JACOBIAN MATRIX AND DETERMINANT
Revision C

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May 26, 2012

Introduction

The Jacobian of a function describes the orientation of a tangent plane to the function at a given point.

Consider a function F given by m real-valued component functions:

$$y_1(x_1, \ldots, x_n), \ldots, y_m(x_1, \ldots, x_n)$$

The Jacobian matrix J of F is

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} \tag{1}$$

Example

Consider a single-degree-of-freedom system with the following differential equation.

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{2}$$

$$\ddot{x} + (c/m)\dot{x} + (k/m)x = (F/m) \tag{3}$$

$$\ddot{x} = -(c/m)\dot{x} - (k/m)x - (F/m) \tag{4}$$

Let

$$y = \dot{x}$$

$$\dot{y} = \ddot{x}$$

The second-order equation is thus transformed into two first-order equations.

$$\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \quad (5)$$

The Jacobian is

$$J = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \quad (6)$$

Other Applications

The Jacobian is also used in geometrical transformation applications, as shown in Appendix A

Reference

1. T. Irvine, The State Space Method for Solving Shock and Vibration Problems, Revision A, Vibrationdata, 2005.

APPENDIX A

Four-Node, Two-Dimensional Isoparametric Plate Element

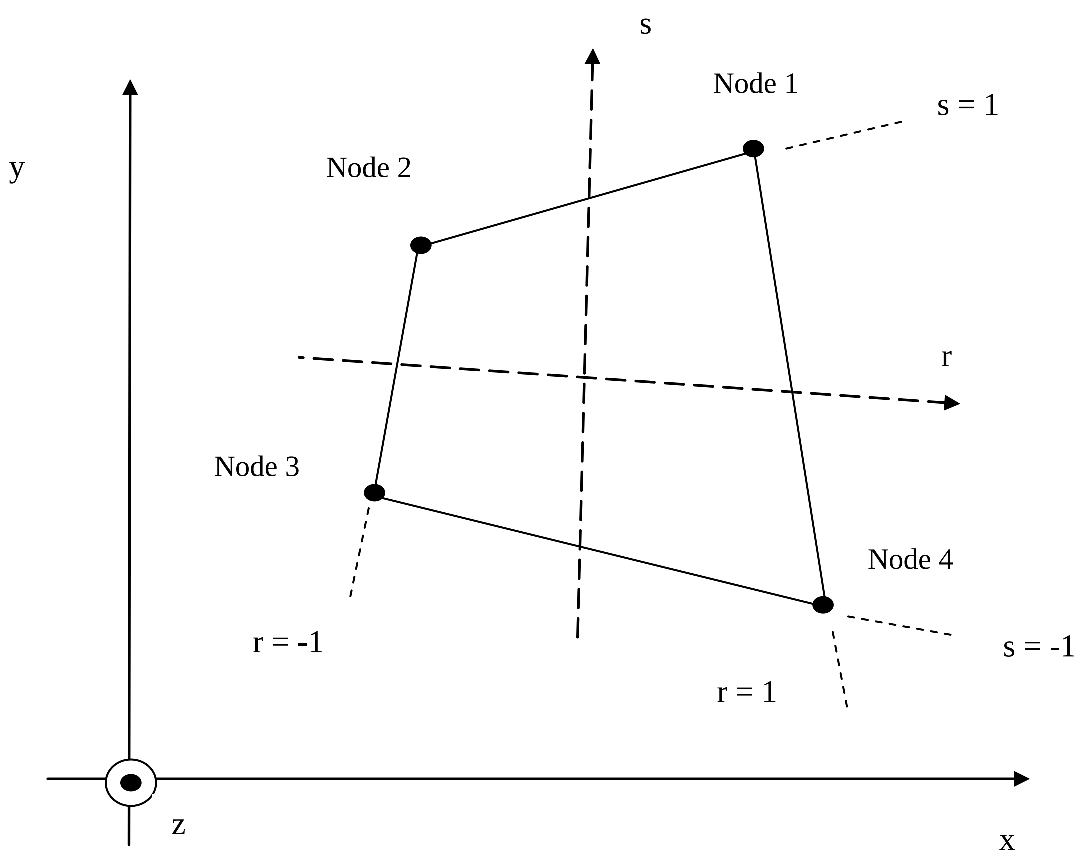

Figure A-1.

Note that

$-1 \leq r \leq +1$

$-1 \leq s \leq +1$

The coordinate interpolation is

$$x = \frac{1}{4}(1+r)(1+s)x_1 + \frac{1}{4}(1-r)(1+s)x_2 + \frac{1}{4}(1-r)(1-s)x_3 + \frac{1}{4}(1+r)(1-s)x_4 \tag{A-1}$$

$$y = \frac{1}{4}(1+r)(1+s)y_1 + \frac{1}{4}(1-r)(1+s)y_2 + \frac{1}{4}(1-r)(1-s)y_3 + \frac{1}{4}(1+r)(1-s)y_4 \tag{A-2}$$

$$\frac{\partial x}{\partial r} = \frac{1}{4}(1+s)x_1 - \frac{1}{4}(1+s)x_2 - \frac{1}{4}(1-s)x_3 + \frac{1}{4}(1-s)x_4 \tag{A-3}$$

$$\frac{\partial x}{\partial s} = \frac{1}{4}(1+r)x_1 + \frac{1}{4}(1-r)x_2 - \frac{1}{4}(1-r)x_3 - \frac{1}{4}(1+r)x_4 \tag{A-4}$$

$$\frac{\partial y}{\partial r} = \frac{1}{4}(1+s)y_1 - \frac{1}{4}(1+s)y_2 - \frac{1}{4}(1-s)y_3 + \frac{1}{4}(1-s)y_4 \tag{A-5}$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}(1+r)x_1 + \frac{1}{4}(1-r)x_2 - \frac{1}{4}(1-r)x_3 - \frac{1}{4}(1+r)x_4 \tag{A-6}$$

The transformation equation is

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial r} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \tag{A-7}$$

The Jacobian matrix J is

$$J = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \tag{A-8}$$

By substitution,

$J =$

$$\begin{bmatrix} [(1+s)x_1 - (1+s)x_2 - (1-s)x_3 + (1-s)x_4]/4 & [(1+s)y_1 - (1+s)y_2 - (1-s)y_3 + (1-s)y_4]/4 \\ \\ [(1+r)x_1 + (1-r)x_2 - (1-r)x_3 - (1+r)x_4]/4 & [(1+r)y_1 + (1-r)y_2 - (1-r)y_3 - (1+r)y_4]/4 \end{bmatrix}$$

(A-9)

The inverse of the Jacobian is evaluated symbolically using the software tool wxMaxima 0.8.7.

Let

$$\hat{J} = J^{-1} \qquad \text{(A-10)}$$

$$\hat{J} =$$

$$\frac{1}{den}\begin{bmatrix} (2r+2)y_4 + (2-2r)y_3 + (2r-2)y_2 + (-2r-2)y_1 & -\left[(2s-2)y_4 + (2-2s)y_3 + (2s+2)y_2 + (-2s-2)y_1\right] \\ -\left[(2s-2)x_4 + (2-2s)x_3 + (2s+2)x_2 + (-2s-2)x_1\right] & (2s-2)x_4 + (2-2s)x_3 + (2s+2)x_2 + (-2s-2)x_1 \end{bmatrix}$$

(A-11)

den = ((s-1) x3+(-s-r) x2+(r+1) x1) y4+((1-s) x4+(r-1) x2+(s-r) x1) y3
+((s+r) x4+(1-r) x3+(-s-1) x1) y2+((-r-1) x4+(r-s) x3+(s+1) x2) y1

(A-12)

The wxMaxima format is

$\hat{J}_{11} =$

((2*r+2)*y4+(2-2*r)*y3+(2*r-2)*y2+(-2*r-2)*y1)/(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1) (A-13)

$\hat{J}_{12} =$

-((2*s-2)*y4+(2-2*s)*y3+(2*s+2)*y2+(-2*s-2)*y1)/(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1) (A-14)

$\hat{J}_{21} =$

-((2*r+2)*x4+(2-2*r)*x3+(2*r-2)*x2+(-2*r-2)*x1)/(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1) (A-15)

$\hat{J}_{22} =$

((2*s-2)*x4+(2-2*s)*x3+(2*s+2)*x2+(-2*s-2)*x1)/(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1) (A-16)

The inverse Jacobian at each of the four nodes is

$\hat{J}_1 =$

$$\frac{1}{(x_2 - x_1)y_4 + (x_1 - x_4)y_2 + (x_4 - x_2)y_1}\begin{bmatrix} -(2y_4 - 2y_1) & 2y_2 - 2y_1 \\ 2x_4 - 2x_1 & -(2x_2 - 2x_1) \end{bmatrix}$$

(A-17)

$\hat{J}_2 =$

$$\frac{1}{(x_2 - x_1)y_3 + (x_1 - x_3)y_2 + (x_3 - x_2)y_1}\begin{bmatrix} -(2y_4 - 2y_1) & 2y_2 - 2y_1 \\ 2x_4 - 2x_1 & -(2x_2 - 2x_1) \end{bmatrix}$$

(A-18)

$\hat{J}_3 =$

$$\frac{1}{(x_3 - x_2)y_4 + (x_2 - x_4)y_3 + (x_4 - x_3)y_2}\begin{bmatrix} -(2y_3 - 2y_2) & -(2y_4 - 2y_3) \\ 2x_3 - 2x_2 & 2x_4 - 2x_3 \end{bmatrix}$$

(A-19)

$\hat{J}_4 =$

$$\frac{1}{(x_3 - x_1)y_4 + (x_1 - x_4)y_3 + (x_4 - x_3)y_1}\begin{bmatrix} -(2y_4 - 2y_1) & -(2y_4 - 2y_3) \\ 2x_4 - 2x_1 & 2x_4 - 2x_3 \end{bmatrix}$$

(A-20)

Note that the Jacobian and its inverse should be evaluated at each node. Furthermore, at node i,

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} = J_i^{-1} \begin{bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial s} \end{bmatrix}, \quad \text{at } r = r_i \text{ and } s = s_j$$

(A-21)

The determinant of the Jacobian is found via wxMaxima.

```
a(r,s):=((1+s)*x1-(1+s)*x2+(-(1-s))*x3+(1-s)*x4)/4;
b(r,s):=((1+s)*y1-(1+s)*y2+(-(1-s))*y3+(1-s)*y4)/4;
c(r,s):=((1+r)*x1+(1-r)*x2+(-(1-r))*x3+(-(1+r))*x4)/4;
d(r,s):=((1+r)*y1+(1-r)*y2+(-(1-r))*y3+(-(1+r))*y4)/4;
ratsimp(determinant(matrix([a(r,s),b(r,s)],[c(r,s),d(r,s)])));
```

(A-22)

det J =

```
-(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+
((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+
((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1)/8
```

(A-23)

Equation (A-23) is left in wxMaxima format because it is convenient to copy and paste into Matlab.

Other useful wxMaxima commands:

```
J(r,s):=matrix([a(r,s),b(r,s)],[c(r,s),d(r,s)]);

detJ(r,s):=ratsimp(determinant(matrix([a(r,s),b(r,s)],[c(r,s),d(r,s)])));

Jinv(r,s):=invert(matrix([a(r,s),b(r,s)],[c(r,s),d(r,s)]));
```