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# COMPONENT MODE SYNTHESIS - A method for efficient dynamic simulation of complex technical systems

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**Technical Report** 

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### SUMMARY

The finite element (FE) method is a general method to model and simulate the physical behavior of bodies with arbitrary shape. There is a desire to create and use FE models early in the design process as well as to use the method detailing of complex artifacts. Modeling of the dynamic behavior of a multi-body system (MBS) is characterized by a composition of rigid bodies, interconnected by joints, springs, dampers, and actuators. The FE method is not directly scalable and MBS modeling is often based on too crude approximations of the properties of the bodies and their interaction. An obvious solution to this dilemma is to integrate FE and MBS technology in a new type of software or to allow condensed elastic submodels to be easily transferred from FE to MBS software and dynamic loads to be transferred from MBS to FE software.

Different condensation methods have been developed in the last decades. They are basically complementary. Three condensation methods are explained, exemplified, and compared below. The technique currently used in the MBS software ADAMS to define flexible bodies from imported condensed FE models is also briefly described and discussed.

The presented work was performed under the VISP research program. VISP is a collaborative project on configuration, modeling, simulation, and visualization, between seven research groups at Royal Institute of Technology, University of Skövde, IVF, and Linköping University, in Sweden. The goal of VISP is to develop an efficient, flexible and industry relevant modeling and simulation methodology, and an information framework aiding integrated realization of customized, modular products- and product-program configured production systems.

Keywords: Component mode synthesis, FEM, MBS

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### 1 INTRODUCTION

Engineering is a highly complex human cognitive activity. The increasing complexity of technical systems offers unique challenges for engineers, and the rapid development of computer-based tools provides new opportunities for engineers to solve engineering problems with the aid of numerical modeling and simulation. A dilemma frequently encountered in modeling of a non-trivial problem is the need to develop a simulation model that is as simple as possible and at the same time as complex as necessary. This dilemma is sometimes addressed by using different tools and classes of models for different problems. The main drawback of such an approach is that islands of automation are created. From a process point of view it is highly necessary to integrate these islands or to enable communication between them.

The finite element (FE) method is a general method to model and simulate the physical behavior of bodies with arbitrary shape. FE simulations have mainly been used as a tool for detailing components, but there is a need to use the FE method in the earlier phases of the design process. There is a trend to qualify detailed behavior of complex artifacts with FE simulations assisted by reduced testing, see for example (Morris and Vignjevic, 1997). The computer resources required for an FE solution grows exponentially with the size of the model. The FE method is thus not directly scalable. This problem is addressed by several modeling methods that are variations on the approach to synthesize models of complex technical systems from condensed FE submodels.

Modeling of the dynamic behavior of a multi-body system (MBS) is characterized by a composition of rigid bodies, interconnected by joints, springs, dampers, and actuators. Force elements such as springs, dampers, and actuators acting at discrete attachment points results in applied forces and torques on rigid bodies. Joints constrain the motion of the bodies in the system. The most widely used software for MBS modeling and simulation is ADAMS from MSC Software. The rigid body assumption is in many cases a too crude approximation. Several MBS softwares have capabilities to import condensed FE models from one or several of the most widely used FE softwares.

There are several complementary condensation methods available for generating a reduced FE problem. The most widely known methods are static condensation and component mode synthesis. The static condensation method and two different methods for component mode synthesis are explained, exemplified, and compared below. The technique currently used in ADAMS to define flexible bodies from imported condensed FE models is also briefly described and discussed.

### 2 FE CONDENSATION METHODS

In finite element analysis of dynamic problems, the primary variable solved for is the generalized displacement  $\mathbf{u}^{T} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$  for all *n* nodal degrees of freedom (DOFs). The set of equations to solve in a dynamic simulation is:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}(\mathbf{t})$$

where  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  are the inertia, damping and stiffness matrices. The load vector  $\mathbf{F}$  is in general a function of time. If the any of the inertia, damping or stiffness matrices depend on the computed state, the problem is referred to as a nonlinear dynamic problem.

In modeling and simulation of large and complex technical system there is a general desire to develop models of subsystems in parallel, and to solve the dynamic problem for a reduced set  $\mathbf{u}_r$  of m < n generalized DOFs, where  $\mathbf{u} = \mathbf{W}\mathbf{u}_r$  and  $\mathbf{W}$  is a set of Ritz vectors that constitute a reduced basis. The equations to solve in the reduced problem are:

$$\mathbf{M}_r \ddot{\mathbf{u}}_r + \mathbf{C}_r \dot{\mathbf{u}}_r + \mathbf{K}_r \mathbf{u}_r = \mathbf{F}_r$$

(2)

(1)

where  $\mathbf{M}_{r} = \mathbf{W}^{T}\mathbf{M}\mathbf{W}$ ,  $\mathbf{C}_{r} = \mathbf{W}^{T}\mathbf{C}\mathbf{W}$ ,  $\mathbf{K}_{r} = \mathbf{W}^{T}\mathbf{K}\mathbf{W}$ ,  $\mathbf{F}_{r} = \mathbf{W}^{T}\mathbf{F}$  are the reduced mass, damping, stiffness, and load matrices, respectively.

There are several complementary methods available for generating the Ritz vectors. The static condensation method and two different methods for component mode synthesis are explained and exemplified below.

#### 2.1 Static condensation

Condensation can employed to reduce the number of DOFs. Static condensation of the stiffness matrix involves no further assumptions besides the ones imposed by the idealization and discretization. Consider the static problem  $\mathbf{Ku} = \mathbf{F}$  and partition the matrices into master and slave sub-matrices:

$$\begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{sm} & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{m} \\ \mathbf{u}_{s} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{m} \\ \mathbf{F}_{s} \end{bmatrix}$$
(3)

where  $\mathbf{u}_{m}$  are the master or retained DOFs and  $\mathbf{u}_{s}$  are the slave or removed DOFs.

This is equal to:

$$\mathbf{K}_{mm}\mathbf{u}_m + \mathbf{K}_{ms}\mathbf{u}_s = \mathbf{F}_m \tag{4}$$

$$\mathbf{K}_{sm}\mathbf{u}_m + \mathbf{K}_{ss}\mathbf{u}_s = \mathbf{F}_s \tag{5}$$

If we reformulate 5 we get for the slave DOFs:

$$\mathbf{u}_{s} = \mathbf{K}_{ss}^{-1}\mathbf{F}_{s} - \mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}\mathbf{u}_{m}$$
(6)

If we substitute 6 into 4 we get for the reduced static problem, i.e. for the master DOFs:

$$\left[\mathbf{K}_{mm} - \mathbf{K}_{ms}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}\right]\mathbf{u}_{m} = \mathbf{F}_{m} - \mathbf{K}_{ms}\mathbf{K}_{ss}^{-1}\mathbf{F}_{s}$$
(7)

which also can be written  $\mathbf{K}_r \mathbf{u}_r = \mathbf{F}_r$ . If we order the master DOFs first, the transformation matrix **W** used in the static condensation procedure can be written as:

$$\mathbf{W} = \begin{bmatrix} \mathbf{I} \\ -\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm} \end{bmatrix}$$
(8)

In Guyan reduction or eigenvalue economization, as independently proposed by Guyan (1965) and Irons (1965, the mass matrix is reduced with the same transformation matrices as in the static condensation of the stiffness matrix. Thus:

$$\mathbf{M}_{r} = \left[\mathbf{M}_{mm} + \mathbf{K}_{ms}\mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}\right]$$
(9)

The principal assumption in this method is that for the lower frequency modes, inertia forces on slave DOFs, i.e. those DOFs being reduced out, are much less important than elastic forces transmitted by the master DOFs. The slave DOFs are thus assumed to move quasi-statically with the master DOFs. Therefore, the total mass of the structure is appositioned among only the master DOFs. This method introduces errors in the inertia terms but the reduced stiffness matrix is exact. Two measures that quantifies the condensation error can be calculate from the solutions to the eigenproblem of the reduced problem defined by  $(\mathbf{K}_r - \lambda_r \mathbf{M}_r)\mathbf{D}_r = \mathbf{0}$  and the full problem defined by  $(\mathbf{K} - \lambda \mathbf{M})\mathbf{D} = \mathbf{0}$ . The relative error  $\varepsilon_i$  in eigenvalue number *i* and the error  $\gamma_i$  in the error in the mode shape are defined as:

$$\varepsilon_i = \frac{\lambda_{r,i} - \lambda_i}{\lambda_{r,i}} \tag{10}$$

$$\gamma_{i} = 1 - \frac{\left| \mathbf{D}_{r,i}^{T} \mathbf{D}_{i} \right|}{\left| \mathbf{D}_{r,i} \right| \left| \mathbf{D}_{i} \right|}$$
(11)

By choosing the most suitable set of master DOFs, the condensation error can be minimized. Henshell and Ong (1975) defined the relation  $k_{ii}/m_{ii}$  as a criterion for ranking the dynamic DOFs as candidates for choosing as master DOFs. Their method is implemented in most commercial FE codes as a tool for automatically selecting the most appropriate master DOFs. Furthermore, Thomas (1982) defined a priori error bounds for the computed eigenvalues.

#### 2.2 Component mode synthesis - the CMS1 methods

The component mode synthesis method (CMS), first proposed by Hurty (1965), has significant condensation advantages, and it is well suited for modeling and simulation of large and complex systems. The basic approach in CMS is to divide the system into k subsystems. In one class of CMS methods, further on referred to as the *CSM1* method, the n mode shapes of the subsystems are used in a Raleigh-Ritz analysis to calculate approximate mode shapes of the complete system with the following assumed load pattern:

$$\mathbf{R} = \begin{bmatrix} \mathbf{D}_{c1} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{I}_{c1,c2} & \mathbf{0} & \dots \\ \mathbf{D}_{c2} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{c2,c3} & \dots \\ \dots & \dots & \dots & \dots \\ \mathbf{D}_{ck} & \mathbf{0} & \mathbf{0} & \dots \end{bmatrix}$$
(12)

where  $I_{ck-1,ck}$  is a unit matrix that has the effect of releasing the attachment DOFs. It is of order equal to the connection DOFs between subsystems *k*-1 and *k*. The Ritz vectors, i.e. the transformation matrix, is created by pre-multiplying the load matrix in 12 with the inverse of the stiffness matrix for the assembled systems model as shown in equation 13

$$\mathbf{W}_{cms1} = \mathbf{K}^{-1}\mathbf{R} \tag{13}$$

In the *CMS1* method, the transformation matrix  $\mathbf{W}_{cms1}$  that defines the reduced base contains thus *n* columns that correspond to the component normal modes and *p* attachment mode shapes, where *p* is the total number of interface DOFs in the model.

#### 2.3 Component mode synthesis - the CMS2 methods

In the Craig-Bampton method, further on referred to as the CSM2 method, the normal modes of the component models, with the attachment DOFs fixed, are used directly as Ritz vectors (Craigh and Bampton, 1968). These vectors are then complemented by *constraint modes*, which are displacement shapes of the assembled model obtained by successively applying a unit displacement to each interface DOF while keeping all other attachment DOFs fixed. For each constraint mode, only the interior DOFs in the components that are directly related to the displacement loaded interface DOF gets a non-zero value in the actual constraint shape. The constraint modes can thus as the normal modes be created on the component level.

$$\mathbf{W}_{cms2} = \begin{bmatrix} \mathbf{D}_{c1} & \mathbf{U}_{c1,c1,c2} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{I}_{c1,c2} & \mathbf{0} & \dots \\ \mathbf{D}_{c2} & \mathbf{U}_{c2,c1,c2} & \mathbf{U}_{c2,c2,c3} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{c2,c3} & \dots \\ \dots & \dots & \dots & \dots \\ \mathbf{D}_{ck} & \mathbf{0} & \mathbf{0} & \dots \end{bmatrix}$$
(14)

### **3 A SIMPLE TECHNICAL SYSTEM**

Consider axial vibration of the assembly shown in figure 1. It consists of two components of different lengths and with different cross sections -A and 2A, respectively. The material in the two components is linear elastic with the Young's modulus *E* and the density  $\rho$ 



Figure 1. A system composed of two components with different cross sections.

### 3.1 A model of the system

An FE discretization of the system with five truss elements of equal length and six nodes is shown in figure 2. Because of the inherent characteristic properties of the truss element, each node in the model has one axial DOF.



Figure 2. System modeled by five truss elements of equal length L.

The stiffness and lumped mass matrices for elements *e1* and *e3* are:

$$\mathbf{K}_{e1} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \ \mathbf{M}_{e1} = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
(15)

$$\mathbf{K}_{e3} = \frac{AE}{L} \begin{bmatrix} 2 & -2\\ -2 & 2 \end{bmatrix}, \ \mathbf{M}_{e3} = \frac{\rho AL}{2} \begin{bmatrix} 2 & 0\\ 0 & 2 \end{bmatrix}$$
(16)

The stiffness and lumped matrices for the complete assembly with node *n6* fixed are thus:

$$\mathbf{K} = \frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 0 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & -2 & 4 \end{bmatrix}, \quad \mathbf{M} = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$
(17)

The five eigenvalues and the corresponding eigenvectors, i.e. the solution to  $(\mathbf{K} - \lambda \mathbf{M})\mathbf{D} = \mathbf{0}$ , are:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_{1} & & & \\ & \lambda_{2} & & \\ & & \lambda_{3} & \\ & & & \lambda_{4} & \\ & & & & \lambda_{5} \end{bmatrix} = \frac{2E}{\rho L^{2}} \begin{bmatrix} 0.0703 & & & & \\ & 0.3790 & & & \\ & & 1.0000 & & \\ & & & 1.6210 & \\ & & & & 1.9297 \end{bmatrix}$$
(18)

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \mathbf{D}_4 & \mathbf{D}_5 \end{bmatrix} = \begin{bmatrix} 0.5990 & 0.7274 & -0.5774 & -0.7274 & -0.5990 \\ 0.5569 & 0.4517 & 0.0000 & 0.4517 & 0.5569 \\ 0.4365 & -0.1664 & 0.5774 & 0.1664 & -0.4365 \\ 0.3303 & -0.3809 & 0.0000 & -0.3809 & 0.3303 \\ 0.1776 & -0.3066 & -0.5774 & 0.3066 & -0.1776 \end{bmatrix}$$
(19)

The five mode shapes in 19 have been normalized to a unith length. If the natural frequencies in Hz are preferred, they are calculated from the eigenvalues  $f_i = \frac{1}{2\pi} \sqrt{\lambda_i}$ .

## 3.2 Static condensation - 1 master DOF

If we study the stiffness and mass matrices in 17, we find that  $k_{ii}/m_{ii} = 2E/\rho L^2$  for all DOFs. All DOFs are thus equally suitable candidates for choosing as dynamic master DOFs.



Figure 3. Truss assembly with a single master DOF at node n3.

If we choose the DOF at node n3 as the single master DOF as indicated in figure 3 and rearrange the matrices we get the following reduced stiffness and mass matrices:

$$\mathbf{K}_{r} = \mathbf{K}_{mm} - \mathbf{K}_{ms} \mathbf{K}_{ss}^{-1} \mathbf{K}_{sm} = \frac{AE}{L} \left[ [3] - \begin{bmatrix} -2 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right] =$$

$$= \frac{AE}{L} \left[ [3] - \begin{bmatrix} -2 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/3 & 1/6 & 0 & 0 \\ 1/6 & 1/3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right] = \frac{AE}{L} [2/3]$$
(20)

$$\mathbf{M}_{r} = \mathbf{M}_{mm} + \mathbf{K}_{ms} \mathbf{K}_{ss}^{-1} \mathbf{M}_{ss} \mathbf{K}_{ss}^{-1} \mathbf{K}_{sm} = \\ = \frac{AA}{2} \left[ (3) + [-2 \quad 0 \quad 0 \quad -1] \begin{bmatrix} 1/3 & 1/6 & 0 & 0 \\ 1/6 & 1/3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/3 & 1/6 & 0 & 0 \\ 1/6 & 1/3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right] = \\ = \frac{AA}{2} [74/9]$$

$$(21)$$

The single eigenvalue of the reduced system defined by 20 and 21 is:

$$\lambda_1 = \frac{2E}{\rho L^2} 0.0811$$
 (22)

The corresponding mode shape for the reduced model  $\mathbf{D}_r^T = [1]$  can be expanded to all five unrestrained DOFs, according to equation 8:

$$\mathbf{D}_{scl}^{T} = \begin{bmatrix} 1 & 1 & 1 & 2/3 & 1/3 \end{bmatrix}$$
(23)

If we compare the eigenvalue in 22 and the corresponding mode shape in 23 to the first mode of the full model, as given in 18 and 19, we get for the condensation errors as defined in 10 and 11:

$$\varepsilon_l = 0.1332$$
 (24)  
 $\gamma_l = 0.0074$  (25)

### 3.3 Static condensation - 3 master DOFs



Figure 4. Truss assembly with three evenly distributed master DOFs.

If we increase the reduced set to three master DOFs, evenly distributed in the geometric space as shown in figure 4, we get the following solution to the reduced eigenvalue problem:

$$\mathbf{\Lambda}_{sc3} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \frac{2E}{\rho L^2} \begin{bmatrix} 0.0728 \\ 0.4434 \\ 1.0000 \end{bmatrix}$$

$$\mathbf{D}_{sc3} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 \end{bmatrix} = \begin{bmatrix} 0.7859 & 0.9017 & -0.5774 \\ 0.6793 & 0.3477 & 0.0000 \\ 0.5727 & -0.2062 & 0.5774 \\ 0.4029 & -0.2932 & 0.0000 \\ 0.2331 & -0.3801 & -0.5774 \end{bmatrix}$$
(26)
$$(27)$$

The condensation errors for the three modes are:.

**-** .

$$\mathbf{E}_{sc3} = \begin{bmatrix} \boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_2 & \boldsymbol{\varepsilon}_3 \end{bmatrix} = \begin{bmatrix} 0.034 & 0.1452 & 0.0000 \end{bmatrix}$$

$$\boldsymbol{\Gamma}_{sc3} = \begin{bmatrix} \boldsymbol{\gamma}_1 & \boldsymbol{\gamma}_2 & \boldsymbol{\gamma}_3 \end{bmatrix} = \begin{bmatrix} 0.0006 & 0.0210 & 0.0000 \end{bmatrix}$$
(28)
(29)

Compared to the model with one master DOF, the condensation error is significantly reduced for the first mode. The error in the second is also low and the third mode is by coincidence equal to the third mode of the full FE model.



Figure 5. Truss assembly with three distributed master DOFs in two variants labeled c3t and c3b.

Two other possible sets of master DOFs are shown in figure 5. If we solve the eigenvalue problems for these two reduced sets, we get for the condensation error:

$E_{c3t} = [0.0317]$	0.2495	$0.4258$ ], $\Gamma_{sc3t} = [0.0013]$	0.0521	0.5693]	(30)
$E_{c3b} = [0.0929]$	0.4459	$0.3638$ ], $\Gamma_{sc3b} = [0.0103]$	0.2664	0.6414]	(31)

As expected, the condensation error depends strongly on the chosen set of master DOFs.

#### 3.4 Component mode synthesis - method CMS1

Consider now the two components, referred to as c1 and c2 in figure 6, independently. Component c1 has one mating feature where it may be connected to another component or to ground. The unconnected component c2 has two mating features.



Figure 6. Two components with three mating features.

In the following section, the Ritz vectors for the discretized component model c1 shown in the left portion of figure 7 is first generated. The Ritz vectors for component model c2 shown in the right portion of figure 7 is then calculated and the two sets of vectors are assembled.



The stiffness and lumped mass matrices for component *c1*, which consists of elements one and two, are:

$$\mathbf{K}_{c1} = \frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \ \mathbf{M}_{c1} = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
(32)

By fixing the attachment node *n3*, the matrices in 32 are reduced to:

$$\mathbf{K}_{cli} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \ \mathbf{M}_{cli} = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$
(33)

For component *c1*, the solution to the eigenproblem defined by  $(\mathbf{K}_{c1i} - \lambda_{c1i} \mathbf{M}_{c1i}) \mathbf{D}_{c1i} = \mathbf{0}$  is:

$$\boldsymbol{\Lambda}_{cli} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{2E}{\rho L^2} \begin{bmatrix} 1 - 1/\sqrt{2} \\ 1 + 1/\sqrt{2} \end{bmatrix}, \quad \boldsymbol{D}_{cli} = \begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{D}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$
(34)

The Ritz vectors for the discretized component model c2 shown in the right portion of figure 7 is generated in the same way as described above for component model c1. In this model there are though two mating features, that are represented by two attachment DOFs at nodes n3 and n6. The stiffness and lumped mass matrices for c2, which consists of elements three, four, and five, are:

$$\mathbf{K}_{c2} = \frac{AE}{L} \begin{bmatrix} 2 & -2 & 0 & 0\\ -2 & 4 & -2 & 0\\ 0 & -2 & 4 & -2\\ 0 & 0 & -2 & 2 \end{bmatrix}, \ \mathbf{M}_{c2} = \frac{\rho AL}{2} \begin{bmatrix} 2 & 0 & 0 & 0\\ 0 & 4 & 0 & 0\\ 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 2 \end{bmatrix}$$
(35)

By fixing the attachment nodes *n6* and *n3* in *c2*, 35 is reduced to:

$$\mathbf{K}_{c2i} = \frac{AE}{L} \begin{bmatrix} 4 & -2\\ -2 & 4 \end{bmatrix}, \ \mathbf{M}_{c2i} = \frac{\rho AL}{2} \begin{bmatrix} 4 & 0\\ 0 & 4 \end{bmatrix}$$
(36)

For component c2, the solution to the reduced eigenproblem is:

$$\Lambda_{c2i} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{2E}{\rho L^2} \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}, \quad \mathbf{D}_{c2i} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{D}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
(37)

According to the CMS1 method, the matrix of Ritz vectors for all unrestrained DOFs is:

$$\mathbf{W}_{cms1} = \mathbf{K}^{-1}\mathbf{R} = \frac{L}{AE} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 0 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & -2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} =$$

$$= \frac{L}{AE} \begin{bmatrix} 6.7678 & 2.2322 & 1.5000 \\ 5.7678 & 1.2322 & 1.5000 \\ 5.7678 & 1.2322 & 1.5000 \\ 4.0607 & 0.9393 & 1.5000 \\ 3.2071 & 0.7929 & 1.0000 \\ 1.8536 & 0.1464 & 0.5000 \end{bmatrix}$$
(38)

The reduced stiffness and mass matrices for the system are thus:

$$\mathbf{K}_{cms1} = \mathbf{W}_{cms1}^{T} \mathbf{K} \mathbf{W}_{cms1} = \frac{AE}{L} \begin{bmatrix} 15.9069 & 4.0429 & 4.0607 \\ 4.0429 & 2.0074 & 0.9393 \\ 4.0467 & 0.9393 & 1.5000 \end{bmatrix}$$
(39)  
$$\mathbf{M}_{cms1} = \mathbf{W}_{cms1}^{T} \mathbf{M} \mathbf{W}_{cms1} = \frac{\rho AL}{2} \begin{bmatrix} 216.6886 & 52.0221 & 62.2635 \\ 52.0221 & 13.2672 & 14.7365 \\ 62.2635 & 14.7365 & 18.5000 \end{bmatrix}$$
(40)

The eigenvalue solution to the reduced problem is:

$$\boldsymbol{\Lambda}_{cms1} = \frac{2E}{\rho L^2} \begin{bmatrix} 0.0706 \\ 0.7733 \\ 1.3960 \end{bmatrix}, \ \boldsymbol{D}_{cms1} = \begin{bmatrix} -4.5815 & -0.5451 & 0.3108 \\ -4.1602 & -0.1465 & -0.1975 \\ -3.2973 & 0.3107 & 0.1166 \\ -2.4427 & 0.0604 & 0.0533 \\ -1.3775 & 0.0095 & -0.2641 \end{bmatrix}$$
(41)

The condensation errors in the three modes of the reduce model are thus:

$$\mathbf{E}_{sc3} = \begin{bmatrix} \boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_2 & \boldsymbol{\varepsilon}_3 \end{bmatrix} = \begin{bmatrix} 0.0043 & 0.5099 & 0.2837 \end{bmatrix}$$
(42)

$$\Gamma_{sc3} = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} = \begin{bmatrix} 0.0001 & 0.1651 & 0.9143 \end{bmatrix}$$
(43)

#### 3.5 Component mode synthesis - method CMS2

In the CMS2 method, the component normal mode shapes, as given in 34 and 37, and the constraint modes are used directly as Ritz vectors. There is one attachment DOF in component model *c1*. The constraint mode shape  $\mathbf{u}_{c1}$  is obtained by solving  $\mathbf{K}_{c1}\mathbf{u}_{c1} = \mathbf{F}_{c1}$  for a prescribed unit deflection at attachment node *n3*, i.e.:

$$\frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f_3 \end{bmatrix}$$
(44)

The solution to equation 44 is:

$$\mathbf{u}_{_{cl}}^{T} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$
(45)

The matrix of Ritz vectors  $\mathbf{W}_{c1}$  for component model c1 is created from the displacement shapes in 34 for the full set of DOFs, i.e.  $\mathbf{D}_{c1,1}^T = \begin{bmatrix} 1 & 1/\sqrt{2} & 0 \end{bmatrix}$  and  $\mathbf{D}_{c1,2}^T = \begin{bmatrix} 1 & -1/\sqrt{2} & 0 \end{bmatrix}$ , and the constraint mode shape defined by 45:

$$\mathbf{W}_{c1} = \begin{bmatrix} 1 & 1 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
(46)

The reduced stiffness and mass matrices for component *c1* are:

$$\mathbf{K}_{cl,r} = \mathbf{W}_{cl}^{T} \mathbf{K}_{cl} \mathbf{W}_{cl} = \frac{AE}{L} \begin{bmatrix} 1 & 1/\sqrt{2} & 0 \\ 1 & -1/\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 1 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \frac{AE}{L} \begin{bmatrix} 2-\sqrt{2} & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}_{cl,r} = \mathbf{W}_{cl}^{T} \mathbf{M}_{cl} \mathbf{W}_{cl} = \frac{\rho AL}{2} \begin{bmatrix} 1 & 1/\sqrt{2} & 0 \\ 1 & -1/\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 1 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \frac{\rho AL}{2} \begin{bmatrix} 2 & 0 & 1+\sqrt{2} \\ 0 & 2 & 1-\sqrt{2} \\ 1+\sqrt{2} & 1-\sqrt{2} & 4 \end{bmatrix}$$

$$(48)$$

Since there are two attachment DOFs, there are also two constraint modes. The first mode shape is obtained by solving for a prescribed unit deflection at attachment node n3, and fixing the attachment node n6, *i.e.*:

$$\frac{AE}{L} \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ u_4 \\ u_5 \\ 0 \end{bmatrix} = \begin{bmatrix} f_3 \\ 0 \\ 0 \\ f_6 \end{bmatrix}$$
(49)

The solution to equation 49 is:

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$$\mathbf{u}_{_{c2,1}}^{T} = \begin{bmatrix} 1 & 2/3 & 1/3 & 0 \end{bmatrix}$$
(50)

The second mode shape is obtained by solving for a prescribed unit deflection at attachment node *n6*, and fixing the attachment node *n3*, i.e.:

$$\frac{AE}{L} \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ u_4 \\ u_5 \\ 1 \end{bmatrix} = \begin{bmatrix} f_3 \\ 0 \\ 0 \\ f_6 \end{bmatrix}$$
(51)

The solution to equation 51 is:

$$\mathbf{u}_{_{c2,2}}^{^{T}} = \begin{bmatrix} 0 & 1/3 & 2/3 & 1 \end{bmatrix}$$
(52)

The Ritz vector  $\mathbf{W}_{c1}$  for component model c2 is created from the displacement shapes in equations 37, 50 and 52:

$$\mathbf{W}_{c2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 2/3 & 1/3 \\ 1 & -1 & 1/3 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
(53)

In the CMS2 method, the reduced stiffness and mass matrices for component *c2* are:

$$\begin{split} \mathbf{K}_{c2,r} &= \mathbf{W}_{c2}^{T} \mathbf{K}_{c2} \mathbf{W}_{c2} = \\ &= \frac{AE}{L} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 0 & 1/3 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 2/3 & 1/3 \\ 1 & -1 & 1/3 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$
(54)  
$$&= \frac{AE}{L} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 2/3 & -2/3 \\ 0 & 0 & -2/3 & 2/3 \end{bmatrix}$$
$$\mathbf{M}_{c2,r} = \mathbf{W}_{c2}^{T} \mathbf{M}_{c2} \mathbf{W}_{c2} = \frac{\rho AL}{2} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 0 & 1/3 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 2/3 & 1/3 \\ 1 & -1 & 1/3 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$
(55)  
$$&= \frac{\rho AL}{2} \begin{bmatrix} 8 & 0 & 4 & 4 \\ 0 & 8 & 4/3 & -4/3 \\ 4 & 4/3 & 38/9 & 16/9 \\ 4 & -4/3 & 16/9 & 38/9 \end{bmatrix}$$

The matrix of Ritz vectors for the complete assembly is an aggregation of the vectors for the components. For the present assembly we get a transformation matrix that consists of two modal vectors and two constraint mode shapes:

$$\mathbf{W}_{cms2} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2/3 & 1/3 \\ 1 & -1 & 1/3 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
(56)

The reduced stiffness and mass matrices for the unrestrained system are thus reduced two four DOFs as shown in 57 and 58.

$$\mathbf{K}_{cms2} = \mathbf{W}_{cms2}^{T} \mathbf{K}_{cl+c2} \mathbf{W}_{cms2} = \mathbf{W}_{cms2}^{T} \frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -2 & 0 & 0 \\ 0 & 0 & -2 & 4 & -2 & 0 \\ 0 & 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix} \mathbf{W}_{cms2} = \mathbf{W}_{cms2}^{T} \begin{bmatrix} 6-\sqrt{2} & 0 & 0 & 0 \\ 0 & 14+\sqrt{2} & 0 & 0 \\ 0 & 0 & 2/3 & -2/3 \\ 0 & 0 & -2/3 & 2/3 \end{bmatrix}$$

$$\mathbf{M}_{cms2} = \mathbf{W}_{cms2}^{T} \mathbf{M}_{cl+c2} \mathbf{W}_{cms2} = \mathbf{W}_{cms2}^{T} \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \mathbf{W}_{cms2} = \begin{bmatrix} \frac{\rho AL}{2} \begin{bmatrix} 10 & 0 & 5+\sqrt{2} & 4 \\ 0 & 10 & -\sqrt{2} + 7/3 & -4/3 \\ 5+\sqrt{2} & -\sqrt{2} + 7/3 & 74/9 & 16/9 \\ 4 & -4/3 & 16/9 & 38/9 \end{bmatrix}$$
(57)

The stiffness matrix of the unrestrained system is not positive definite. In 57 we can also see that the two last rows of the reduced stiffness matrix are not unique. With four DOFs, the unrestrained system has eigenvalues including a rigid body mode, i.e. a zero-value eigenvalue. Fixing the n6 DOF to ground can be done by excluding the second constraint mode for component c2 or alternatively by removing the fourth column and row in the

reduced stiffness and mass matrices for component c2. We thus get the following reduced stiffness and mass matrices for the restrained system:

$$\mathbf{K}_{cms2} = \frac{AE}{L} \begin{bmatrix} 6 - \sqrt{2} & 0 & 0\\ 0 & 14 + \sqrt{2} & 0\\ 0 & 0 & 2/3 \end{bmatrix}$$
(59)

$$\mathbf{M}_{cms2} = \frac{\rho AL}{2} \begin{bmatrix} 10 & 0 & 5 + \sqrt{2} \\ 0 & 10 & -\sqrt{2} + 7/3 \\ 5 + \sqrt{2} & -\sqrt{2} + 7/3 & 74/9 \end{bmatrix}$$
(60)

The eigenvalue solution to the reduced model of the restrained system is:

$$\Lambda_{cms2} = \frac{2E}{\rho L^2} \begin{bmatrix} 0.0739 \\ 0.9854 \\ 1.6076 \end{bmatrix}$$
(61)

$$\mathbf{D}_{cms1} = \mathbf{W}_{cms1} \begin{bmatrix} 0.1224 & -0.7640 & -0.3445 \\ 0.0046 & 0.1037 & -0.8567 \\ 0.9925 & 0.6368 & 0.3839 \end{bmatrix} = \begin{bmatrix} -1.1194 & -0.0235 & -0.8173 \\ -1.0757 & 0.0232 & 0.7461 \\ -0.9925 & 0.6368 & 0.3839 \\ -0.7886 & -0.2357 & -0.9453 \\ -0.44865 & -0.6555 & 0.6402 \end{bmatrix}$$
(62)

The condensation errors in the three modes of the reduced model are:

$$\mathbf{E}_{sc3} = \begin{bmatrix} \boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_2 & \boldsymbol{\varepsilon}_3 \end{bmatrix} = \begin{bmatrix} 0.0487 & 0.6154 & 0.3780 \end{bmatrix}$$
(63)  
$$\boldsymbol{\Gamma}_{sc3} = \begin{bmatrix} \boldsymbol{\gamma}_1 & \boldsymbol{\gamma}_2 & \boldsymbol{\gamma}_3 \end{bmatrix} = \begin{bmatrix} 0.0054 & 0.8113 & 0.8020 \end{bmatrix}$$
(64)

#### 4 THE CONCEPT OF MATING FEATURE MODES

### 4.1 Ortogonalization of the constraint modes

In the CMS2 or the Craig-Bampton method rigid body motion is a linear combination of the constraint modes. If we use the constraint mode to reduce the full stiffness and mass matrices of the component *c1* we get:

$$\mathbf{K}_{c1\_cm} = \frac{AE}{L} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 0 \end{bmatrix}$$
(65)

$$\mathbf{M}_{c1\_cm} = \frac{\rho AL}{2} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{\rho AL}{2} \begin{bmatrix} 4 \end{bmatrix}$$
(66)

It can be observed that the reduced mass in 66 for the rigid body mode is equal to the mass of component cl. The solution to the eigenvalue problem for the mating feature DOF of component cl is:

$$\lambda_{c1\_mf} = 0, \ \mathbf{D}_{c1\_mf}^{T} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$
(67)

If we perform the same operations for component c2 we get:

$$\mathbf{K}_{c2\_cm} = \frac{AE}{L} \begin{bmatrix} 1 & 2/3 & 1/3 & 0 \\ 0 & 1/3 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2/3 & 1/3 \\ 1/3 & 2/3 \\ 0 & 1 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 2/3 & -2/3 \\ -2/3 & 2/3 \end{bmatrix}$$
(68)  
$$\mathbf{M}_{c2\_cm} = \frac{\rho AL}{2} \begin{bmatrix} 1 & 2/3 & 1/3 & 0 \\ 0 & 1/3 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2/3 & 1/3 \\ 1/3 & 2/3 \\ 0 & 1 \end{bmatrix} = \frac{\rho AL}{2} \begin{bmatrix} 38/9 & 16/9 \\ 16/9 & 38/9 \end{bmatrix}$$
(69)

If we add the rows in the reduced mass matrix in 69 we get the total mass of component *c2* which is  $m_{c2} = 3\rho AL$ . The solution to the eigenvalue problem for the mating feature DOFs of component *c2* is:

$$\Lambda_{c2\_mf} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{2E}{\rho L^2} \begin{bmatrix} 0 \\ 0.5455 \end{bmatrix}$$
(70)

$$\mathbf{D}_{c_{2}_{mf}} = \mathbf{W}_{c_{2}_{mf}} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2/3 & 1/3 \\ 1/3 & 2/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/3\sqrt{2} \\ 1/\sqrt{2} & 1/3\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$
(71)

The modes in 67 and 71 are here referred to as *mating feature modes*. Mating feature modes are orthogonal and they have eigenvalues associated to them. The number of DOFs per node determines the number of rigid body mating feature modes for a component. In our example where each node had one DOF, the first calculated mode is a rigid body mode.

### **5 COMPONENT MODES IN ADAMS**

The first attempt of Mechanical Dynamics Inc. (MDI), the company that initially developed Adams, to interface with FE software was a product called ADAMS/FEA. ADAMS/FEA could import statically condensed stiffness and mass matrices from FE software, such as ANSYS and MSC/NASTRAN. Each master node was represented by an ADAMS PART element and the condensed stiffness matrices were captured correctly in by the ADAMS NFORCE element. The requirement to represent the total mass, the center-of-gravity, the moments of inertia, and the frequency content of the flexible component was often difficult to satisfy with the condensed mass matrix.

In 1995, MDI added modal flexibility support and a new inertia element, the FLEX\_BODY, to the ADAMS/FEA version 8.1 software module as an alternative to the discrete flexibility approach. The main assumption behind FLEX\_BODY is that only small, linear deformations relative to a local frame of reference, that is undergoing large nonlinear global motion. With release 8.2 modal flexibility was expanded to include constraint modes, resulting in an MBS-FEA interface based on the Craig-Bampton approach to component mode synthesis. Today, the discrete flexibility approach has been removed and the CMS2 method has been repacked into a product called ADAMS/Flex. Since the ADAMS FLEX\_BODY element provides its own large-motion DOF there is a need to disconnect the rigid body modes, which are embedded in the constraint modes. This is the main reason for the present release of ADAMS/Flex to utilize the Craig-Bampton, or CMS2, approach and mating feature modes, i.e. orthogonalized constraint modes, which in ADAMS/Flex are referred to as *boundary eigenvectors* (ADAMS, 1998). The natural and constraint modes are imported from a modal neutral-format file (MNF). ANSYS, MSC/NASTRAN and I-DEAS Master Series can create the binary MNF-files.



Figure 8. The position vector to a deformed point P' on a flexible body, relative to a local B and a global G reference frame.

The generalized coordinates of an ADAMS flexible body are:

$$\boldsymbol{\xi}^{T} = \begin{bmatrix} \mathbf{x} & \boldsymbol{\varphi} & \mathbf{q} \end{bmatrix}$$
(72)

where  $\mathbf{x}^T = \begin{bmatrix} x & y & z \end{bmatrix}$  is the position vector from the ground origin to the origin of the local body reference frame,  $\mathbf{\phi}^T = \begin{bmatrix} \varphi & \theta & \phi \end{bmatrix}$  is a body fixed set of Euler angles that define the

orientation of the local body reference frame with respect to ground, and  $\mathbf{q}^T = [q_1 \quad q_2 \quad \dots \quad q_m]$  are the *m* modal coordinates of the flexible body.

The instantaneous location of an FE node P on a flexible body B expressed in the global or ground coordinate system, as shown in figure 8, is:

$$\mathbf{r}_{P} = \mathbf{x} + \mathbf{s}_{P} + \mathbf{u}_{P} \tag{73}$$

where  $\mathbf{s}_P$  is the position vector from the local body frame of reference *B* to the point *P*, expressed in the local body coordinate system and  $\mathbf{u}_P$  is the translational deformation vector expressed in the local body coordinate system. The deformation vector is a modal superposition:

$$\mathbf{u}_P = \mathbf{D}_P \mathbf{q} \tag{74}$$

where  $\mathbf{D}_P$  is the part of modal matrix that corresponds to the translational DOFs of point *P*. Expressed in global ground coordinates, equation 74 is:

$$\mathbf{r}_{p} = \mathbf{x} + {}^{G} \mathbf{A}^{B} (\mathbf{s}_{p} + \mathbf{u}_{p})$$
(75)

where  ${}^{G}\mathbf{A}^{B}$  is the transformation matrix from the body fixed coordinate system *B* to the global coordinate system *G*.

As the flexible body deforms, the flexible marker rotates through small angles relative to its local reference frame. These small angles are also obtained from a modal superposition:

$$\boldsymbol{\theta}_{P} = \mathbf{D}_{P}^{*} \mathbf{q}$$
(76)

where  $\mathbf{D}_{P}^{*}$  is the part of modal matrix that corresponds to the rotational DOFs of point *P*.

Translational velocities of flexible markers are obtained by differentiating 80 with respect to time. from transformations of 74 and 75 the global reference frame. Rotational velocities of flexible markers is the sum of the rotational velocities of body and the rotational velocity due to deformation.

In the Craig-Bampton approach to component mode synthesis, the physical DOFs of a flexible body is condensed to an set of physical DOFs that represent the external mating features and a set of modal DOFs that represent the internal dynamic properties of the body. The mating features of two component models that have been condensed with nthe CMS2 approach can thus be connected without any manipulation.

The position, velocity and acceleration of *flexible markers*, referred to as marker kinematics in ADAMS (1998), are required to satisfy constraints, e.g. those imposed by joints, as well as to project point forces applied at markers on generalized coordinates of the flexible body. ADAMS does not, however, allow joints to be connected to flexible markers - joints must be connected to rigid bodies. To overcome this restriction, the recommended method is to connect a rigid "dummy"-body to a flexible marker, and then connect the joint to this rigid body. The reason for the concept used in ADAMS to treat equations of motions for a system of multiple bodies, which is embodied in the required properties of the ADAMS modal neutral file. The required orthogonalization of the constraint modes makes all DOFs in the

condensed model modal. The instantaneous position of any marker on a flexible body can only be obtained from modal superposition.

#### 6 COMPARISON OF THE CONDENSATION METHODS

Condensation methods can be compared is several different ways. There can be a focus on the *accuracy* of the methods. Conceptual *similarities* and differences between the methods can be basis for another type of comparison. In the design context it is important to compare the ability to treat models of large and complex systems, i.e. the *scalability* of the methods. With the accuracy, similarity, and scalability of each method properly assessed, it is possible to define the purpose of each method and to decide on if some methods are complementary or not.

#### 6.1 Similarities

If we use the CMS1 attachment and the CMS2 constraint modes as the only basis for reduction, we get a model that is condensed to the interface DOF(s). For the present system we get:

$$\mathbf{K}_{cms1,i\_f} = \begin{bmatrix} 1.5 & 1.5 & 1.5 & 1 & 0.5 \end{bmatrix} \mathbf{K} \begin{bmatrix} 1.5 \\ 1.5 \\ 1.5 \\ 1 \\ 0.5 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1.5 \end{bmatrix}, \ \mathbf{M}_{cms1,i\_f} = \frac{\rho AL}{2} \begin{bmatrix} 18.5 \end{bmatrix}$$
(77)
$$\mathbf{K}_{cms2,i\_f} = \frac{AE}{L} \begin{bmatrix} 1 & 1 & 1 & 2/3 & 1/3 \end{bmatrix} \mathbf{K} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2/3 \\ 1/3 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 2/3 \end{bmatrix}, \ \mathbf{M}_{cms2,i\_f} = \frac{\rho AL}{2} \begin{bmatrix} 74/9 \end{bmatrix}$$
(78)

Solving the two eigenvalue problems for the matrices in for 77 and 78 and expanding the mode shape to all unrestrained DOF gives:

$$\lambda_{cms1,i_f} = \frac{2E}{\rho L^2} 0.0811, \ \mathbf{D}_{cms1,i_f}^T = \begin{bmatrix} 1.5 & 1.5 & 1.5 & 1 & 0.5 \end{bmatrix}$$
(79)

$$\lambda_{cms2,i_f} = \frac{2E}{\rho L^2} 0.0811 \,\mathbf{D}_{cms2,i_f}^T = \begin{bmatrix} 1 & 1 & 1 & 2/3 & 1/3 \end{bmatrix}$$
(80)

One observation is that the eigenvectors in 79 and 80 are equal to the attachment and constraint mode shapes respectively. Another observation is that the condensed stiffness and mass matrices in 67 are completely equal to the reduced matrices in the static condensation case with one MDOF, see 20 and 21. A third observation is that the solutions in 79 and 80 are equivalent to the result for the 1 DOF static condensation problem given in 22 and 23.

Using the constraint modes as the only Ritz vectors is thus in fact a static condensation, with the interface DOFs as the masters and all internal DOFs as the slaves.

### 6.2 Accuracy

The numerical accuracy in terms of the errors in the eigenvalues and in the mode shapes of the different condensation methods are collected in tables 1 and 2. It can be seen that static condensation has the potential to model the dynamic behavior of a system of submodels with good accuracy. The quality of the condensed systems model is although pathologically sensitive to the master degrees of freedom (MDOFs) selected. The CMS1 method is also slightly more accurate than the CMS2 method. The component modes used by ADAMS are equaivalent with models condensed with the CMS2 method.

Table 1. Condensation errors in the calculated eigenvalues.							
Model	<i>E</i> <sub>1</sub>	<i>E</i> <sub>2</sub>	E3	E4	$\mathcal{E}_5$		
Full model (5 MDOFs)	0	0	0	0	0		
Static condensation (1 MDOF)	0.1332	-	-	-	-		
Cms1 (1 MDOF)	0.1332	-	-	-	-		
Cms2 (1 MDOF)	0.1332	-	-	-	-		
Static condensation (3 MDOFs)	0.0034-0.0929	0.1452-0.4459	0.0000-0.4258	-	-		
Cms1 (3 MDOFs)	0.0043	0.5099	0.2837	-	-		
Cms2 (3 MDOFs)	0.0487	0.6154	0.3780	-	-		
ADAMS/Flex (3 MDOFs)	0.0487	0.6154	0.3780	-	-		

Table 2. Condensation errors in the calculated eigenvectors.							
Model	<i>Y</i> 1	γ2	γз	<i>Y</i> 4	γ5		
Full model (5 MDOFs)	0	0	0	0	0		
Static condensation (1 MDOF)	0.0074	-	-	-	-		
Cms1 (1 MDOF)	0.0074	-	-	-	-		
Cms2 (1 MDOF)	0.0074	-	-	-	-		
Static condensation (3 MDOFs)	0.0006-0.0103	0.0210-0.2664	0.0000-0.6414	-	-		
Cms1 (3 MDOFs)	0.0001	0.1651	0.9143	-	-		
Cms2 (3 MDOFs)	0.0054	0.8113	0.8020	-	-		
ADAMS/Flex (3 MDOFs)	0.0054	0.8113	0.8020	-	-		

## 6.3 Scalability

The scalability of a modeling method is here viewed as the multidimensional ability to deal with complexity in the model. The first dimension referred to as *spatial scalability* in table 3, is the ability to handle models composed from many components. The *multiphysics scalability* dimension is the ability to expand the number of physical domains, e.g. from a model in the mechanical domain to a thermomechanical model. The capability to model a detailed *contact* condition is treated here as the third dimension.

Table 3. Scalability of models for different methods.						
Model	spatial	multiphysics	contact			
Full model	low	high	high			
Static condensation	high	some	high			
Component mode synthesis (Cms1)	medium	low	high			
Component mode synthesis (Cms2)	high	low	high			
ADAMS/Flex (modified Cms2)	medium	low	low			

The scalability differs considerably between the methods. A full FE model can treat multiphysics behavior and detailed contact conditions, but a large model in terms of the number of DOFs easily outgrows the computer resources that are available. Static condensation allows systems models to be synthesized from very compact submodels. Static condensation may be performed for other physical matrices than stiffness and mass, e.g. conductivity and specific heat matrices. Some limited multiphysics modeling may be performed by overlapping condensed models that represent different physical domains. Complex contact regions can be defined between statically condensed models, as long as the DOFs on the interacting surfaces as retained as master DOFs in the condensed models. Component mode synthesis is targeted for dynamic simulations in the mechanical domain. The two CMS methods are thus not suitable for treating other physical domains. Since all master DOFs at the mating surfaces are treated as ordinary translational and rotational DOFs, the both CMS methods has the same capability as the static condensation method to model contact between complex regions on the systems level. The CMS1 method requires an inversion of the stiffness matrix of the system to calculate the attachment modes, whereas the CMS2 method allows complete submodels to be created on the component level. The CMS2 method is thus highly spatially scalable and the CMS1 method is far less scalable in the spatial domain. The modified CMS2 method used by ADAMS/Flex has exactly the same accuracy as the CMS2 method, but since all DOFs retained at the mating surfaces are converted to modal DOFs, all interaction between interacting submodels require a transformation from modal DOFs to physical DOFs. This limits the spatial scalability as well as it severely limits the ability to treat contact interaction between flexible surfaces.

## 7 ACKNOWLEDGEMENTS

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