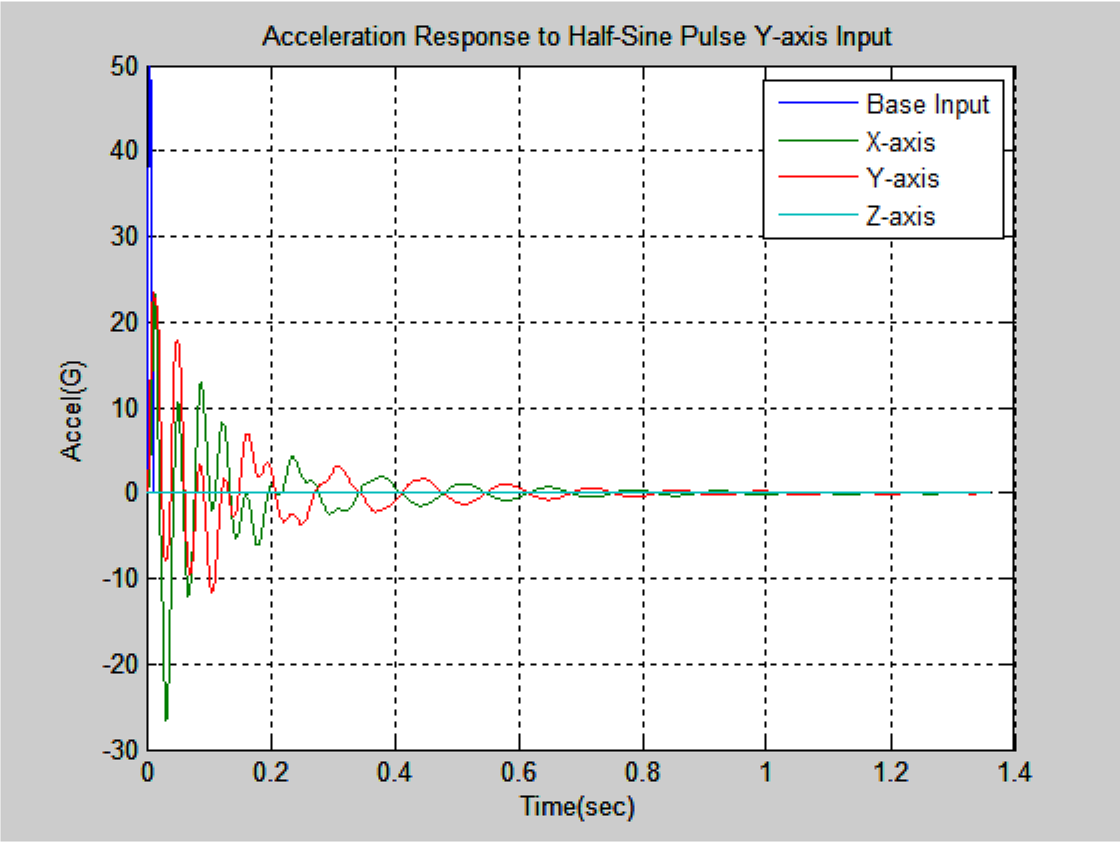

Acceleration Response to Half-Sine Pulse Y-axis Input
Base Input
X-axis
Y-axis
Z-axis
Accel(G)
50
40
30
20
10
0
-10
-20
-30
0
0.2
0.4
0.6
0.8
1
1.2
1.4
Time(sec)

Figure A-9.

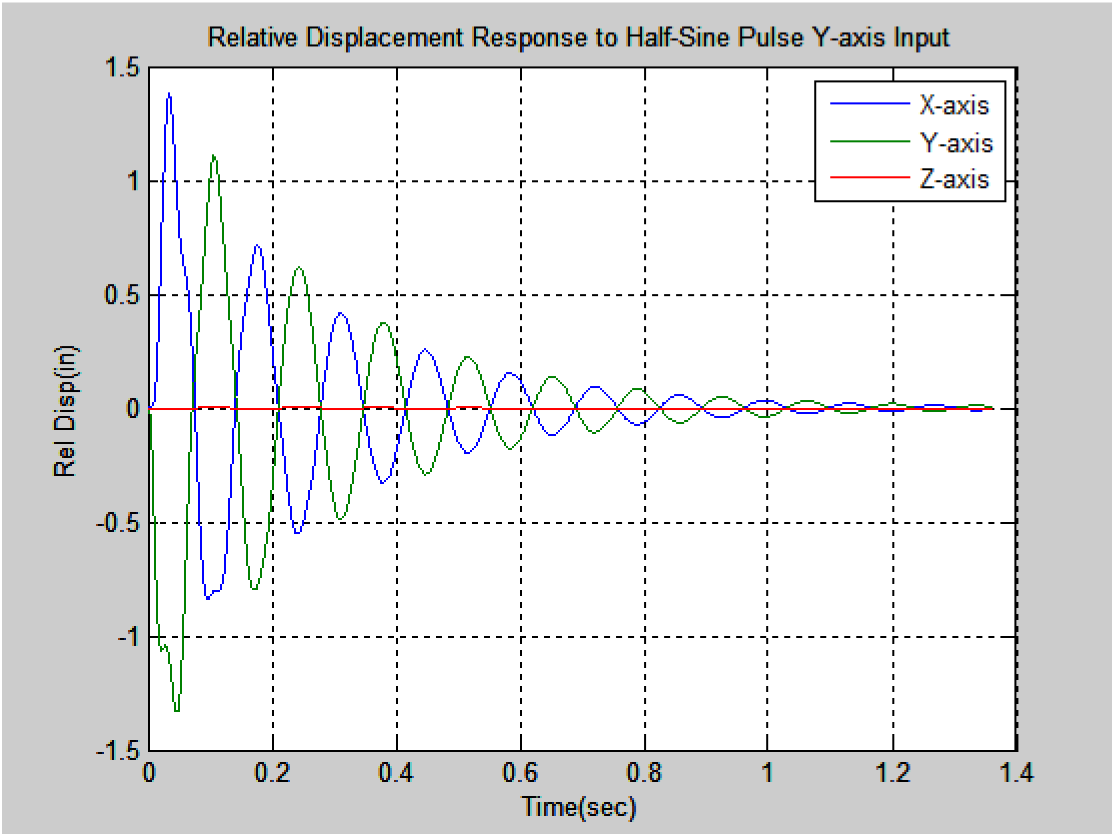


Figure A-10.

The resulting displacement is rather high, nearly 1.4 inches maximum. The isolators must be able to withstand this displacement without bottoming out. But this displacement requirement is too high for most isolator models. Sway space and clearance are also concerns.

Thus, stiffer isolators may be necessary in order to reduce the peak displacement.

Arbitrary Base Input

Apply a simulated pyrotechnic shock pulse as follows.

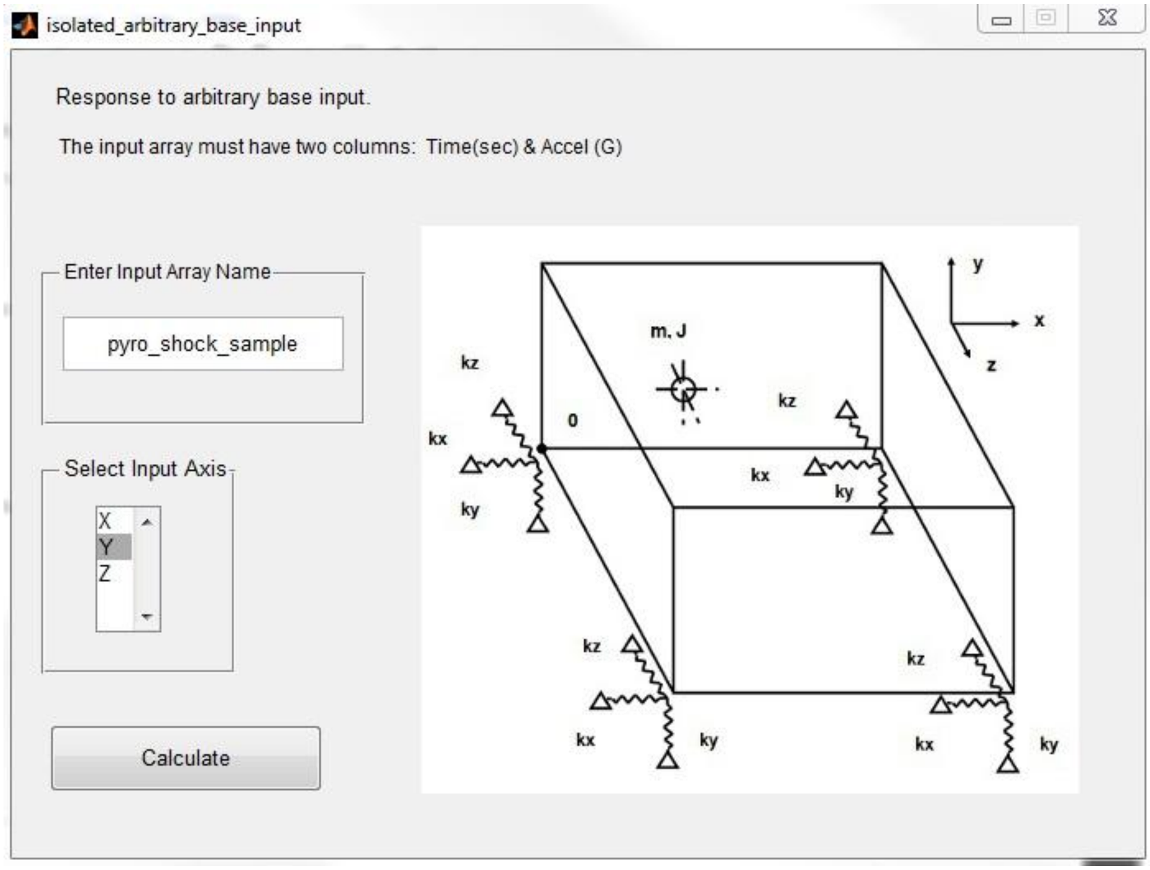


Figure A-11.

```
Y-axis input

  Acceleration Response (G)
               max        min
  X-axis:    16.31      -15.4
  Y-axis:    16.27     -7.529
  Z-axis:        0          0

  Relative Displacement Response (inch)
               max       min
  X-axis:   0.7749     -0.611
  Y-axis:    0.661    -0.7777
  Z-axis:        0          0
```

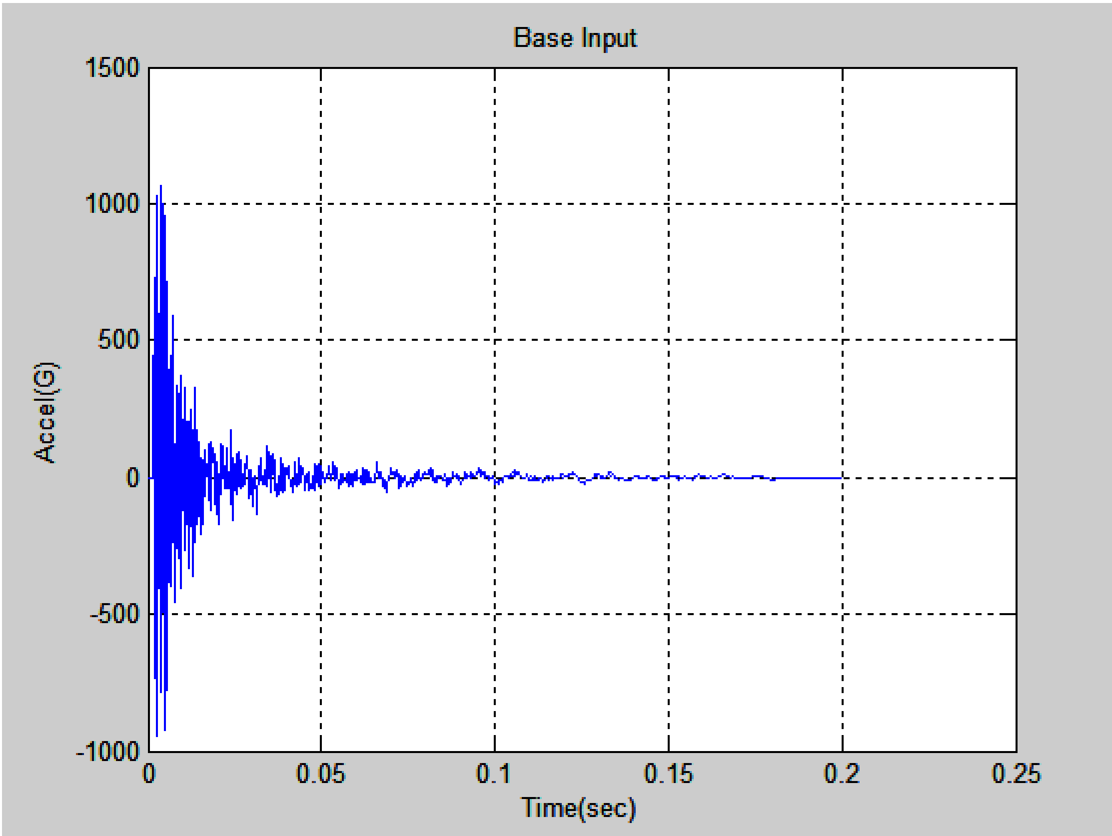
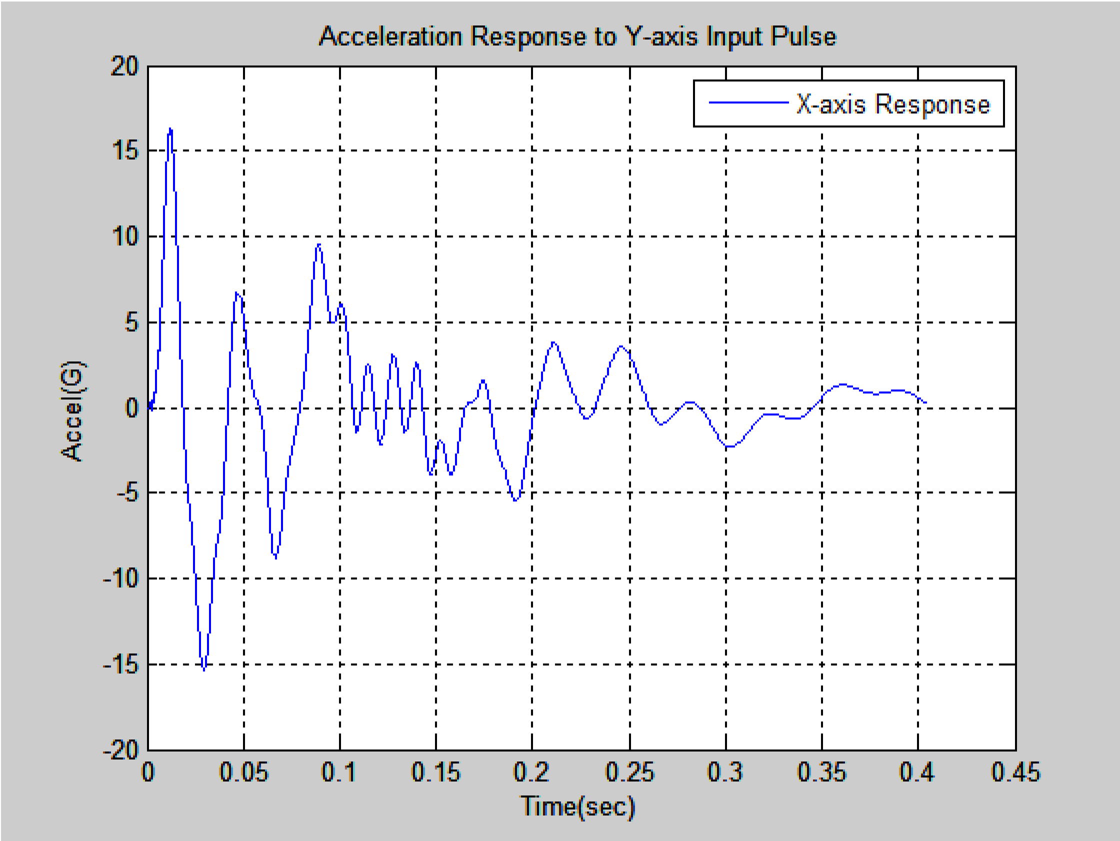


Figure A-12.

The shock data is from: pyro_shock_sample.txt



Acceleration Response to Y-axis Input Pulse
X-axis Response
Accel(G)
Time(sec)
20
15
10
5
0
-5
-10
-15
-20
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45

Figure A-13.

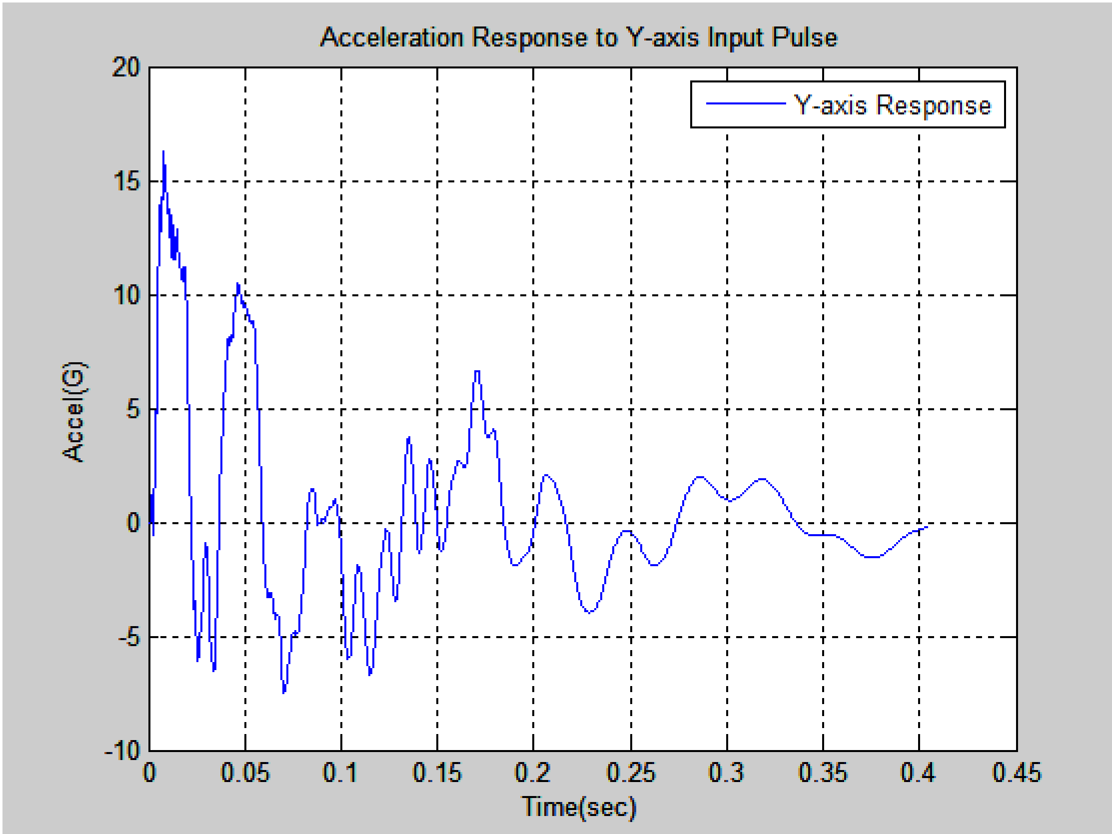

Acceleration Response to Y-axis Input Pulse
Y-axis Response
Accel(G)
Time(sec)
20
15
10
5
0
-5
-10
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45

Figure A-14.

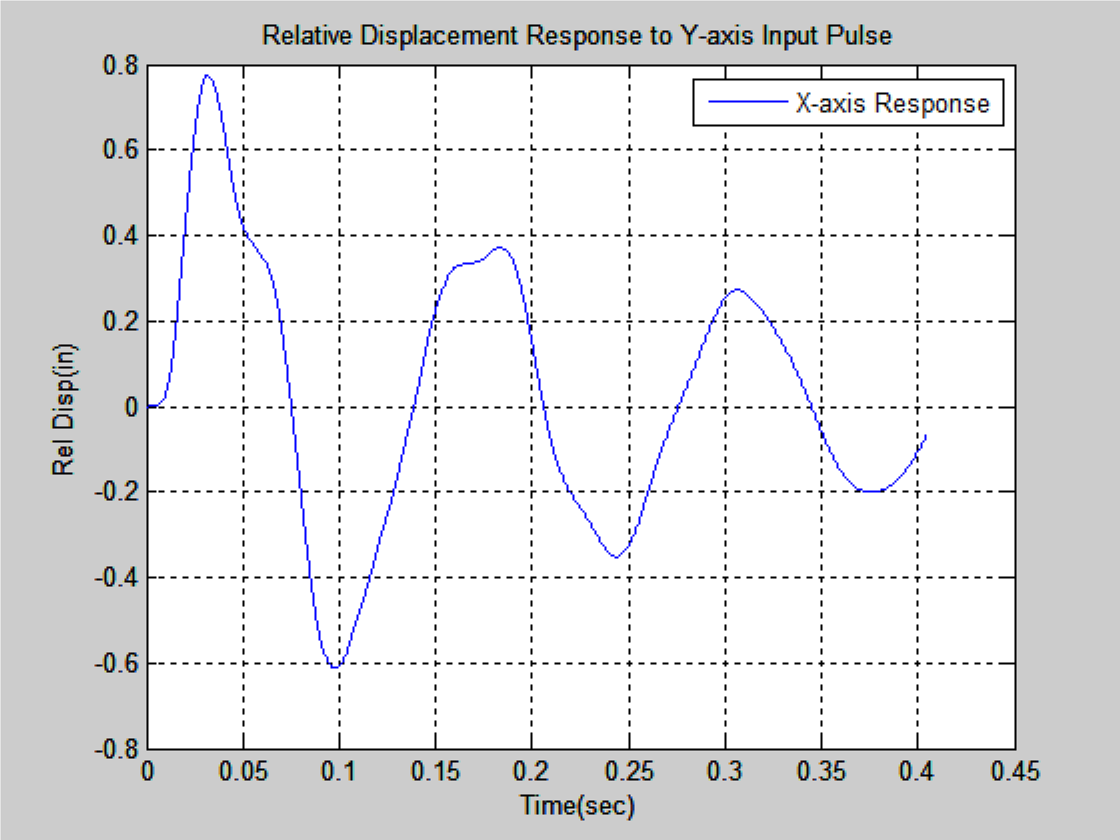


Figure A-15.

Again, the relative displacement may beyond the isolators' allowable limit.

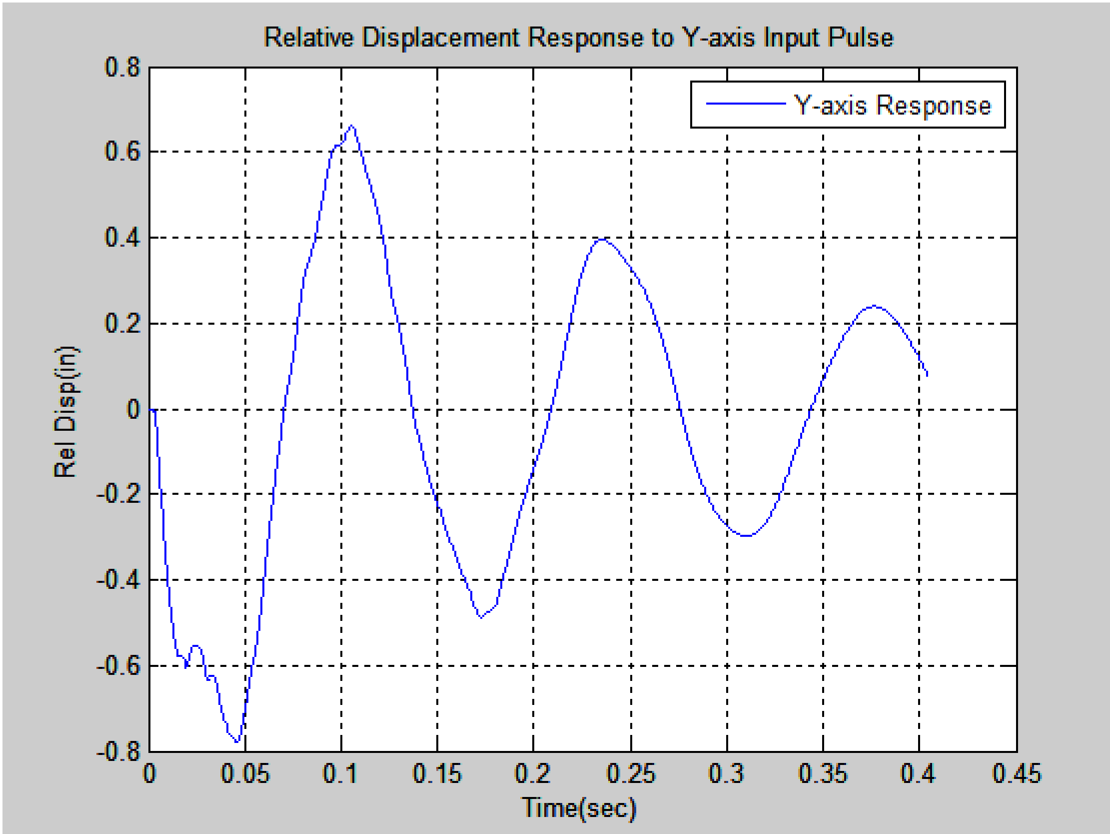
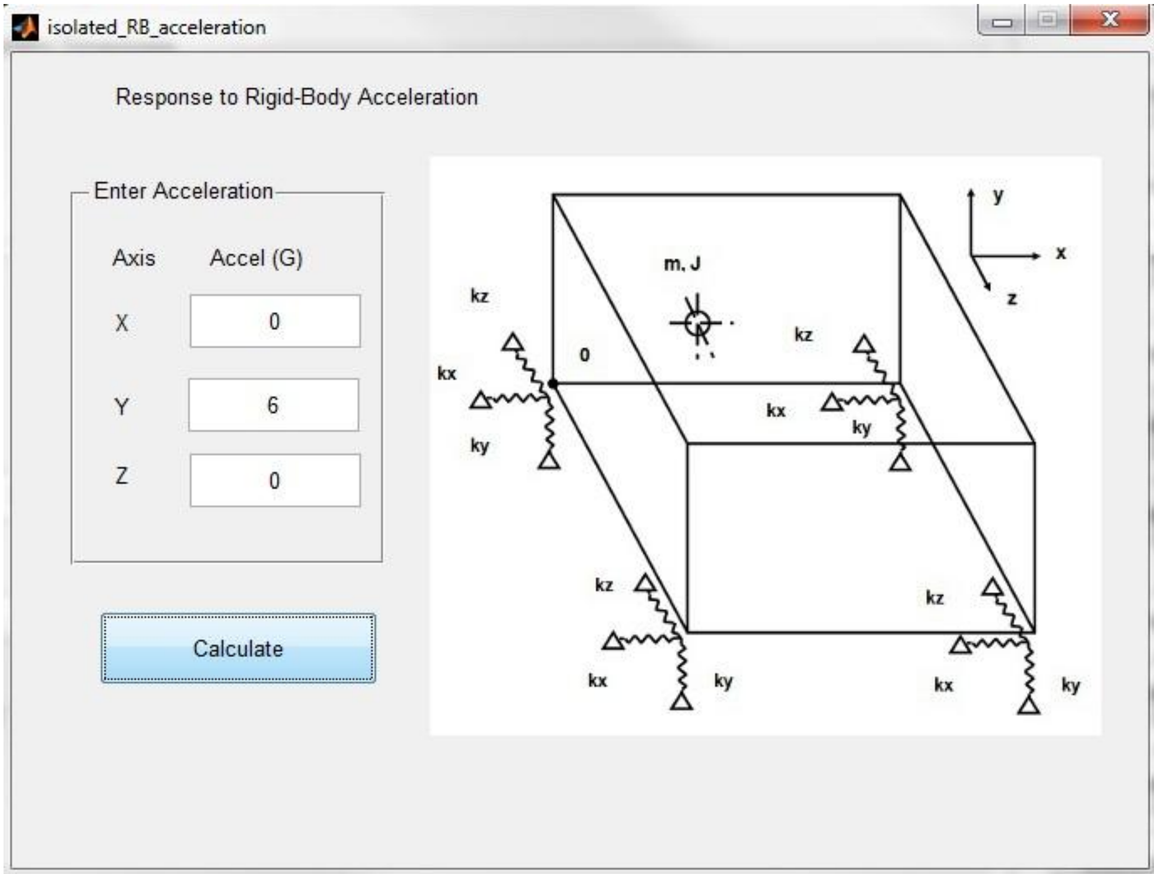


Figure A-16.

Note that the Z-axis response plots are omitted because the response in this axis was effectively zero.

Rigid-body Acceleration

Apply a rigid-body acceleration as follows.




```
Rigid-Body Acceleration (G)
     x          y          z
      0          6          0


   C.G. Displacement
     x          y          z
    (in)       (in)       (in)
 -0.447      0.595         0


   C.G. Rotation
   theta-x      theta-y     theta-z
    (rad)        (rad)       (rad)
                      0          0         0.116
```

APPENDIX B

<u>Perpendicular Axis Theorem</u>

By the perpendicular axis theorem, the following equation relates J_z to the area moments of inertia about the other two mutually perpendicular axes:

$$J_Z = I_{xx} + I_{yy}$$

Reference: http://en.wikipedia.org/wiki/Polar_moment_of_inertia