

RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A
TERMINAL SAWTOOTH PULSE BASE EXCITATION
Revision C

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Introduction

Consider the single-degree-of-freedom system in Figure 1.

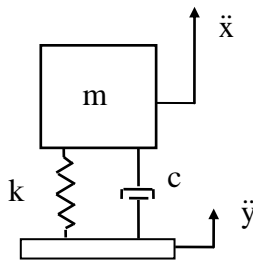


Figure 1.

where

- m is the mass
- c is the viscous damping coefficient
- k is the stiffness
- x is the absolute displacement of the mass
- y is the base input displacement

A free-body diagram is shown in Figure 2.

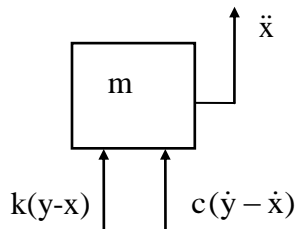


Figure 2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \quad (1)$$

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \quad (2)$$

Let $z = x - y$ (relative displacement)

$$\dot{z} = \dot{x} - \dot{y}$$

$$\ddot{z} = \ddot{x} - \ddot{y}$$

$$\ddot{x} = \ddot{z} + \ddot{y}$$

Substituting the relative displacement terms into equation (2) yields

$$m(\ddot{z} + \ddot{y}) = -c\dot{z} - kz \quad (3)$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \quad (4)$$

Dividing through by mass yields

$$\ddot{z} + (c/m)\dot{z} + (k/m)z = -\ddot{y} \quad (5)$$

By convention,

$$(c/m) = 2\xi\omega_n$$

$$(k/m) = \omega_n^2$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substitute the convention terms into equation (5).

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2z = -\ddot{y} \quad (6)$$

Terminal Sawtooth Pulse

Consider the pulse given by equation (7).

$$\ddot{y}(t) = \begin{cases} A\left(\frac{t}{T}\right), & 0 \leq t \leq T \\ 0, & t > T \end{cases} \quad (7)$$

The equation of motion becomes

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z = -A\left(\frac{t}{T}\right), \quad 0 \leq t \leq T \quad (8)$$

Now take the Laplace transform.

$$L\{\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z\} = L\{-At/T\} \quad (9)$$

$$\begin{aligned} s^2 Z(s) - sz(0) - \dot{z}(0) \\ + 2\xi\omega_n s Z(s) - 2\xi\omega_n \dot{z}(0) \\ + \omega_n^2 Z(s) &= \frac{-A/T}{s^2} \end{aligned} \quad (10)$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Z(s) + \{-1\}\dot{z}(0) + \{-s - 2\xi\omega_n\}z(0) = \frac{-A/T}{s^2} \quad (11)$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Z(s) = \dot{z}(0) + \{s + 2\xi\omega_n\}z(0) - \frac{A/T}{s^2} \quad (12)$$

$$Z(s) = \left\{ \frac{\dot{z}(0) + \{s + 2\xi\omega_n\}z(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} - \left\{ \frac{A/T}{s^2} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (13)$$

Let

$$Z(s) = Z_n(s) + Z_f(s) \quad (14)$$

where

$$Z_n(s) = \left\{ \frac{\dot{z}(0) + \{s + 2\xi\omega_n\}z(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (15)$$

$$Z_f(s) = - \left\{ \frac{A/T}{s^2} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (16)$$

Consider the denominator term,

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2 - (\xi\omega_n)^2 \quad (17)$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2) \quad (18)$$

Now define the damped natural frequency,

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \quad (19)$$

Substitute equation (19) into (18),

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \quad (20)$$

Substitute equation (20) into (16).

$$Z_n(s) = \left\{ \frac{\dot{z}(0) + \{s + 2\xi\omega_n\}z(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad (21)$$

Rearrange the terms into a convenient format prior to the inverse Laplace transform.

$$Z_n(s) = \left\{ \frac{(s + \xi\omega_n)z(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad (22)$$

$$Z_n(s) = \left\{ \frac{(s + \xi\omega_n)z(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} + \left\{ \frac{\left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \omega_d}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad (23)$$

Take the inverse Laplace transform using Reference 1.

$$z_n(t) = z(0) \exp(-\xi\omega_n t) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \exp(-\xi\omega_n t) \sin(\omega_d t) \quad (24)$$

$$z_n(t) = \exp(-\xi\omega_n t) \left\{ z(0) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} \quad (25)$$

Take the first derivative to determine the relative velocity.

$$\begin{aligned} \dot{z}_n(t) = & -\xi\omega_n \exp(-\xi\omega_n t) \left\{ z(0) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} \\ & + \exp(-\xi\omega_n t) \left\{ -\omega_d z(0) \sin(\omega_d t) + \left\{ \dot{z}(0) + (\xi\omega_n)z(0) \right\} \cos(\omega_d t) \right\} \end{aligned} \quad (26)$$

$$\begin{aligned} \dot{z}_n(t) = & \exp(-\xi\omega_n t) \left\{ -\xi\omega_n z(0) \cos(\omega_d t) - \xi\omega_n \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} \\ & + \exp(-\xi\omega_n t) \left\{ -\omega_d z(0) \sin(\omega_d t) + \left\{ \dot{z}(0) + (\xi\omega_n)z(0) \right\} \cos(\omega_d t) \right\} \end{aligned} \quad (27)$$

$$\begin{aligned}
\dot{z}_n(t) = & \\
& \exp(-\xi\omega_n t) \{ \{-\xi\omega_n z(0) + \dot{z}(0) + (\xi\omega_n)z(0)\} \cos(\omega_d t) \} \\
& + \exp(-\xi\omega_n t) \left\{ \left\{ -\omega_d z(0) - \xi\omega_n \left[\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right] \right\} \sin(\omega_d t) \right\}
\end{aligned} \tag{28}$$

$$\begin{aligned}
\dot{z}_n(t) = & \exp(-\xi\omega_n t) \{ \dot{z}(0) \cos(\omega_d t) \} \\
& + \exp(-\xi\omega_n t) \left\{ \frac{1}{\omega_d} \left\{ -\omega_d^2 z(0) - \xi\omega_n [\dot{z}(0) + (\xi\omega_n)z(0)] \right\} \sin(\omega_d t) \right\}
\end{aligned} \tag{29}$$

$$\begin{aligned}
\dot{z}_n(t) = & \exp(-\xi\omega_n t) \{ \dot{z}(0) \cos(\omega_d t) \} \\
& + \exp(-\xi\omega_n t) \left\{ \frac{1}{\omega_d} \left\{ -\omega_d^2 z(0) - \xi\omega_n \dot{z}(0) - (\xi\omega_n)^2 z(0) \right\} \sin(\omega_d t) \right\}
\end{aligned} \tag{30}$$

$$\begin{aligned}
\dot{z}_n(t) = & \exp(-\xi\omega_n t) \{ \dot{z}(0) \cos(\omega_d t) \} \\
& + \exp(-\xi\omega_n t) \left\{ \frac{1}{\omega_d} \left\{ -\omega_n^2 (1 - \xi^2) z(0) - \xi\omega_n \dot{z}(0) - (\xi\omega_n)^2 z(0) \right\} \sin(\omega_d t) \right\}
\end{aligned} \tag{31}$$

$$\begin{aligned}
\dot{z}_n(t) = & \exp(-\xi\omega_n t) \{ \dot{z}(0) \cos(\omega_d t) \} \\
& + \exp(-\xi\omega_n t) \left\{ \frac{1}{\omega_d} \left\{ (-\omega_n^2 + \xi^2 \omega_n^2) z(0) - \xi\omega_n \dot{z}(0) - (\xi\omega_n)^2 z(0) \right\} \sin(\omega_d t) \right\}
\end{aligned} \tag{32}$$

$$\begin{aligned} \dot{z}_n(t) = & \exp(-\xi\omega_n t) \{ \dot{z}(0) \cos(\omega_d t) \} \\ & + \exp(-\xi\omega_n t) \left\{ \frac{1}{\omega_d} \{ -\omega_n^2 z(0) - \xi\omega_n \dot{z}(0) \} \sin(\omega_d t) \right\} \end{aligned} \quad (33)$$

$$\dot{z}_n(t) = \exp(-\xi\omega_n t) \left\{ \dot{z}(0) \cos(\omega_d t) + \frac{1}{\omega_d} \{ -\omega_n^2 z(0) - \xi\omega_n \dot{z}(0) \} \sin(\omega_d t) \right\} \quad (34)$$

$$\dot{z}_n(t) = \exp(-\xi\omega_n t) \left\{ \dot{z}(0) \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \{ -\omega_n z(0) - \xi \dot{z}(0) \} \sin(\omega_d t) \right\} \quad (35)$$

Take the second derivative to determine the acceleration.

$$\begin{aligned} \ddot{z}_n(t) = & -\xi\omega_n \exp(-\xi\omega_n t) \left\{ \dot{z}(0) \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \{ -\omega_n z(0) - \xi \dot{z}(0) \} \sin(\omega_d t) \right\} \\ & + \exp(-\xi\omega_n t) \{ -\omega_d \dot{z}(0) \sin(\omega_d t) + \omega_n \{ -\omega_n z(0) - \xi \dot{z}(0) \} \cos(\omega_d t) \} \end{aligned} \quad (36)$$

$$\begin{aligned} \ddot{z}_n(t) = & \exp(-\xi\omega_n t) \left\{ -\xi\omega_n \dot{z}(0) \cos(\omega_d t) - \frac{\xi\omega_n^2}{\omega_d} \{ -\omega_n z(0) - \xi \dot{z}(0) \} \sin(\omega_d t) \right\} \\ & + \exp(-\xi\omega_n t) \{ -\omega_d \dot{z}(0) \sin(\omega_d t) + \omega_n \{ -\omega_n z(0) - \xi \dot{z}(0) \} \cos(\omega_d t) \} \end{aligned} \quad (37)$$

$$\begin{aligned}\ddot{z}_n(t) &= \exp(-\xi\omega_n t)\{-\xi\omega_n \dot{z}(0) + \omega_n\{-\omega_n z(0) - \xi \dot{z}(0)\}\}\cos(\omega_d t) \\ &+ \exp(-\xi\omega_n t)\left\{-\omega_d \dot{z}(0) - \frac{\xi\omega_n^2}{\omega_d}\{-\omega_n z(0) - \xi \dot{z}(0)\}\right\}\sin(\omega_d t)\end{aligned}\tag{38}$$

$$\begin{aligned}\ddot{z}_n(t) &= -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t) \\ &+ \frac{1}{\omega_d} \exp(-\xi\omega_n t)\left\{-\omega_d^2 \dot{z}(0) - \xi\omega_n^2\{-\omega_n z(0) - \xi \dot{z}(0)\}\right\}\sin(\omega_d t)\end{aligned}\tag{39}$$

$$\begin{aligned}\ddot{z}_n(t) &= -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t) \\ &+ \frac{1}{\omega_d} \exp(-\xi\omega_n t)\left\{-\omega_d^2 \dot{z}(0) + \xi\omega_n^3 z(0) + \xi^2\omega_n^2 \dot{z}(0)\right\}\sin(\omega_d t)\end{aligned}\tag{40}$$

$$\begin{aligned}\ddot{z}_n(t) &= -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t) \\ &+ \frac{1}{\omega_d} \exp(-\xi\omega_n t)\left\{-\omega_n^2(1-\xi^2)\dot{z}(0) + \xi\omega_n^3 z(0) + \xi^2\omega_n^2 \dot{z}(0)\right\}\sin(\omega_d t)\end{aligned}\tag{41}$$

$$\begin{aligned}\ddot{z}_n(t) &= -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t) \\ &+ \frac{\omega_n^2}{\omega_d} \exp(-\xi\omega_n t)\left\{-\left(1-\xi^2\right)\dot{z}(0) + \xi\omega_n z(0) + \xi^2 \dot{z}(0)\right\}\sin(\omega_d t)\end{aligned}\tag{42}$$

$$\begin{aligned}\ddot{z}_n(t) &= -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t) \\ &+ \frac{\omega_n^2}{\omega_d} \exp(-\xi\omega_n t)\left\{\xi\omega_n z(0) + \left(-1 + 2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t)\end{aligned}\tag{43}$$

$$\begin{aligned}\ddot{z}_n(t) = & -\omega_n \exp(-\xi\omega_n t) \{ \omega_n z(0) + 2\xi \dot{z}(0) \} \cos(\omega_d t) \\ & - \frac{\omega_n^2}{\omega_d} \exp(-\xi\omega_n t) \left\{ -\xi\omega_n z(0) + (1-2\xi^2) \dot{z}(0) \right\} \sin(\omega_d t)\end{aligned}\quad (44)$$

$$\begin{aligned}\ddot{z}_n(t) = & \\ & - \exp(-\xi\omega_n t) \left\{ \omega_n [\omega_n z(0) + 2\xi \dot{z}(0)] \cos(\omega_d t) + \frac{\omega_n^2}{\omega_d} \left[-\xi\omega_n z(0) + (1-2\xi^2) \dot{z}(0) \right] \sin(\omega_d t) \right\}\end{aligned}\quad (45)$$

$$\begin{aligned}\ddot{z}_n(t) = & \\ & - \omega_n \exp(-\xi\omega_n t) \left\{ [\omega_n z(0) + 2\xi \dot{z}(0)] \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \left[-\xi\omega_n z(0) + (1-2\xi^2) \dot{z}(0) \right] \sin(\omega_d t) \right\}\end{aligned}\quad (46)$$

Recall equation (16).

$$Z_f(s) = - \left\{ \frac{A/T}{s^2} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (47)$$

Expand into partial fractions using Appendix A.

$$\left\{ \frac{1}{s^2} \right\} \left\{ \frac{1}{s^2 + \alpha s + \beta} \right\} = \left\{ \frac{1}{\beta^2} \right\} \left\{ \frac{-\alpha s + \beta}{s^2} \right\} + \left\{ \frac{1}{\beta^2} \right\} \left\{ \frac{\alpha s + \alpha^2 - \beta}{s^2 + \alpha s + \beta} \right\} \quad (48)$$

$$\alpha = 2\xi\omega_n \quad (49)$$

$$\beta = \omega_n^2 \quad (50)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^4} \right\} \left\{ \frac{-2\xi\omega_n s + \omega_n^2}{s^2} + \frac{2\xi\omega_n s + (2\xi\omega_n)^2 - \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (51)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^3} \right\} \left\{ \frac{-2\xi s + \omega_n}{s^2} + \frac{2\xi s + (2\xi)^2 \omega_n - \omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (52)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^3} \right\} \left\{ \frac{-2\xi}{s} + \frac{\omega_n}{s^2} + \frac{2\xi s + (2\xi)^2 \omega_n - \omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (53)$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \quad (54)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^3} \right\} \left\{ \frac{-2\xi}{s} + \frac{\omega_n}{s^2} + \frac{2\xi s + (2\xi)^2 \omega_n - \omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad (55)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^3} \right\} \left\{ \frac{-2\xi}{s} + \frac{\omega_n}{s^2} + \frac{2\xi s + \omega_n \left[(2\xi)^2 - 1 \right]}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad (56)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^3} \right\} \left\{ \frac{-2\xi}{s} + \frac{\omega_n}{s^2} + [2\xi] \left[\frac{s + \omega_n \left[(2\xi) - \frac{1}{2\xi} \right]}{(s + \xi\omega_n)^2 + \omega_d^2} \right] \right\} \quad (57)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^3} \right\} \left\{ \frac{-2\xi}{s} + \frac{\omega_n}{s^2} + [2\xi] \left[\frac{s + \frac{\omega_n}{2\xi} [(2\xi)^2 - 1]}{(s + \xi\omega_n)^2 + \omega_d^2} \right] \right\} \quad (58)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^3} \right\} \left\{ \frac{-2\xi}{s} + \frac{\omega_n}{s^2} + [2\xi] \left[\frac{s + \frac{\omega_n}{2\xi} [(2\xi)^2 - 1]}{(s + \xi\omega_n)^2 + \omega_d^2} \right] \right\} \quad (59)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^3} \right\} \left\{ \frac{-2\xi}{s} + \frac{\omega_n}{s^2} + 2\xi \left[\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right] + \left[\frac{-\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right] \right\} \quad (60)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^3} \right\} \left\{ \frac{-2\xi}{s} + \frac{\omega_n}{s^2} + 2\xi \left[\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right] + \left[\frac{-\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right] \right\} \quad (61)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^3} \right\} \left\{ \frac{-2\xi}{s} + \frac{\omega_n}{s^2} + 2\xi \left[\frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right] + \left[\frac{2\xi^2\omega_n - \omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right] \right\} \quad (62)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^3} \right\} \left\{ \frac{-2\xi}{s} + \frac{\omega_n}{s^2} + 2\xi \left[\frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right] + \omega_n \left[\frac{2\xi^2 - 1}{(s + \xi\omega_n)^2 + \omega_d^2} \right] \right\} \quad (63)$$

$$Z_f(s) = -\left\{ \frac{A/T}{\omega_n^2} \right\} \left\{ \frac{-2\xi}{\omega_n s} + \frac{1}{s^2} + \left[\frac{2\xi}{\omega_n} \right] \left[\frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right] + \left[\frac{2\xi^2 - 1}{(s + \xi\omega_n)^2 + \omega_d^2} \right] \right\} \quad (64)$$

$$Z_f(s) = \left\{ \frac{A/T}{\omega_n^2} \right\} \left\{ \frac{2\xi}{\omega_n s} - \frac{1}{s^2} - \left[\frac{2\xi}{\omega_n} \right] \left[\frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right] - \left[\frac{2\xi^2 - 1}{(s + \xi\omega_n)^2 + \omega_d^2} \right] \right\} \quad (65)$$

Take the inverse Laplace transform using Reference 1. The relative displacement is

$$z_f(t) = \frac{A}{\omega_n^2 T} \left\{ \frac{2\xi}{\omega_n} - t - \exp(-\xi\omega_n t) \left[\frac{2\xi}{\omega_n} \cos(\omega_d t) + \frac{1}{\omega_d} (2\xi^2 - 1) \sin(\omega_d t) \right] \right\} \quad (66)$$

$$z_n(t) = \exp(-\xi\omega_n t) \left\{ z(0) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} \quad (67)$$

The total relative displacement is

$$z(t) = \exp(-\xi\omega_n t) \left\{ z(0) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} + \frac{A}{\omega_n^2 T} \left\{ \frac{2\xi}{\omega_n} - t - \exp(-\xi\omega_n t) \left[\frac{2\xi}{\omega_n} \cos(\omega_d t) + \frac{1}{\omega_d} (2\xi^2 - 1) \sin(\omega_d t) \right] \right\}, \quad 0 \leq t \leq T \quad (68)$$

The relative velocity is

$$\begin{aligned}
 \dot{z}(t) = & \exp(-\xi\omega_n t) \left\{ \dot{z}(0) \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \{-\omega_n z(0) - \xi \dot{z}(0)\} \sin(\omega_d t) \right\} \\
 & - \frac{A}{\omega_n^2 T} \\
 & + \frac{A}{\omega_n^2 T} \left\{ \xi\omega_n \exp(-\xi\omega_n t) \left[\frac{2\xi}{\omega_n} \cos(\omega_d t) + \frac{1}{\omega_d} (2\xi^2 - 1) \sin(\omega_d t) \right] \right\} \\
 & + \frac{A}{\omega_n^2 T} \left\{ -\exp(-\xi\omega_n t) \left[\frac{-2\xi\omega_d}{\omega_n} \sin(\omega_d t) + (2\xi^2 - 1) \cos(\omega_d t) \right] \right\} \\
 & \qquad \qquad \qquad 0 \leq t \leq T
 \end{aligned}
 \tag{69}$$

Note the previously derived relative velocity natural response term was applied in the derivation of equation (69) for simplicity.

The relative acceleration is

$$\begin{aligned}
\ddot{z}(t) = & -\omega_n \exp(-\xi\omega_n t) \{ \omega_n z(0) + 2\xi \dot{z}(0) \} \cos(\omega_d t) \\
& - \frac{\omega_n^2}{\omega_d} \exp(-\xi\omega_n t) \left\{ -\xi\omega_n z(0) + (1 - 2\xi^2) \dot{z}(0) \right\} \sin(\omega_d t) \\
& - \frac{\xi A}{\omega_n^2 T} \left\{ \xi\omega_n \exp(-\xi\omega_n t) \left[\frac{2\xi}{\omega_n} \cos(\omega_d t) + \frac{1}{\omega_d} (2\xi^2 - 1) \sin(\omega_d t) \right] \right\} \\
& - \frac{\xi A}{\omega_n^2 T} \left\{ -\exp(-\xi\omega_n t) \left[\frac{-2\xi\omega_d}{\omega_n} \sin(\omega_d t) + (2\xi^2 - 1) \cos(\omega_d t) \right] \right\} \\
& + \frac{A\omega_d}{\omega_n^2 T} \left\{ \xi\omega_n \exp(-\xi\omega_n t) \left[\frac{-2\xi}{\omega_n} \sin(\omega_d t) + \frac{1}{\omega_d} (2\xi^2 - 1) \cos(\omega_d t) \right] \right\} \\
& + \frac{A\omega_d}{\omega_n^2 T} \left\{ -\exp(-\xi\omega_n t) \left[\frac{-2\xi\omega_d}{\omega_n} \cos(\omega_d t) - (2\xi^2 - 1) \sin(\omega_d t) \right] \right\}
\end{aligned}$$

$0 \leq t \leq T$

(70)

Note the previously derived relative acceleration natural response term was applied in the derivation of equation (70) for simplicity.

The absolute acceleration is

$$\begin{aligned}
\ddot{x}(t) = & -\omega_n \exp(-\xi\omega_n t) \{ \omega_n z(0) + 2\xi \dot{z}(0) \} \cos(\omega_d t) \\
& - \frac{\omega_n^2}{\omega_d} \exp(-\xi\omega_n t) \left\{ -\xi\omega_n z(0) + (1 - 2\xi^2) \dot{z}(0) \right\} \sin(\omega_d t) \\
& - \frac{\xi A}{\omega_n^2 T} \left\{ \xi\omega_n \exp(-\xi\omega_n t) \left[\frac{2\xi}{\omega_n} \cos(\omega_d t) + \frac{1}{\omega_d} (2\xi^2 - 1) \sin(\omega_d t) \right] \right\} \\
& - \frac{\xi A}{\omega_n^2 T} \left\{ -\exp(-\xi\omega_n t) \left[\frac{-2\xi\omega_d}{\omega_n} \sin(\omega_d t) + (2\xi^2 - 1) \cos(\omega_d t) \right] \right\} \\
& + \frac{A\omega_d}{\omega_n^2 T} \left\{ \xi\omega_n \exp(-\xi\omega_n t) \left[\frac{-2\xi}{\omega_n} \sin(\omega_d t) + \frac{1}{\omega_d} (2\xi^2 - 1) \cos(\omega_d t) \right] \right\} \\
& + \frac{A\omega_d}{\omega_n^2 T} \left\{ -\exp(-\xi\omega_n t) \left[\frac{-2\xi\omega_d}{\omega_n} \cos(\omega_d t) - (2\xi^2 - 1) \sin(\omega_d t) \right] \right\} + A \left(\frac{t}{T} \right), \\
& \qquad \qquad \qquad 0 \leq t \leq T
\end{aligned} \tag{71}$$

The solution for $t > T$ is the free vibration solution referenced to time T .

As an aside, recall

$$\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n^2 z = -\ddot{y}(t), \qquad 0 \leq t \leq T \tag{72}$$

$$\ddot{z} = -\ddot{y}(t) - 2\xi\omega_n\dot{z} - \omega_n^2 z \quad (73)$$

Also,

$$\ddot{x} = \ddot{z} + \ddot{y} \quad (74)$$

Substitute equation (73) into (74).

$$\ddot{x} = -2\xi\omega_n\dot{z} - \omega_n^2 z \quad (75)$$

Equation (80) is simple for computer programming purposes.

Note that equation (75) is valid for both the base excitation duration and the free vibration duration.

References

1. T. Irvine, Partial Fractions in Shock and Vibration Analysis Rev K, Vibrationdata, 2013.
2. T. Irvine, Table of Laplace Transforms Rev J, Vibrationdata, 2011.

APPENDIX A

Partial Fraction Expansion

$$\left\{ \frac{1}{s^2} \right\} \left\{ \frac{1}{s^2 + \alpha s + \beta} \right\} = \frac{as + b}{s^2} + \frac{cs + d}{s^2 + \alpha s + \beta} \quad (\text{A-1})$$

$$1 = [as + b][s^2 + \alpha s + \beta] + [cs + d][s^2] \quad (\text{A-2})$$

$$1 = [as^3 + a\alpha s^2 + a\beta s] + [bs^2 + b\alpha s + b\beta] + [cs^3 + ds^2] \quad (\text{A-3})$$

$$1 = [a + c]s^3 + [a\alpha + b + d]s^2 + [a\beta + b\alpha]s + b\beta \quad (\text{A-4})$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \alpha & 1 & 0 & 1 \\ \beta & \alpha & 0 & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (\text{A-5})$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -\alpha & 1 \\ 0 & \alpha & -\beta & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (\text{A-6})$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & \alpha & -\beta & 0 \\ 0 & 1 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (\text{A-7})$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -\beta/\alpha & 0 \\ 0 & 1 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\beta \\ 0 \\ 0 \end{bmatrix} \quad (\text{A-8})$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\beta/\alpha & 0 \\ 0 & 0 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\beta \\ -1/\beta \\ -1/\beta \end{bmatrix} \quad (\text{A-9})$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/\alpha \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\beta \\ \alpha/\beta^2 \\ 1/(\alpha\beta) \end{bmatrix} \quad (\text{A-10})$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1/\alpha \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -\alpha/\beta^2 \\ 1/\beta \\ \alpha/\beta^2 \\ -[\alpha/\beta^2] + [1/(\alpha\beta)] \end{bmatrix} \quad (\text{A-11})$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -\alpha/\beta^2 \\ 1/\beta \\ \alpha/\beta^2 \\ [\alpha^2/\beta^2] - [1/\beta] \end{bmatrix} \quad (\text{A-12})$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -\alpha \\ \beta \\ \alpha \\ \alpha^2 - \beta \end{bmatrix} \begin{bmatrix} 1 \\ \beta^2 \end{bmatrix} \quad (\text{A-13})$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -\alpha & 1 \\ 0 & \alpha & -\beta & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (\text{A-14})$$

$$\left\{ \frac{1}{s^2} \right\} \left\{ \frac{1}{s^2 + \alpha s + \beta} \right\} = \left\{ \frac{1}{\beta^2} \right\} \left\{ \frac{-\alpha s + \beta}{s^2} \right\} + \left\{ \frac{1}{\beta^2} \right\} \left\{ \frac{\alpha s + \alpha^2 - \beta}{s^2 + \alpha s + \beta} \right\} \quad (\text{A-15})$$