HANNING WINDOW COMPENSATION FACTOR Revision A

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August 18, 2008

A Fourier transform may have a leakage error, whereby energy is smeared across adjacent frequency bands.

The leakage error can be reduced by subjecting the time history to a window, such a Hanning window. The Hanning window is given in equation (1).

$$w(t) = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi t}{T}\right), \qquad 0 \le t \le T \tag{1}$$

The Hanning window forces the signal to start and stop at zero amplitude as shown in Figures 1 and 2.

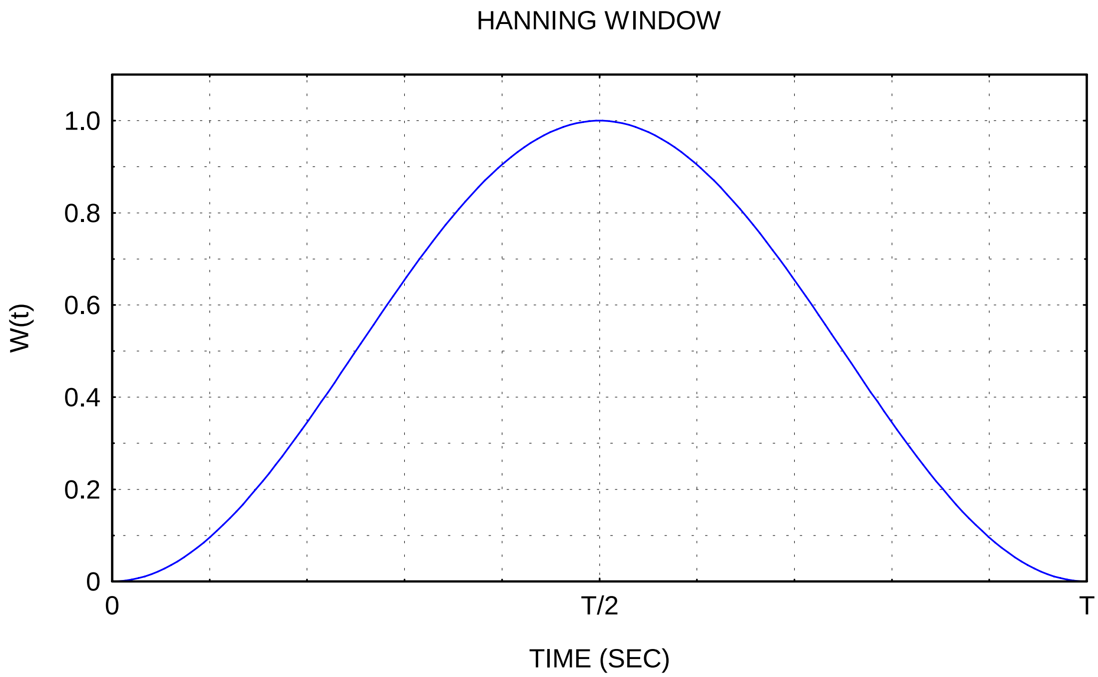

Figure 1.

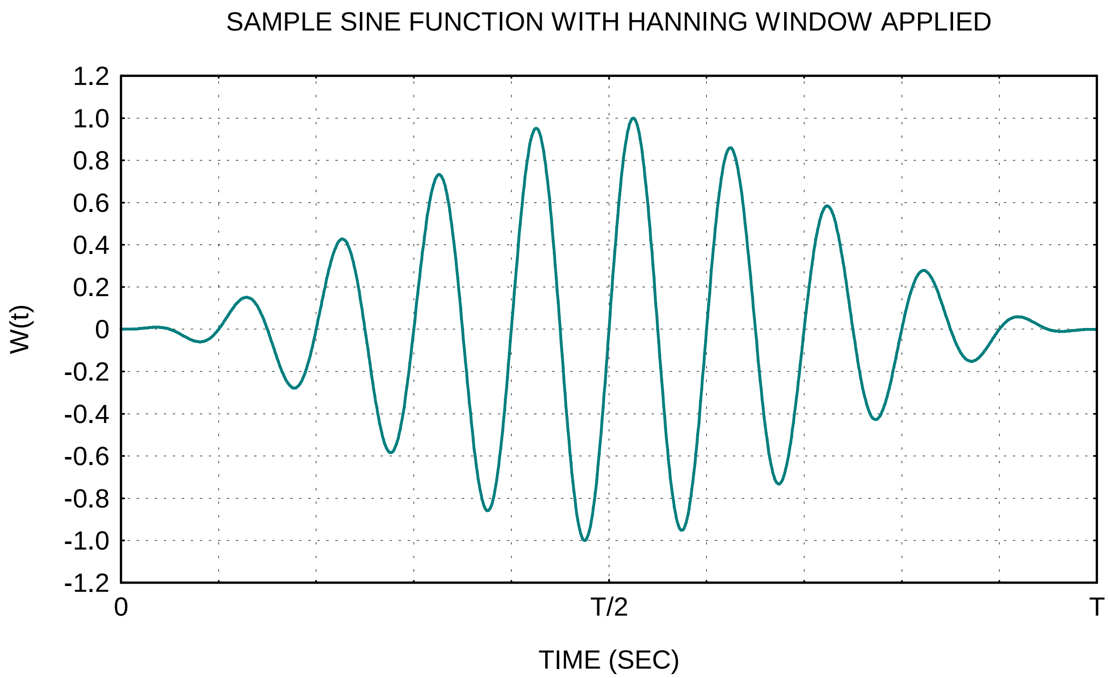

Figure 2.

A compensation factor must be applied to the windowed signal so that the RMS value remains the same as the original signal.

Derive the compensation factor.

Consider a sample sine function.

$$y(t) = A \sin\left(2\pi f\, t\right) \tag{2}$$

Assume that the duration is such that an integer number of cycles occur, for simplicity.

Let $y_{h,rms}$ be the rms value of the signal with the Hanning windowed applied.

$$y_{h,rms}{}^{2} = \frac{1}{T}\int_{0}^{T} \{w(t)y(t)\}^{2}\, dt \tag{3}$$

$$y_{h,rms}{}^2 = \frac{1}{T}\int_0^T \left\{\left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi t}{T}\right)\right]\left[A\sin(2\pi f\ t)\right]\right\}^2 dt \qquad (4)$$

Note that the total signal duration T is such that

$$f = \frac{n}{T} \ , \qquad n = \text{integer} \geq 2 \qquad (5)$$

Again, equation (5) is a simplifying assumption.

$$y_{h,rms}{}^2 = \frac{1}{T}\int_0^T \left\{\left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi t}{T}\right)\right]\left[A\sin\left(\frac{2\pi n\ t}{T}\right)\right]\right\}^2 dt \qquad (6)$$

$$y_{h,rms}{}^2 = \frac{1}{T}\int_0^T \left\{\frac{1}{2}A\sin\left(\frac{2\pi n\ t}{T}\right) - \frac{1}{2}A\sin\left(\frac{2\pi n\ t}{T}\right)\cos\left(\frac{2\pi t}{T}\right)\right\}^2 dt \qquad (7)$$

$$y_{h,rms}{}^2 = \frac{A^2}{4T}\int_0^T \left\{\sin\left(\frac{2\pi n\ t}{T}\right) - \sin\left(\frac{2\pi n\ t}{T}\right)\cos\left(\frac{2\pi t}{T}\right)\right\}^2 dt \qquad (8)$$

$$y_{h,rms}{}^2 = \frac{A^2}{4T}\int_0^T \left\{\sin\left(\frac{2\pi n\ t}{T}\right) - \frac{1}{2}\sin\left(\frac{2\pi(n+1)t}{T}\right) - \frac{1}{2}\sin\left(\frac{2\pi(n-1)t}{T}\right)\right\}^2 dt \qquad (9)$$

$$
\begin{aligned}
{y_{h,rms}}^2 = \; & \frac{A^2}{4T}\int_0^T \sin^2\left(\frac{2\pi n t}{T}\right)dt \\
& + \frac{A^2}{16T}\int_0^T \sin^2\left(\frac{2\pi(n+1)t}{T}\right)dt \\
& + \frac{A^2}{16T}\int_0^T \sin^2\left(\frac{2\pi(n-1)t}{T}\right)dt \\
& + \frac{A^2}{4T}\int_0^T \left\{-\sin\left(\frac{2\pi n t}{T}\right)\sin\left(\frac{2\pi(n+1)t}{T}\right)\right\}dt \\
& + \frac{A^2}{4T}\int_0^T \left\{-\sin\left(\frac{2\pi n t}{T}\right)\sin\left(\frac{2\pi(n-1)t}{T}\right)\right\}dt \\
& + \frac{A^2}{8T}\int_0^T \left\{\sin\left(\frac{2\pi(n+1)t}{T}\right)\sin\left(\frac{2\pi(n-1)t}{T}\right)\right\}dt
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
{y_{h,rms}}^2 = \; & \frac{A^2}{4T}\int_0^T \left\{\frac{1}{2}-\frac{1}{2}\cos\left(\frac{4\pi n t}{T}\right)\right\}dt \\
& + \frac{A^2}{16T}\int_0^T \left\{\frac{1}{2}-\frac{1}{2}\cos\left(\frac{4\pi(n+1)t}{T}\right)\right\}dt \\
& + \frac{A^2}{16T}\int_0^T \left\{\frac{1}{2}-\frac{1}{2}\cos\left(\frac{4\pi(n-1)t}{T}\right)\right\}dt \\
& - \frac{A^2}{8T}\int_0^T \left\{-\cos\left(\frac{2\pi(2n+1)t}{T}\right)+\cos\left(\frac{2\pi t}{T}\right)\right\}dt \\
& - \frac{A^2}{8T}\int_0^T \left\{-\cos\left(\frac{2\pi(2n-1)t}{T}\right)+\cos\left(\frac{2\pi t}{T}\right)\right\}dt \\
& + \frac{A^2}{16T}\int_0^T \left\{-\cos\left(\frac{2\pi n t}{T}\right)+\cos\left(\frac{4\pi t}{T}\right)\right\}dt
\end{aligned}
\tag{11}
$$

The last three terms of equation (11) are zero by inspection given that an integer number of cycles occur with $n \geq 2$.

$$y_{h,rms}^2 = \frac{A^2}{4T}\int_0^T \left\{\frac{1}{2}-\frac{1}{2}\cos\left(\frac{4\pi n t}{T}\right)\right\}dt + \frac{A^2}{16T}\int_0^T \left\{\frac{1}{2}-\frac{1}{2}\cos\left(\frac{4\pi(n+1)t}{T}\right)\right\}dt + \frac{A^2}{16T}\int_0^T \left\{\frac{1}{2}-\frac{1}{2}\cos\left(\frac{4\pi(n-1)t}{T}\right)\right\}dt \tag{12}$$

The respective integrals of the cosine terms of equation (12) are zero by inspection given that an integer number of cycles occur with $n \geq 2$.

$$y_{h,rms}^2 = \frac{A^2}{4T}\int_0^T \left\{\frac{1}{2}\right\}dt + \frac{A^2}{16T}\int_0^T \left\{\frac{1}{2}\right\}dt + \frac{A^2}{16T}\int_0^T \left\{\frac{1}{2}\right\}dt \tag{13}$$

$$y_{h,rms}^2 = \frac{A^2}{8}+\frac{A^2}{32}+\frac{A^2}{32} \tag{14}$$

$$y_{h,rms}^2 = \frac{4A^2}{32}+\frac{A^2}{32}+\frac{A^2}{32} \tag{15}$$

$$y_{h,rms}^2 = \frac{6A^2}{32} \tag{16}$$

$$y_{h,rms}^2 = \frac{3}{16}A^2 \tag{17}$$

The rms value of the data with the Hanning window applied is

$$y_{h,rms} = \sqrt{\frac{3}{16}}\,A \tag{18}$$

The rms value of the original data is

$$y_{,rms} = \frac{1}{\sqrt{2}} A \qquad (19)$$

Define a compensation factor as

$$\alpha = \frac{y_{,rms}}{y_{h,rms}} \qquad (20)$$

$$\alpha = \frac{\frac{1}{\sqrt{2}}}{\sqrt{\frac{3}{16}}} \qquad (21)$$

$$\alpha = \sqrt{\frac{16}{6}} \qquad (22)$$

The compensation factor is thus

$$\alpha = \sqrt{\frac{8}{3}} \qquad (23)$$

The modified Hanning window equation is thus

$$\hat{w}(t) = \sqrt{\frac{8}{3}} \left\{ \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi t}{T}\right) \right\}, \qquad 0 \le t \le T \qquad (24)$$