THE STEADY-STATE VIBRATION RESPONSE OF A BAFFLED PLATE SIMPLY-SUPPORTED ON ALL SIDES SUBJECTED TO HARMONIC PRESSURE WAVE EXCITATION AT OBLIQUE INCIDENCE

Revision F

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The baffled, simply-supported plate in Figure 1 is subjected to an oblique pressure wave on one side. Only a side view along the length is shown because the pressure is assumed to be uniform with width. This diagram and the corresponding pressure field equation are taken from Reference 1.

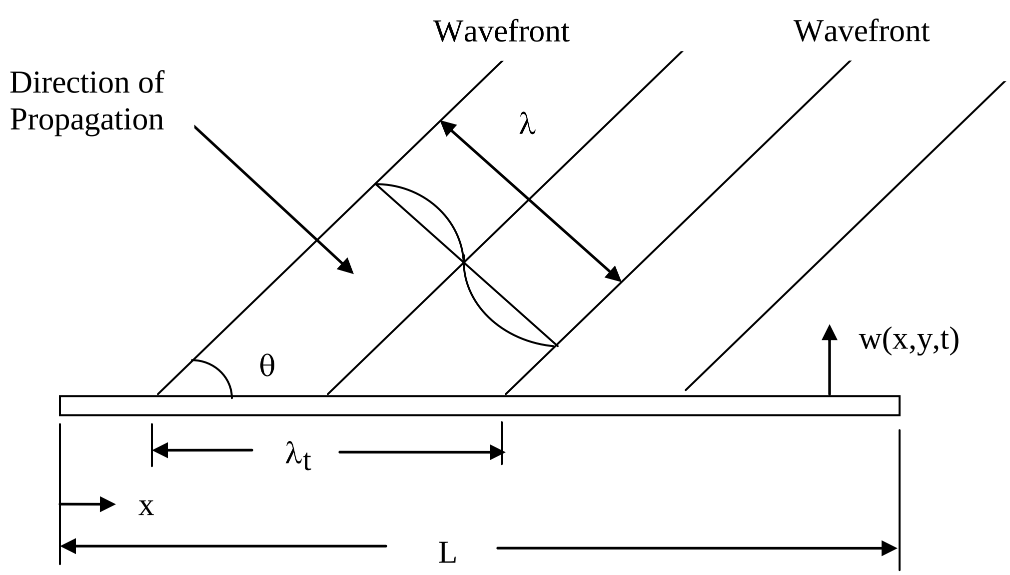

Figure 1.

λ is the acoustic wavelength. λ_t is the trace wavelength.

$$\lambda_t = \lambda / \sin\theta \tag{1}$$

The governing differential equation from Reference 2 is

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2 z}{\partial t^2}=P(x,t) \tag{2}$$

The plate stiffness factor D is given by

$$D=\frac{Eh^3}{12\left(1-\mu^2\right)} \tag{3}$$

where

E	is the modulus of elasticity
μ	Poisson's ratio
H	is the thickness
ρ	is the mass density (mass/volume)
P	is the applied pressure

Now assume that the displacement can separated into spatial and temporal components such that

$$P(x,t)=p_o\cos\left(\omega t-\frac{2\pi x}{\lambda_t}+\phi\right) \tag{4}$$

The trace wave number k_t is

$$k_t=\frac{2\pi}{\lambda_t} \tag{5}$$

$$P(x,t)=p_o\cos\left(\omega t-k_t x+\phi\right) \tag{6}$$

The differential equation becomes

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2 w}{\partial t^2}=P(x,t) \tag{7}$$

Let

$$w(x,y,t) = \Phi(x,y)\,u(t) \tag{8}$$

By substitution,

$$D\left(\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4}\right)u + \rho h\,\Phi\,\frac{\partial^2 u}{\partial t^2} = P(x,t) \tag{9}$$

The homogeneous equation is

$$D\left(\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4}\right)u + \rho h\,\Phi\,\frac{\partial^2 u}{\partial t^2} = 0 \tag{10}$$

$$\rho h\,\Phi\,\frac{\partial^2 u}{\partial t^2} = -D\left(\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4}\right)u \tag{11}$$

$$\frac{\ddot{u}}{u} = -\frac{D}{\rho h\Phi}\left(\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4}\right) = -c^2 \tag{12}$$

$$\ddot{u} + c^2 u = 0 \tag{13}$$

$$D\left(\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4}\right)u - c^2\Phi = 0 \tag{14}$$

$$\left(\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4}\right)u - c^2\,\frac{\rho h}{D}\,\Phi = 0 \tag{15}$$

The mass-normalized mode shapes are

$$\Phi_{mn} = \frac{2}{\sqrt{\rho a\, b\, h}} \sin\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right) \tag{16}$$

$$\frac{\partial}{\partial x}\Phi_{mn} = \frac{2}{\sqrt{\rho a\, b\, h}} \left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right) \tag{17}$$

$$\frac{\partial^2}{\partial x^2}\Phi_{mn} = -\frac{2}{\sqrt{\rho a\, b\, h}} \left(\frac{m\pi}{a}\right)^2 \sin\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right) \tag{18}$$

$$\frac{\partial}{\partial y}\Phi_{mn} = \frac{2}{\sqrt{\rho a\, b\, h}} \left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi\, x}{a}\right) \cos\left(\frac{n\pi\, y}{b}\right) \tag{19}$$

$$\frac{\partial^2}{\partial y^2}\Phi_{mn} = -\frac{2}{\sqrt{\rho a\, b\, h}} \left(\frac{n\pi}{b}\right)^2 \sin\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right) \tag{20}$$

The natural frequencies are

$$c^2 = \omega_{mn}{}^2 = \frac{D}{\rho h}\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)^2 \tag{21}$$

$$\omega_{mn} = \sqrt{\frac{D}{\rho h}}\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right) \tag{22}$$

Let

$$\beta_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{23}$$

Thus

$$\omega_{mn} = \beta_{mn}{}^2 \sqrt{\frac{D}{\rho h}} \tag{24}$$

$$c^2 = \omega_{mn}{}^2 = \beta_{mn}{}^4 \frac{D}{\rho h} \tag{25}$$

$$\left(\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4}\right) u - \beta_{mn}{}^4 \Phi = 0 \tag{26}$$

Recall

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \rho h \frac{\partial^2 w}{\partial t^2} = P(x,t) \tag{27}$$

Let

$$w(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Phi_{mn}(x,y) u_{mn}(t) \tag{28}$$

By substitution,

$$D\left(\frac{\partial^4}{\partial x^4}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Phi_{mn} u_{mn}\right) + 2\frac{\partial^4 w}{\partial x^2 \partial y^2}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Phi_{mn} u_{mn}\right) + \frac{\partial^4 w}{\partial y^4}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Phi_{mn} u_{mn}\right)\right)$$
$$+ \rho h \frac{\partial^2}{\partial t^2}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Phi_{mn} u_{mn}\right) = P(x,t) \tag{29}$$

$$D\left(\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\partial^4}{\partial x^4} \Phi_{mn} u_{mn}\right) + 2\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\partial^4}{\partial x^2 \partial y^2} \Phi_{mn} u_{mn}\right) + \left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\partial^4}{\partial y^4} \Phi_{mn} u_{mn}\right)\right)$$
$$+ \rho h \left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Phi_{mn} \frac{\partial^2}{\partial t^2} u_{mn}\right) = P(x,t) \tag{30}$$

$$D\left(\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_{mn}\frac{\partial^4}{\partial x^4}\Phi_{mn}\right)+2\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_{mn}\frac{\partial^4}{\partial x^2\partial y^2}\Phi_{mn}\right)+\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_{mn}\frac{\partial^4}{\partial y^4}\Phi_{mn}\right)\right)$$

$$+\rho h\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Phi_{mn}\frac{d^2}{dt^2}u_{mn}\right)=P(x,t) \tag{31}$$

$$D\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_{mn}\left(\frac{\partial^4}{\partial x^4}\Phi_{mn}+2\frac{\partial^4}{\partial x^2\partial y^2}\Phi_{mn}+\frac{\partial^4}{\partial y^4}\Phi_{mn}\right)\right)+\rho h\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Phi_{mn}\frac{d^2}{dt^2}u_{mn}\right)=P(x,t) \tag{32}$$

Note that

$$\left(\frac{\partial^4\Phi_{mn}}{\partial x^4}+2\frac{\partial^4\Phi_{mn}}{\partial x^2\partial y^2}+\frac{\partial^4\Phi_{mn}}{\partial y^4}\right)u=\beta_{mn}{}^4\Phi_{mn} \tag{33}$$

By substitution,

$$D\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_{mn}{}^4\, u_{mn}\Phi_{mn}\right)+\rho h\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Phi_{mn}\frac{d^2}{dt^2}u_{mn}\right)=P(x,t) \tag{34}$$

Multiply each term by Φ_{pq}.

$$D\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_{mn}{}^4\, u_{mn}\Phi_{pq}\Phi_{mn}\right)+\rho h\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Phi_{pq}\Phi_{mn}\frac{d^2}{dt^2}u_{mn}\right)=\Phi_{pq}P(x,t) \tag{35}$$

Integrate with respect to surface area. Note that the eigenvectors Φ_{pq},Φ_{mn} are functions of area, but the argument is omitted for brevity.

$$D\int_0^a\int_0^b\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_{mn}{}^4\,u_{mn}\Phi_{pq}\Phi_{mn}\right\}dxdy+\int_0^a\int_0^b\rho h\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Phi_{pq}\Phi_{mn}\frac{d^2}{dt^2}u_{mn}\right)dxdy$$

$$=\int_0^a\int_0^b\Phi_{pq}P(x,t)dxdy \tag{36}$$

$$D\int_0^a\int_0^b\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_{mn}{}^4\,u_{mn}\Phi_{pq}\Phi_{mn}\right\}dxdy+\rho h\frac{d^2}{dt^2}u_{mn}\int_0^a\int_0^b\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Phi_{pq}\Phi_{mn}\right)dxdy$$

$$=\int_0^a\int_0^b\Phi_{pq}P(x,t)dxdy \tag{37}$$

$$D\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\beta_{mn}{}^4\,u_{mn}\int_0^a\int_0^b\Phi_{pq}\Phi_{mn}dxdy\right\}+\rho h\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{d^2u_{mn}}{dt^2}\int_0^a\int_0^b\Phi_{pq}\Phi_{mn}dxdy\right\}$$

$$=\int_0^a\int_0^b\Phi_{pq}P(x,t)dxdy \tag{38}$$

The eigenvectors are orthogonal such that

$\rho h\int_0^a\int_0^b\Phi_{pq}\Phi_{mn}dxdy=0$ if any of the indices (m, n, p, q) is unique with respect to the others

$\rho h\int_0^a\int_0^b\Phi_{pq}\Phi_{mn}dxdy=1$ if $m=n=p=q$

Thus

$$\frac{d^2u_{mn}}{dt^2}+\frac{D\beta_{mn}{}^4}{\rho h}u_{mn}=\int_0^a\int_0^b\Phi_{pq}P(x,t)dxdy \tag{39}$$

$$\omega_{mn}{}^2 = \beta_{mn}{}^4 \frac{D}{\rho h} \tag{40}$$

$$\frac{d^2u_{mn}}{dt^2} + \omega_{mn}{}^2\, u_{mn} = \int_0^a \int_0^b \Phi_{mn} P(x,t) dxdy \tag{41}$$

Add a modal damping term.

$$\frac{d^2u_{mn}}{dt^2} + 2\xi_{mn}\omega_{mn} \frac{du_{mn}}{dt} + \omega_{mn}{}^2\, u_{mn} = \int_0^a \int_0^b \Phi_{mn} P(x,t) dxdy \tag{42}$$

$$P(x,t) = p_o \cos(\omega t - k_t x + \phi) \tag{43}$$

$$\Phi_{mn} = \frac{2}{\sqrt{\rho a\, b\, h}} \sin\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{n\pi\, y}{b}\right) \tag{44}$$

Let

$$\hat{A} = \frac{2}{\sqrt{\rho abh}} \tag{44}$$

$$\Phi_{mn} = \hat{A} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{45}$$

$$\frac{d^2u_{mn}}{dt^2} + 2\,\xi_{mn}\omega_{mn} \frac{du_{mn}}{dt} + \omega_{mn}{}^2\, u_{mn} = \hat{A}\, p_o \int_0^b \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos(\omega t - k_t x + \phi) dxdy$$

(46)

$$\frac{d^2u_{mn}}{dt^2} + 2\,\xi_{mn}\omega_{mn} \frac{du_{mn}}{dt} + \omega_{mn}{}^2\, u_{mn} = L_{mn}(t) \tag{47}$$

The generalized force is

$$L_{mn}(t) = \hat{A}\,p_o \int_0^b \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos(\omega t - k_t x + \phi)\,dx\,dy \tag{48}$$

$$L_{mn}(t) = \frac{b\,\hat{A}\,p_o}{n\pi} \left\{ \int_0^a \sin\left(\frac{m\pi x}{a}\right) \cos(\omega t - k_t x + \phi)\,dx \right\} \left\{ -\cos\left(\frac{n\pi y}{b}\right) \Big|_0^b \right\} \tag{49}$$

$$L_{mn}(t) = \frac{b[1-\cos(n\pi)]\hat{A}\,p_o}{n\pi} \left\{ \int_0^a \sin\left(\frac{m\pi x}{a}\right) \cos(\omega t - k_t x + \phi)\,dx \right\} \tag{50}$$

The phase angle is arbitrary for the steady-state analysis. Set $\phi = 0$.

$$L_{mn}(t) = \frac{b[1-\cos(n\pi)]\hat{A}\,p_o}{n\pi} \left\{ \int_0^a \sin\left(\frac{m\pi x}{a}\right) \cos(\omega t - k_t x)\,dx \right\} \tag{51}$$

Equation (51) is evaluated in Appendices A and B. There are three possible results as follows.

For $k_t \neq \dfrac{m\pi}{a}$,

$$L_{mn}(t) = \frac{2\hat{A}\,p_o[1-\cos(n\pi)]\,a\,b}{\pi^2 mn\left(\left(\dfrac{\lambda_m}{\lambda_t}\right)^2 - 1\right)} \cos\left[\frac{m\pi}{2}\frac{\lambda_m}{\lambda_t}\right] \sin(\omega t + \varepsilon) \quad \text{, for m=1,3,5…} \tag{52}$$

and

$$L_{mn}(t) = \frac{2\hat{A}\,p_o[1-\cos(n\pi)]\,a\,b}{\pi^2 mn\left(\left(\dfrac{\lambda_m}{\lambda_t}\right)^2 - 1\right)} \sin\left[\frac{m\pi}{2}\frac{\lambda_m}{\lambda_t}\right] \sin(\omega t + \varepsilon) \quad \text{, for m=2,4,6…} \tag{53}$$

For $k_t = \dfrac{m\pi}{a}$,

$$L_{mn}(t) = \frac{\hat{A}\, p_o a\, b}{2\pi n}\left[\cos(n\pi) - 1\right]\sin(\omega t) \tag{54}$$

Only the steady-state solution is needed. So define a participation factor Γ_{mn} and represent the time varying term as a harmonic excitation function.

$$L_{mn}(t) = \Gamma_{mn}\, p(t) \tag{55}$$

where

$$p(t) = p_o \exp(j\omega t) \tag{56}$$

There are three participation factor cases.

For $k_t \neq \dfrac{m\pi}{a}$,

$$\Gamma_{mn} = \frac{2\hat{A}\left[1 - \cos(n\pi)\right] a\, b}{\pi^2 mn\left(\left(\dfrac{\lambda_m}{\lambda_t}\right)^2 - 1\right)} \cos\left[\frac{m\,\pi}{2}\,\frac{\lambda_m}{\lambda_t}\right] \text{ , for m=1,3,5…} \tag{57}$$

and

$$\Gamma_{mn} = \frac{2\hat{A}\left[1 - \cos(n\pi)\right] a\, b}{\pi^2 mn\left(\left(\dfrac{\lambda_m}{\lambda_t}\right)^2 - 1\right)} \sin\left[\frac{m\,\pi}{2}\,\frac{\lambda_m}{\lambda_t}\right] \text{ , for m=2,4,6…} \tag{58}$$

For $k_t = \frac{m\pi}{a}$,

$$\Gamma_{mn} = \frac{\hat{A}\,a\,b}{2\pi n}\left[\cos(n\pi) - 1\right] \tag{59}$$

Define a joint acceptance function J_{mn}.

$$J_{mn} = \hat{A}\,a\,b\,\Gamma_{mn} \tag{60}$$

$$\Gamma_{mn} = \frac{1}{\hat{A}\,a\,b} J_{mn} \tag{61}$$

There are three joint acceptance cases.

For $k_t \neq \frac{m\pi}{a}$,

$$J_{mn} = \frac{2\left[1 - \cos(n\pi)\right]}{\pi^2 mn\left(\left(\frac{\lambda_m}{\lambda_t}\right)^2 - 1\right)} \cos\left[\frac{m\pi}{2}\frac{\lambda_m}{\lambda_t}\right] \text{ , for m=1,3,5…} \tag{62}$$

and

$$J_{mn} = \frac{2\left[1 - \cos(n\pi)\right]}{\pi^2 mn\left(\left(\frac{\lambda_m}{\lambda_t}\right)^2 - 1\right)} \sin\left[\frac{m\pi}{2}\frac{\lambda_m}{\lambda_t}\right] \text{ , for m=2,4,6…} \tag{63}$$

For $k_t = \frac{m\pi}{a}$,

$$J_{mn} = \frac{1}{2\pi n}\left[\cos(n\pi) - 1\right] \tag{64}$$

The equation of motion for the modal coordinates is

$$\frac{d^2 u_{mn}}{dt^2} + 2\xi_{mn}\omega_{mn}\frac{du_{mn}}{dt} + \omega_{mn}{}^2 u_{mn} = L(t) \tag{65}$$

$$\frac{d^2 u_{mn}}{dt^2} + 2\xi_{mn}\omega_{mn}\frac{du_{mn}}{dt} + \omega_{mn}{}^2 u_{mn} = \Gamma_{mn}\, p(t) \tag{66}$$

$$\frac{d^2 u_{mn}}{dt^2} + 2\xi_{mn}\omega_{mn}\frac{du_{mn}}{dt} + \omega_{mn}{}^2 u_{mn} = \hat{A}\, J_{mn}\, p(t) \tag{67}$$

The temporal variable response to the applied force transposed to the frequency domain is

$$U_{mn}(\omega) = \left[\frac{\Gamma_{mn}}{\left(\omega_{mn}{}^2 - \omega^2\right) + j2\xi_{mn}\omega\,\omega_{mn}}\right] P(\omega) \tag{68}$$

$$U_{mn}(\omega) = \left[\frac{\hat{A}\, a\, b\, J_{mn}}{\left(\omega_{mn}{}^2 - \omega^2\right) + j2\xi_{mn}\omega\,\omega_{mn}}\right] P(\omega) \tag{69}$$

Recall

$$w(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Phi_{mn}(x,y)\, u_{mn}(t) \tag{70}$$

Transpose to the frequency domain.

$$W(x, y, \omega) = P(\omega) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Gamma_{mn} \Phi_{mn}(x, y)}{\left(\omega_{mn}{}^2 - \omega^2\right) + j 2 \xi_{mn} \omega \omega_{mn}} \tag{71}$$

$$W(x, y, \omega) = P(\omega) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\hat{A}\, a\, b\, J_{mn} \Phi_{mn}(x, y)}{\left(\omega_{mn}{}^2 - \omega^2\right) + j 2 \xi_{mn} \omega \omega_{mn}} \tag{72}$$

The frequency response function relating the displacement to the oblique pressure field is

$$\frac{W(x, y, \omega)}{P(\omega)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Gamma_{mn} \Phi_{mn}(x, y)}{\left(\omega_{mn}{}^2 - \omega^2\right) + j 2 \xi_{mn} \omega \omega_{mn}} \tag{73}$$

$$\frac{W(x, y, \omega)}{P(\omega)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\hat{A}\, a\, b\, J_{mn} \Phi_{mn}(x, y)}{\left(\omega_{mn}{}^2 - \omega^2\right) + j 2 \xi_{mn} \omega \omega_{mn}} \tag{74}$$

The bending moments are

$$M_{xx}(x, y, \omega) = -D\left(\frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2}\right) W(x, y, \omega) \tag{75}$$

$$M_{yy}(x, y, \omega) = -D\left(\frac{\partial^2}{\partial y^2} + \mu \frac{\partial^2}{\partial x^2}\right) W(x, y, \omega) \tag{76}$$

The bending stresses from Reference 3 are

$$\sigma_{xx}(x,y,\omega) = -\frac{E\,\hat{z}}{1-\mu^2}\left(\frac{\partial^2}{\partial x^2} + \mu\frac{\partial^2}{\partial y^2}\right) W(x,y,\omega) \tag{77}$$

$$\sigma_{yy}(x,y,\omega) = -\frac{E\,\hat{z}}{1-\mu^2}\left(\frac{\partial^2}{\partial y^2} + \mu\frac{\partial^2}{\partial x^2}\right) W(x,y,\omega) \tag{78}$$

$$\tau_{xy}(x,y,\omega) = -\frac{E\,\hat{z}}{1+\mu}\left(\frac{\partial^2}{\partial x \partial y} W(x,y,\omega)\right) \tag{79}$$

$\hat{z}$ is the distance from the centerline in the vertical axis

References

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5. D. Segalman, C. Flucher, G. Reese, R Field; An Efficient Method for Calculating RMS von Mises Stress in a Random Vibration Environment, Sandia Report: SAND98-0260, UC-705, 1998.

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APPENDIX A

Case 1: $k_t \neq \frac{m\pi}{a}$

$$L_{mn}(t) = \frac{b[1-\cos(n\pi)]\hat{A}\,p_o}{n\pi}\left\{\int_0^a \sin\left(\frac{m\pi x}{a}\right)\cos(\omega t - k_t x)dx\right\} \tag{A-1}$$

$$L_{mn}(t) = \frac{b[1-\cos(n\pi)]\hat{A}\,p_o}{n\pi}\left\{\sin(\omega t)\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin(k_t x)dx\right.$$
$$\left. + \cos(\omega t)\int_0^a \sin\left(\frac{m\pi x}{a}\right)\cos(k_t x)dx\right\} \tag{A-2}$$

The definite integrals are evaluated using the wxMaxima software program, with the following command:

```
ratsimp(trigexpand(integrate(A*sin(m*%pi*x/a)*sin(k*x)+B*sin(m*%pi*x/a)*cos(k*x),x,0,a)));
positive;
```

By substitution of the results,

$$L_{mn}(t) =$$

$$\frac{b[1-\cos(n\pi)]\hat{A}\,p_o}{n\pi\left(\pi^2 m^2 - a^2 k_t^{\,2}\right)}\left\{\sin(\omega t)\left[-a^2 k_t \cos(k_t a)\sin(m\pi) + m\pi a\cos(k_t a)\cos(m\pi)\right]\right.$$
$$\left. + \cos(\omega t)\left[a^2 k_t \sin(k_t a)\sin(m\pi) + m\pi a\cos(k_t a)\cos(m\pi) - m\pi a\right]\right\} \tag{A-3}$$

Note that for all m values,

$$\sin(m\pi) = 0 \tag{A-4}$$

$L_{mn}(t) =$

$$\frac{b[1-\cos(n\pi)]\hat{A}\,p_o}{n\pi\left(\pi^2 m^2 - a^2 k_t^{\,2}\right)}\{\sin(\omega t)[m\pi a\cos(k_t a)\cos(m\pi)] + \cos(\omega t)[m\pi a\cos(k_t a)\cos(m\pi) - m\pi a]\}$$

(A-5)

$L_{mn}(t) =$

$$\frac{[1-\cos(n\pi)]\hat{A}\,p_o\,a\,b}{\pi^2 mn\left(1-\left(\frac{a\,k_t}{\pi m}\right)^2\right)}\{\sin(\omega t)[\cos(k_t a)\cos(m\pi)] + \cos(\omega t)[\cos(k_t a)\cos(m\pi) - 1]\}$$

(A-6)

Convert the trigonometric term to magnitude and phase format.

$$\{\sin(\omega t)[\sin(k_t a)\cos(m\pi)] + \cos(\omega t)[\cos(k_t a)\cos(m\pi) - 1]\}$$

$$= \sqrt{\left\{[\cos(m\pi)]^2 - 2\cos(k_t a)\cos(m\pi) + 1\right\}}\,\sin(\omega t + \varepsilon)$$

(A-7)

Note that for all m values,

$$[\cos(m\pi)]^2 = 1 \qquad \text{(A-8)}$$

$$\{\sin(\omega t)[\sin(k_t a)\cos(m\pi)] + \cos(\omega t)[\cos(k_t a)\cos(m\pi) - 1]\}$$

$$= \sqrt{\{2 - 2\cos(k_t a)\cos(m\pi)\}}\,\sin(\omega t + \varepsilon)$$

(A-9)

For odd m,

$$\left\{\sin(\omega t)\left[\sin(k_t a)\cos(m\pi)\right]+\cos(\omega t)\left[\cos(k_t a)\cos(m\pi)-1\right]\right\}$$

$$=\sqrt{2\left\{1+\cos(k_t a)\right\}}\sin(\omega t+\varepsilon),\ \text{for } m=1,3,5,\ldots \tag{A-10}$$

$$\left\{\sin(\omega t)\left[\sin(k_t a)\cos(m\pi)\right]+\cos(\omega t)\left[\cos(k_t a)\cos(m\pi)-1\right]\right\}$$

$$=\sqrt{4\cos^2(k_t a/2)}\sin(\omega t+\varepsilon),\ \text{for } m=1,3,5,\ldots \tag{A-11}$$

$$\left\{\sin(\omega t)\left[\sin(k_t a)\cos(m\pi)\right]+\cos(\omega t)\left[\cos(k_t a)\cos(m\pi)-1\right]\right\}$$

$$=2\cos(k_t a/2)\sin(\omega t+\varepsilon),\ \text{for } m=1,3,5,\ldots \tag{A-12}$$

$$L_{mn}(t)=\frac{2\hat{A}\,p_o\left[1-\cos(n\pi)\right]a\,b}{\pi^2 mn\left(1-\left(\frac{a\,k_t}{\pi m}\right)^2\right)}\cos\left[\frac{m\pi}{2}\frac{\lambda_m}{\lambda_t}\right]\sin(\omega t+\varepsilon)\quad \text{, for m=1,3,5\ldots} \tag{A-13}$$

Note the following relationship for the modal wavelength λ_m.

$$\frac{\lambda_m}{2}=\frac{a}{m} \tag{A-14}$$

$$a=\frac{m\lambda_m}{2} \tag{A-15}$$

$$\left(\frac{a\,k_t}{\pi m}\right) = \left(\frac{2\pi a}{\pi m \lambda_t}\right) = \left(\frac{\lambda_m}{\lambda_t}\right) \tag{A-16}$$

$$L_{mn}(t) = \frac{2\hat{A}\,p_o\left[1-\cos(n\pi)\right]a\,b}{\pi^2 mn\left(\left(\frac{\lambda_m}{\lambda_t}\right)^2 - 1\right)} \cos\left[\frac{m\,\pi}{2}\frac{\lambda_m}{\lambda_t}\right]\sin(\omega t + \varepsilon) \quad \text{, for m=1,3,5\ldots}$$

(A-17)

Note that the denominator of the right hand side of equation (A-17) was multiplied by -1 for consistency with other reference. This is acceptable because the phase angle is arbitrary.

For even m,

$$\left\{\sin(\omega t)\left[\sin(k_t a)\cos(m\pi)\right] + \cos(\omega t)\left[\cos(k_t a)\cos(m\pi) - 1\right]\right\}$$

$$= \sqrt{2\left\{1-\cos(k_t a)\right\}}\,\sin(\omega t + \varepsilon), \text{ for } m = 2,4,6,\ldots \tag{A-18}$$

$$\left\{\sin(\omega t)\left[\sin(k_t a)\cos(m\pi)\right] + \cos(\omega t)\left[\cos(k_t a)\cos(m\pi) - 1\right]\right\}$$

$$= \sqrt{4\sin^2(k_t a/2)}\,\sin(\omega t + \varepsilon), \text{ for } m = 2,4,6,\ldots \tag{A-19}$$

$$\left\{\sin(\omega t)\left[\sin(k_t a)\cos(m\pi)\right] + \cos(\omega t)\left[\cos(k_t a)\cos(m\pi) - 1\right]\right\}$$

$$= 2\sin(k_t a/2)\sin(\omega t + \varepsilon), \text{ for } m = 2,4,6,\ldots \tag{A-20}$$

$$L_{mn}(t) = \frac{2\hat{A}\,p_o\left[1-\cos(n\pi)\right]a\,b}{\pi^2 mn\left(1-\left(\frac{a\,k_t}{\pi m}\right)^2\right)}\sin\left[\frac{m\pi}{2}\frac{\lambda_m}{\lambda_t}\right]\sin(\omega t+\varepsilon) \quad \text{, for m=2,4,6…}$$

(A-21)

$$L_{mn}(t) = \frac{2\hat{A}\,p_o\left[1-\cos(n\pi)\right]a\,b}{\pi^2 mn\left(\left(\frac{\lambda_m}{\lambda_t}\right)^2-1\right)}\sin\left[\frac{m\pi}{2}\frac{\lambda_m}{\lambda_t}\right]\sin(\omega t+\varepsilon) \quad \text{, for m=2,4,6…}$$

(A-22)

Note that the denominator of the right hand side of equation (A-22) was multiplied by -1 for consistency with other reference.

APPENDIX B

Case 2: $k_t = \frac{m\pi}{a}$

$$L_{mn}(t) = \frac{b[1-\cos(n\pi)]\hat{A}\,p_o}{n\pi}\left\{\sin(\omega t)\int_0^a \sin^2\left(\frac{m\pi x}{a}\right)dx\right.$$

$$\left.+\cos(\omega t)\int_0^a \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)dx\right\}$$

(B-1)

The definite integrals were evaluated using wxMaxima with the following commands.

```
integrate((sin(m*%pi*x/a))^2,x,0,a);positive;
integrate((sin(m*%pi*x/a))*(cos(m*%pi*x/a)),x,0,a);positive;
```

$$L_{mn}(t) = \frac{b[1-\cos(n\pi)]\hat{A}\,p_o}{n\pi}\left\{\left(-\frac{a}{2}\right)\sin(\omega t)+\left[\frac{a\left(1-\cos^2(\pi m)\right)}{2\pi m}\right]\cos(\omega t)\right\} \quad \text{(B-2)}$$

$$L_{mn}(t) = \frac{[1-\cos(n\pi)]\hat{A}\,p_o ab}{\pi^2\, m\, n}\left\{(-\pi m)\sin(\omega t)+\left(1-\cos^2(\pi m)\right)\cos(\omega t)\right\} \quad \text{(B-3)}$$

$$L_{mn}(t) = \frac{\hat{A}\,p_o a\, b}{2\pi n}[\cos(n\pi)-1]\sin(\omega t) \quad \text{(B-4)}$$

APPENDIX C

<u>Example</u>

Consider a rectangular plate with the following properties:

Boundary Conditions	Simply Supported on All Sides
Material	Aluminum

Thickness	h	=	0.125 inch
Length	a	=	10 inch
Width	b	=	8 inch
Elastic Modulus	E	=	10E+06 lbf/in^2
Mass per Volume	ρ_V	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)
Mass per Area	ρ	=	0.0125 lbm / in^2 (3.24E-05 lbf sec^2/in^3)
Viscous Damping Ratio	ξ	=	0.05

The normal modes and frequency response function analysis are performed via a Matlab script.

The normal modes results are:

Table C-1. Natural Frequency Results, Plate Simply-Supported on all Sides		
fn (Hz)	m	n
302	1	1
656	2	1
855	1	2
1209	2	2
1246	3	1
1777	1	3
1799	3	2
2072	4	1
2131	2	3
2625	4	2
2721	3	3
3067	1	4
3134	5	1
3421	2	4

Now apply a sound pressure wave with variable frequency at an angle of 90 degrees. This is actually a special case called grazing incidence where the propagation vector is parallel to the plate surface.

The fundamental mode shape is shown in Figure C-1. The resulting displacement and transfer functions magnitudes are shown in Figures C-2 and C-3, respectively.

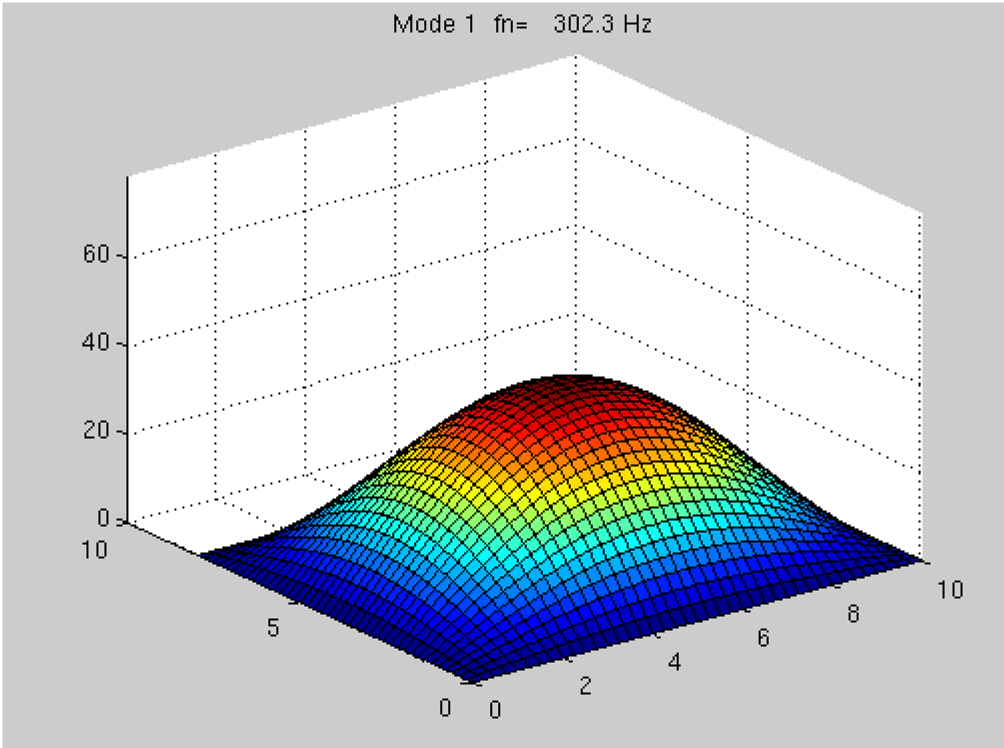


Figure C-1.

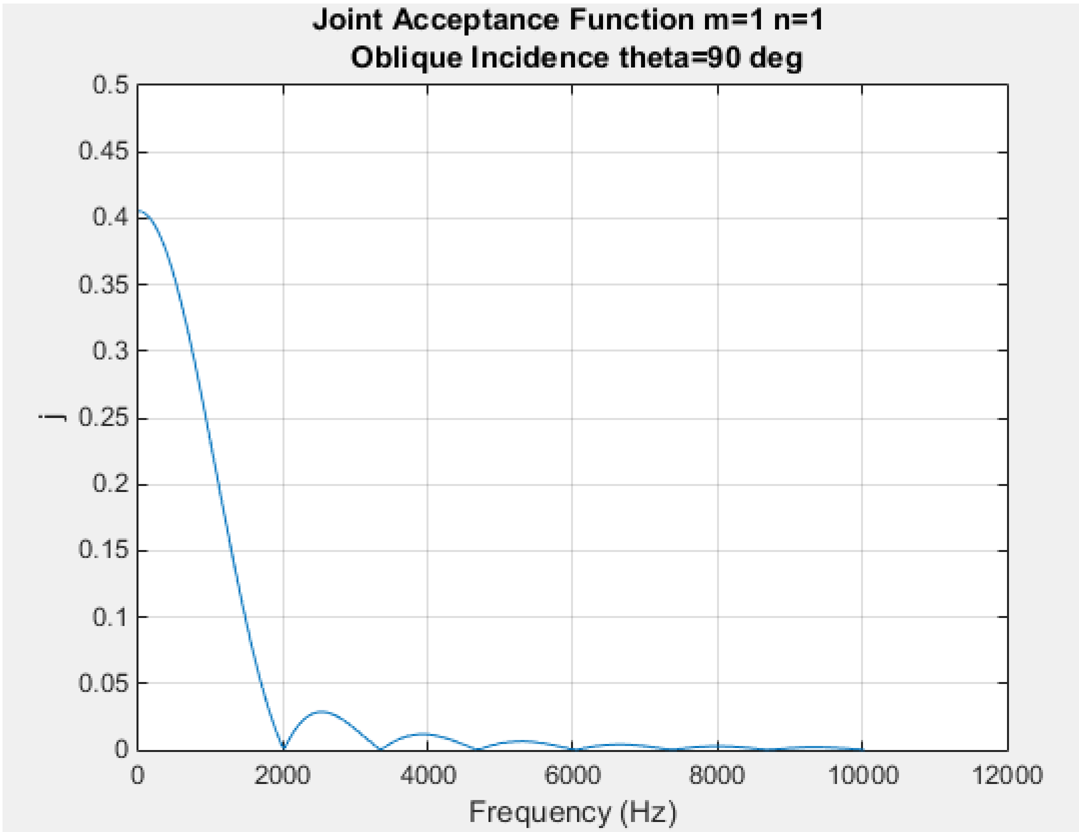

Joint Acceptance Function m=1 n=1
Oblique Incidence theta=90 deg
j
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
2000
4000
6000
8000
10000
12000
Frequency (Hz)

Figure C-2.

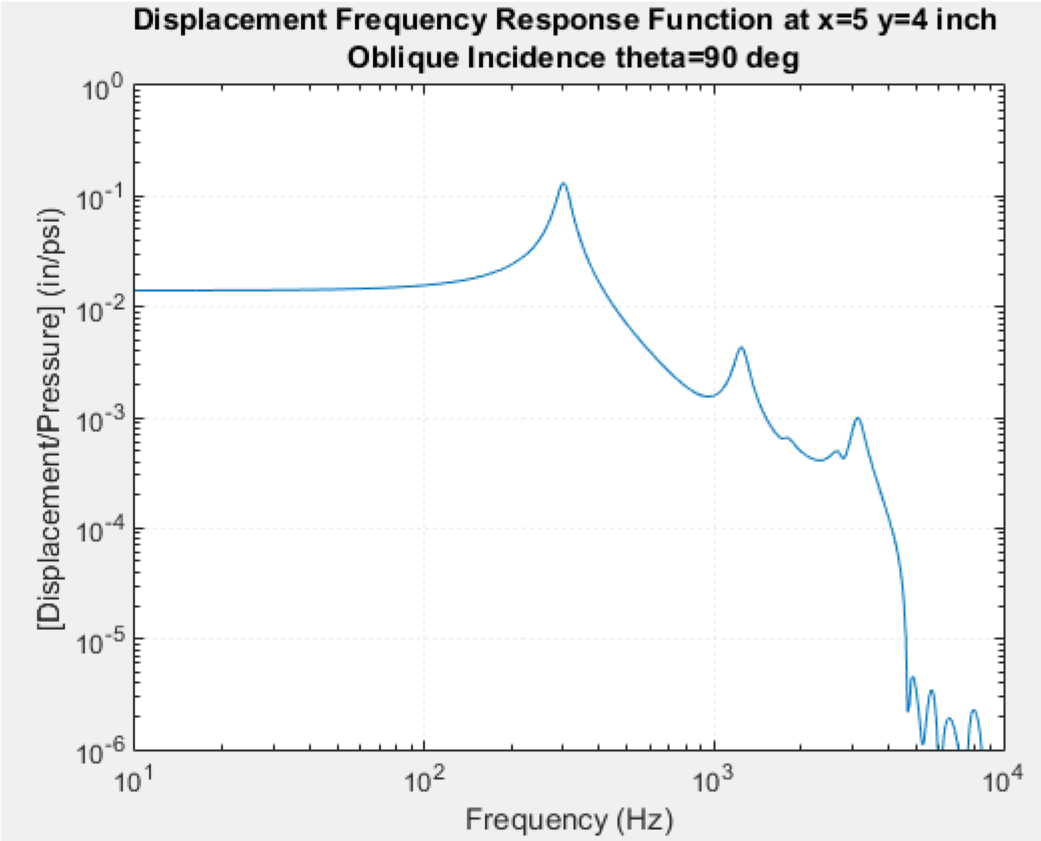


Figure C-3.

The maximum displacement response for oblique incidence at 90 degrees is:

max = 0.132 in/psi at 300.1 Hz

The same plate was subjected to uniform, normal pressure in Reference 6. The maximum displacement for the uniform case was:

max = 0.138 in/psi at 300.1 Hz

Note that the acoustic wavelength at this frequency is about 45 inches. The bending mode wavelength for the fundamental mode is 20 inches, twice the plate length.

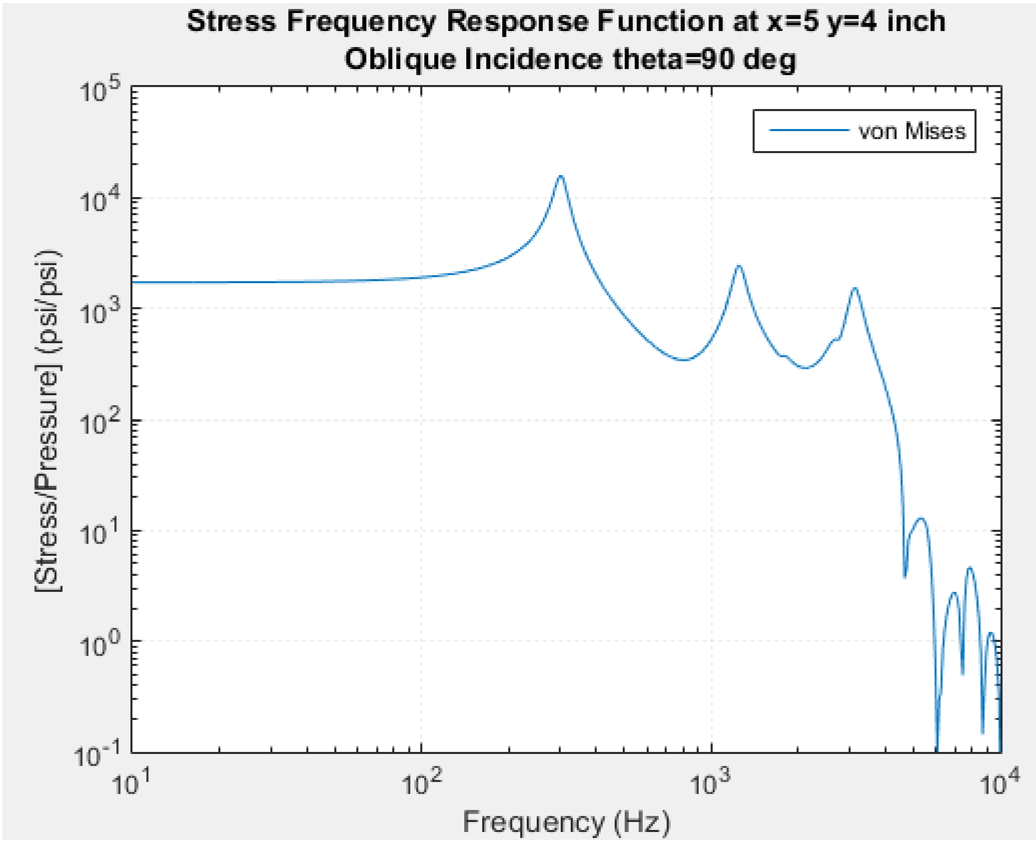


Figure C-4.

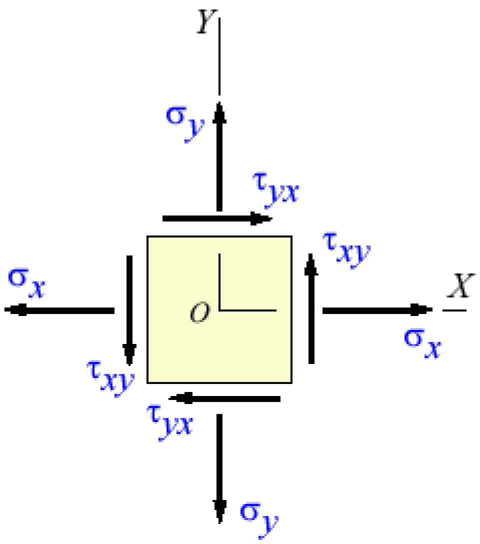
The maximum von Mises stress response for the oblique case is:

max = 15,850 (psi/psi) at 300.1 Hz

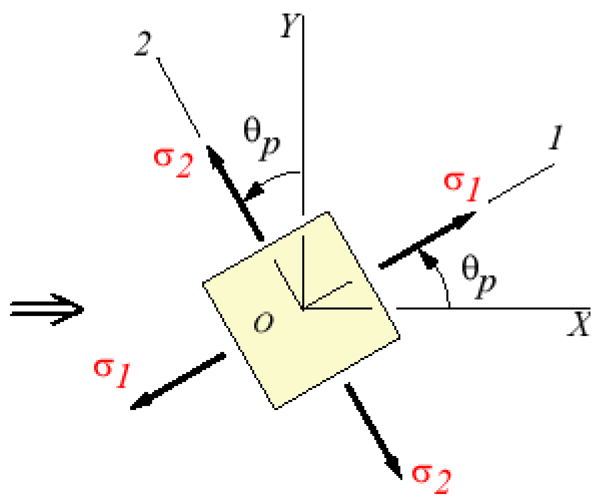
The maximum displacement for the uniform case was:

max = 16,580 in/psi at 300.1 Hz

Principal Stress



Stresses in Given Coordinate System



Principal Stresses

The diagrams are taken from Reference 3.

The principle stresses are

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \qquad \text{(D-1)}$$

The angle at which the shear stress becomes zero is

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{(D-2)}$$

The von Mises stress σ_e is

$$\sigma_e = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} \quad \text{(D-3)}$$

The von Mises stress is used to predict yielding of materials under any loading condition from results of simple uniaxial tensile tests. The von Mises stress satisfies the property that two stress states with equal distortion energy have equal von Mises stress.

An alternate formula from Reference 4 is

$$\sigma_e = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2} \quad \text{(D-4)}$$