The simply-supported plate in Figure 1 is subjected to a uniform pressure.

The following equations are taken from Reference 1.

The governing differential equation is

\[
D \left( \frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} \right) + \rho h \frac{\partial^2 z}{\partial t^2} = P(x,y,t)
\]  

(1)
The plate stiffness factor $D$ is given by

$$D = \frac{Eh^3}{12(1-\mu^2)}$$  \hspace{1cm} (2)

where

- $E$ is the modulus of elasticity
- $\mu$ is Poisson’s ratio
- $h$ is the thickness
- $\rho$ is the mass density (mass/volume)
- $P$ is the applied pressure

Now assume that the pressure field is uniform such that

$$W(t) = P(x, y, t)$$  \hspace{1cm} (2)

The differential equation becomes

$$D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + \rho h \frac{\partial^2 w}{\partial t^2} = W(t)$$  \hspace{1cm} (3)

The mass-normalized mode shapes are

$$Z_{mn} = \frac{2}{\sqrt{\rho a b h}} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right)$$  \hspace{1cm} (4)

$$\frac{\partial}{\partial x} Z_{mn} = \frac{2}{\sqrt{\rho a b h}} \left( \frac{m\pi}{a} \right) \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right)$$  \hspace{1cm} (5)
\[
\frac{\partial^2}{\partial x^2} Z_{mn} = -\frac{2}{\sqrt{\rho a b h}} \left( \frac{m\pi}{a} \right)^2 \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \tag{6}
\]

\[
\frac{\partial}{\partial y} Z_{mn} = -\frac{2}{\sqrt{\rho a b h}} \left( \frac{n\pi}{b} \right) \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) \tag{7}
\]

\[
\frac{\partial^2}{\partial y^2} Z_{mn} = -\frac{2}{\sqrt{\rho a b h}} \left( \frac{n\pi}{b} \right)^2 \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \tag{8}
\]

The natural frequencies are

\[
\omega_{mn} = \sqrt{\frac{D}{\rho h}} \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right) \tag{9}
\]

The participation factors for constant mass density are

\[
\Gamma_{mn} = \rho h \int_0^b \int_0^a Z_{mn}(x, y) \, dx \, dy \tag{10}
\]

\[
\Gamma_{mn} = \left( \frac{2\sqrt{\rho a b h}}{mn \pi^2} \right) [\cos(n\pi) - 1][\cos(m\pi) - 1] \tag{11}
\]
The displacement response $Z(x, y, \omega)$ to the applied force is

$$Z(x, y, \omega) = \frac{1}{\rho h} W(\omega) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\Gamma_{mn} Z_{mn}(x, y)}{\left( \omega_{mn}^2 - \omega^2 \right) + j 2 \xi_{mn} \omega \omega_{mn}} \right\}$$  \hfill (12)

$$Z(x, y, \omega) = \frac{1}{\rho h} W(\omega) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{2 \sqrt{\rho a b h}}{mn \pi^2} \left[ \cos(n\pi) - 1 \right] \left[ \cos(m\pi) - 1 \right] \frac{2}{\sqrt{\rho ab h}} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \right\} \frac{\left( \omega_{mn}^2 - \omega^2 \right) + j 2 \xi_{mn} \omega \omega_{mn}} \right\}$$  \hfill (13)

$$Z(x, y, \omega) = \frac{4}{\rho h \pi^2} W(\omega) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{1}{mn} \left[ \cos(n\pi) - 1 \right] \left[ \cos(m\pi) - 1 \right] \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \right\} \frac{\left( \omega_{mn}^2 - \omega^2 \right) + j 2 \xi_{mn} \omega \omega_{mn}} \right\}$$  \hfill (14)

The bending moments are

$$M_{xx}(x, y, \omega) = -D \left\{ \frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2} \right\} Z(x, y, \omega)$$  \hfill (15)

$$M_{yy}(x, y, \omega) = -D \left\{ \frac{\partial^2}{\partial y^2} + \mu \frac{\partial^2}{\partial x^2} \right\} Z(x, y, \omega)$$  \hfill (16)
The bending stresses from Reference 2 are

\[
\sigma_{xx}(x, y, \omega) = -\frac{E\hat{z}}{1-\mu^2} \left( \frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2} \right) Z(x, y, \omega)
\]  \hspace{1cm} (17)

\[
\sigma_{yy}(x, y, \omega) = -\frac{E\hat{z}}{1-\mu^2} \left( \frac{\partial^2}{\partial y^2} + \mu \frac{\partial^2}{\partial x^2} \right) Z(x, y, \omega)
\]  \hspace{1cm} (18)

\[
\tau_{xy}(x, y, \omega) = -\frac{E\hat{z}}{1+\mu} \frac{\partial^2}{\partial x \partial y} Z(x, y, \omega)
\]  \hspace{1cm} (19)

\(\hat{z}\) is the distance from the centerline in the vertical axis

References

1. T. Irvine, Natural Frequencies of Rectangular Plate Bending Modes, Revision B, Vibrationdata, 2011.


APPENDIX A

Example

Consider a rectangular plate with the following properties:

<table>
<thead>
<tr>
<th>Boundary Conditions</th>
<th>Simply Supported on All Sides</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>Aluminum</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Thickness</th>
<th>h</th>
<th>0.125 inch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>a</td>
<td>10 inch</td>
</tr>
<tr>
<td>Width</td>
<td>b</td>
<td>8 inch</td>
</tr>
<tr>
<td>Elastic Modulus</td>
<td>E</td>
<td>10E+06 lbf/in^2</td>
</tr>
<tr>
<td>Mass per Volume</td>
<td>( \rho_v )</td>
<td>0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)</td>
</tr>
<tr>
<td>Mass per Area</td>
<td>( \rho )</td>
<td>0.0125 lbm / in^2 (3.24E-05 lbf sec^2/in^3)</td>
</tr>
<tr>
<td>Viscous Damping Ratio</td>
<td>( \xi )</td>
<td>0.05</td>
</tr>
</tbody>
</table>

The normal modes and frequency response function analysis are performed via a Matlab script.
The normal modes results are:

<table>
<thead>
<tr>
<th>fn (Hz)</th>
<th>m</th>
<th>n</th>
<th>Participation Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>302</td>
<td>1</td>
<td>1</td>
<td>0.04126</td>
</tr>
<tr>
<td>656</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>855</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1209</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1246</td>
<td>3</td>
<td>1</td>
<td>0.01375</td>
</tr>
<tr>
<td>1777</td>
<td>1</td>
<td>3</td>
<td>0.01375</td>
</tr>
<tr>
<td>1799</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2072</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2131</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2625</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2721</td>
<td>3</td>
<td>3</td>
<td>0.004584</td>
</tr>
<tr>
<td>3067</td>
<td>1</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>3134</td>
<td>5</td>
<td>1</td>
<td>0.008251</td>
</tr>
<tr>
<td>3421</td>
<td>2</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that the mode shape and participation factors are considered as dimensionless, but they must be consistent with respect to one another.

The resulting displacement and transfer functions magnitudes are shown in Figures A-1 and A-2, respectively.
Figure A-1.

The maximum displacement response is: \( \text{max} = 0.138 \text{ in/psi at } 300.1 \text{ Hz} \)
Figure A-2.

The maximum von Mises stress response is:  max = 1.658e+04 (psi/psi) at  300.1 Hz
**Principal Stress**

The diagrams are taken from Reference 3.

The principle stresses are

\[
\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{B-1}
\]

The angle at which the shear stress becomes zero is

\[
\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{B-2}
\]
The von Mises stress $\sigma_e$ is

$$\sigma_e = \sqrt{\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2}$$  \hspace{1cm} (B-3)

The von Mises stress is used to predict yielding of materials under any loading condition from results of simple uniaxial tensile tests. The von Mises stress satisfies the property that two stress states with equal distortion energy have equal von Mises stress.

An alternate formula from Reference 4 is

$$\sigma_e = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx} \sigma_{yy} + 3 \tau_{xy}^2}$$  \hspace{1cm} (B-4)