Shock Response Spectrum Calculation – An Improvement of the Smallwood Algorithm

Kjell Ahlin
Ingemansson Technology
Box 47321, S-100 74 Stockholm, Sweden
+46 8 744 57 80 kjell.ahlin@ingemansson.se

The Smallwood algorithm for calculation of Shock Response Spectrum is perhaps the most widely used. It is based on the ramp invariant method for digital filter design. As such, it has a built-in low-pass characteristic, which limits its usability. A simple and computationally efficient pre-filtering of the signal under analysis is proposed that extends the usable frequency region of the algorithm substantially.

NOTATIONS

a	=	acceleration [m/s^2]
m	=	mass [kg]
c	=	damping constant [Ns/m]
k	=	spring constant [N/m]
s	=	laplace variable, complex frequency [rad/s]
f_0	=	resonance frequency [Hz]
ω_0	=	angular resonance frequency [rad/s]
Q	=	resonance gain, quality factor
T	=	sampling interval [s]
z	=	z-transform variable

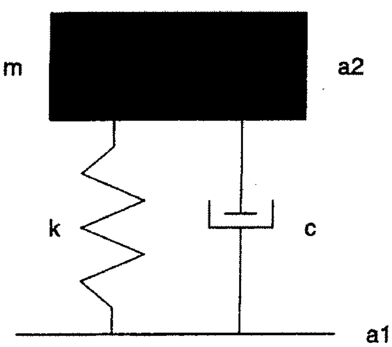
INTRODUCTION

The Smallwood algorithm is thought to be the most commonly used method to calculate the shock response spectrum, SRS. The algorithm was introduced in 1980 by David Smallwood in the 51st Shock and Vibration Bulletin, [1].

The basic part of a SRS algorithm is the filtering mechanism of a damped single degree of freedom, SDOF, mechanical system. The signal under analysis is fed to the SDOF filter and the maximum output is determined. Then the SDOF resonance frequency is changed, the signal is filtered again, and a new maximum value is determined. The plot of the maximum values versus the SDOF resonance frequencies makes up the SRS. The problem of how to determine the maximum value in a sampled system is not discussed here.

SDOF SYSTEM

gree of freedom mechanical system is shown in figure 1 with the usual notations.



1. Single degree of freedom system.

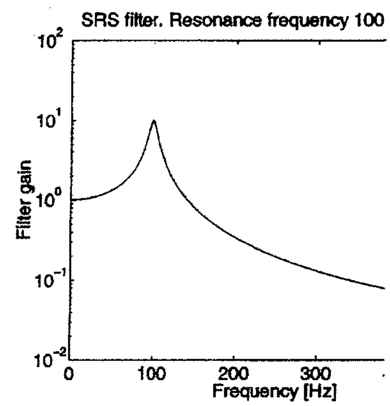


Figure 2. Transfer function for SI

Transfer function from the applied base acceleration a_1 to the mass acceleration a_2 could be calcula

$$H(s) = \frac{a_2}{a_1} = \frac{cs + k}{ms^2 + cs + k}$$

is often characterized by its resonance frequency f_0 and its resonance gain Q:

$$f_0 = \frac{1}{2\pi} \cdot \sqrt{\frac{k}{m}} \text{ [Hz]}$$

$$\omega_0 = 2\pi f_0 \text{ [rad/s]}$$

$$Q = \frac{2\pi f_0 m}{c}$$

:

$$H(s) = \frac{a_2}{a_1} = \frac{\frac{\omega_0 s}{Q} + \omega_0^2}{s^2 + \frac{\omega_0 s}{Q} + \omega_0^2}$$

Magnitude of the transfer function H(s) with f_0 = 100 Hz and Q = 10 is given in figure 2. Please note
ction value for zero Hz is exactly one, and that the gain at resonance is (approximately) Q.
reduction

DIGITAL FILTER DESIGN, IMPULSE INVARIANT METHOD

There are many ways to construct a digital filter, once the analog function H(s) is given. One simple metho "Impulse Invariant Transform". In this method, a digital filter is designed that has the same (sampled) Impulse response as the analog filter. The scheme for the design is as follows: (a simple example is given)

- Start with H(s) $\qquad \dfrac{1}{s+a}$
- Calculate the impulse response $\qquad e^{-at}$
- Change the time t to a sampled version, nT $\qquad e^{-anT}$
- Take the z-transform with respect to n $\qquad \dfrac{z}{z-e^{-aT}}$
- Multiply by the sampling interval T and simplify $\qquad \dfrac{T}{1-z^{-1}\cdot e^{-aT}}$

We now have arrived at a digital filter that could be realized in the usual manner with a recursive digital filter algorithm. The mechanics of the filter design could easily be performed for instance in Mathcad. The multiplying with T is necessary to get the right scaling of the filter.

If we apply this technique to the SDOF filter in (3), we get:

$$H(z)=\frac{b_0+b_1\cdot z^{-1}}{1+a_1\cdot z^{-1}+a_2\cdot z^{-2}}$$

where

$$b_0=2A$$
$$b_1=2A\cdot e^{-A}\cdot(C\cdot\sin B-\cos B)$$
$$a_1=-2e^{-A}\cdot\cos B$$
$$a_2=e^{-2A}$$

and

$$A=\frac{\omega_0 T}{2Q}$$
$$B=\omega_0 T\cdot\sqrt{1-\frac{1}{4Q^2}}$$
$$C=\frac{2Q^2-1}{\sqrt{4Q^2-1}}$$

To get the DC (zero frequency) response, one has to set z = 1. This gives

$$H(1) = \frac{b_0 + b_1}{1 + a_1 + a_2}$$
$$H(1) = \frac{2A + 2A \cdot e^{-A} \cdot (C \cdot \sin B - \cos B)}{1 - 2e^{-A} \cdot \cos B + e^{-2A}} \tag{7}$$

which is not identically equal to one. This is a major drawback of the impulse invariant method, and stems from the fact that the impulse response is not filtered to avoid aliasing. If the sampling frequency is high compared to the resonance frequency of the SDOF, H(1) tends to one. The DC response for the impulse invariant filter as a function of the SDOF resonance frequency is given in figure 3.

The high frequency part of the SRS comes from SDOF systems with high resonance frequencies. In many cases the frequency content of the studied signal is then below the resonance frequency, and the SRS value is determined by the low frequency behavior of the SDOF system. So, the bad behavior of the low frequency part of the impulse invariant realization of the SDOF filter results in errors in the high frequency part of the SRS.

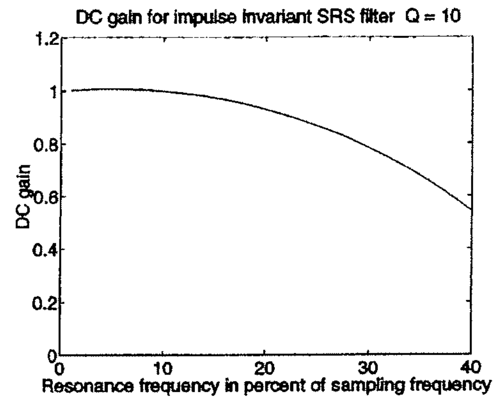


Figure 3. DC gain for impulse invariant method.

RAMP INVARIANT METHOD

One way to get rid of the problem with the impulse invariant method is to use the ramp invariant method instead. In this method, one can imagine that the samples of the impulse response of the SDOF filter are connected by straight lines. This has a low-pass filtering effect, which helps with the aliasing problem. To make a ramp invariant filter, the following procedure should be followed (the same example as for the impulse invariant method is used):

- Start with $H(s)/s^2$ $\qquad \frac{1}{(s+a)s^2}$
- Calculate the impulse response $\qquad \frac{1}{a^2} \cdot (e^{-at} - 1) + \frac{t}{a}$
- Change the time t to a sampled version, nT $\qquad \frac{1}{a^2} \cdot (e^{-anT} - 1) + \frac{nT}{a}$
- Take the z-transform with respect to n $\qquad \frac{z}{(z-1)^2} \cdot \frac{1 - aTe^{-aT} - e^{-aT} + z(-1 + aT + e^{-aT})}{a^2(z - e^{-aT})}$
- Multiply by $\frac{(z-1)^2}{Tz}$ and simplify $\qquad \frac{-1 + aT + e^{-aT} + z^{-1}(1 - aTe^{-aT} - e^{-aT})}{a^2T(1 - z^{-1}e^{-aT})}$

The last multiplication takes away the z-transform of the ramp $1/s^2$ that has been introduced and scales the filter properly. Details of the mathematics behind the method may be found in the literature.

One finds that the ramp invariant method introduces a lot more algebra to calculate the filter coefficients. For general second order filters (for instance SDOF relative displacement or filters defined by poles and residues) it's a formidable task. David Smallwood writes in his paper "(after much algebra)" which is an understatement. Today it is much simpler with programs like Mathcad, but still one has to very careful. The complexity of the calculation depends heavily on the choice of parameters. With the selection of intermediate parameters A and B in this paper, the work for the SRS is quite straightforward.

The ramp invariant SDOF filter turns out to be:

$$H(z)=\frac{b_0+b_1\cdot z^{-1}+b_2\cdot z^{-2}}{1+a_1\cdot z^{-1}+a_2\cdot z^{-2}} \tag{8}$$

where

$$\begin{aligned} b_0 &= 1-\frac{1}{B}e^{-A}\sin B \\ b_1 &= 2e^{-A}\cdot\left(\frac{1}{B}\cdot\sin B-\cos B\right) \\ b_2 &= e^{-2A}-\frac{1}{B}e^{-A}\sin B \\ a_1 &= -2e^{-A}\cdot\cos B \\ a_2 &= e^{-2A} \end{aligned} \tag{9}$$

and

$$\begin{aligned} A &= \frac{\omega_0 T}{2Q} \\ B &= \omega_0 T\cdot\sqrt{1-\frac{1}{4Q^2}} \end{aligned} \tag{10}$$

as before.

We find that the denominator is the same as for the impulse invariant filter, and that one extra term is added to the numerator. The DC response is identical to one for all parameter combinations. This means that the SRS for high frequencies, where the frequency content of the analyzed signal is much less than the SDOF resonance frequency, will be error-free.

What is then the price we have to pay?

As mentioned before, the ramp invariant method is equivalent to connecting the samples with straight lines. That is again equivalent with convolving the sampled time signal with a triangle, see figure 4. That means that the spectrum of the signal is the multiplication of the signal spectrum with the Fourier transform of the triangle, see figure 5. The expression for the Fourier transform of the triangle is:

$$\left(\frac{\sin(\pi fT)}{\pi fT}\right)^2 \tag{11}$$

This is of course equivalent to a low-pass filter. Figure 6 shows the filter function for several SDOF systems with different resonance frequencies, computed with the ramp invariant method. Q = 10 in all cases. The curves are enveloped with the function in (11) above. If we allow 10% error in the filter peak, we can only use a frequency span up to some 17% of the sampling frequency!

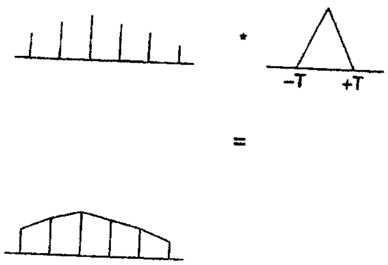


Figure 4. Convolution with a triangle

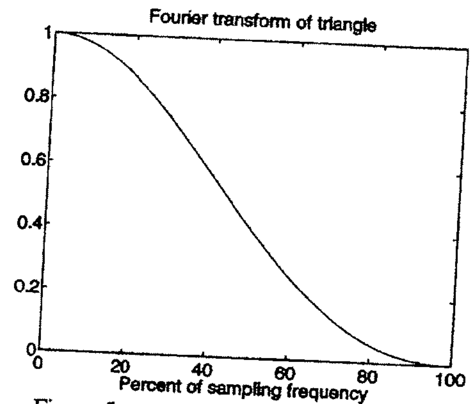


Figure 5. Fourier transform of triangle

Figure 7 shows the result of a simulation. The test signal is an exponentially damped sine, which frequency is allowed to vary. The SRS is computed and the SRS maximum value is saved and then plotted as a function of the test signal frequency. As a comparison, the same procedure is performed with ten times higher sampling frequency. The ramp invariant SRS algorithm is used in both cases, and the result is normalized to the value for low frequencies. As expected, we find an error in the calculation that is explained by the introduced low-pass filter.

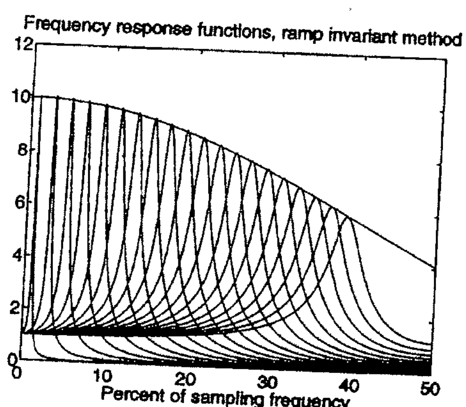


Figure 6. Filter shapes for ramp invariant method

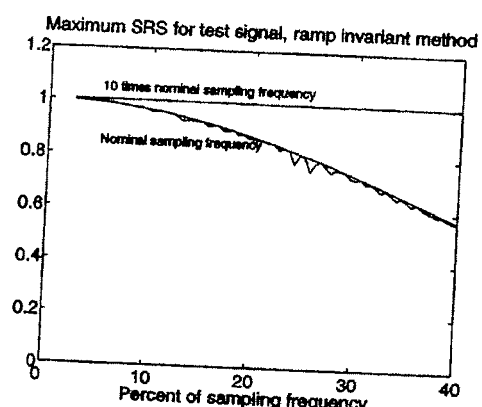


Figure 7. Simulation result

PRE-FILTER OPTIMIZATION

If the operations involved in the calculations are linear, we may introduce a pre-filter, acting on the input signal, to compensate the low-pass filter effect. A low order filter has been chosen, and the parameters of the filter were given as a simple optimization, performed in MATLAB. The strategy was to use a Butterworth low-pass filter as a start and then vary the parameters. The maximum deviation from the inverted filter characteristics in (11) up to 40% of the sampling frequency was minimized. Figure 8 shows the result of the process.

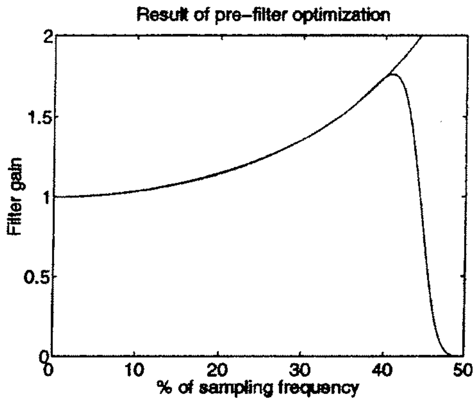


Figure 8. Result of pre-filter optimization.

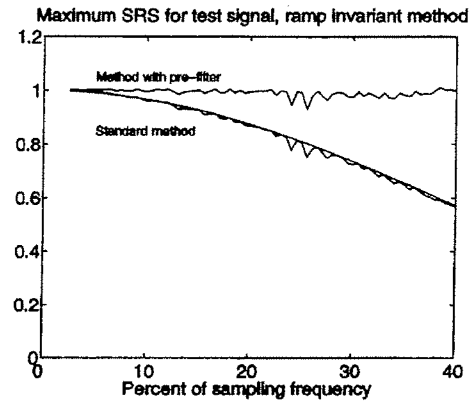


Figure 9. Simulation with pre-filter.

Figure 9 shows the result of the same simulation as above, this time the comparison is made between standard ramp invariant method and the version with pre-filter. We find that we no longer have the bias from (11), the only errors are basically coming from the maximum detection problem. The useful frequency range is now extended to 40% of the sampling frequency. The pre-filter from the optimization turns out to be:

$$H_p(z)=\frac{0.8767+1.7533z^{-1}+0.8767z^{-2}}{1+1.6296z^{-1}+0.8111z^{-2}+0.0659z^{-3}} \tag{12}$$

The filter is applied in MATLAB with the filtfilt function, which means that we have no phase distortion of the input signal. From the MATLAB manual:

"After filtering in the forward direction, the filtered sequence is then reversed and run back through the filter; The result is the time reverse of the output of the second filtering operation. The result has precisely zero phase distortion and magnitude modified by the square of the filter's magnitude response."

As the low-pass filter in (11) is independent of the SDOF parameters and only determined by the sampling frequency, we may use the same pre-filter for all SDOF filters. So, the input signal is pre-filtered with the filter in (12) and then the usual method is used. This adds only a small extra computation time to the process, as it is only performed once.

DISCUSSION AND CONCLUSION

A pre-filter has been introduced, that substantially expands the useful frequency range of the ramp invariant method for SRS calculations. If possible, one should use a sampling frequency that is at least ten times the highest frequency of interest. This makes the errors small when using the standard (Smallwood) ramp invariant method and it also makes the maximum detection problem easy to handle. If for some reason one has to analyze a recorded signal in a frequency range that is above some 10 % of the sampling frequency, the proposed method may be used. It eliminates the low-pass problem of the standard method and the SRS may be calculated up to 40% of the sampling frequency.

REFERENCES

Smallwood, David, "An Improved Recursive Formula for Calculating Shock Response Spectra", 51st Shock and Vibration Bulletin (1980).