LONGITUDINAL NATURAL FREQUENCIES OF RODS
AND RESPONSE TO INITIAL CONDITIONS

Revision B

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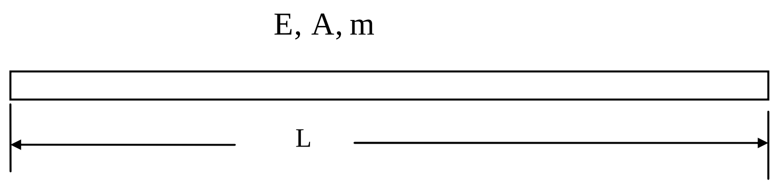
Consider a thin rod.




E is the modulus of elasticity.
A is the cross-section area.
m is the mass per unit length.

The longitudinal displacement u(x, t) is governed by the equation

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right] = m(x)\frac{\partial^2 u}{\partial t^2} \tag{1}$$

This equation is taken from Reference 1.

For a uniform cross-section, the governing equation simplifies to

$$EA(x)\frac{\partial^2 u}{\partial x^2} = m(x)\frac{\partial^2 u}{\partial t^2} \tag{2}$$

Consider a beam with uniform mass density. The governing equation simplifies to

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\rho}{E}\right)\frac{\partial^2 u}{\partial t^2} \tag{3}$$

where

$\rho$ is the mass per unit volume.

Let

$$c = \sqrt{\frac{E}{\rho}} \tag{4}$$

Note that c is the longitudinal wave velocity. Substitute equation (4) into (3).

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{1}{c^2}\right)\frac{\partial^2 u}{\partial t^2} \tag{5}$$

Separate the variables. Let

$$u(x,t) = U(x)T(t) \tag{6}$$

Substitute equation (6) into (5).

$$\frac{\partial^2}{\partial x^2}\left[U(x)T(t)\right] = \left(\frac{1}{c^2}\right)\frac{\partial^2}{\partial t^2}\left[U(x)T(t)\right] \tag{7}$$

Perform the partial differentiation.

$$U''(x)T(t) = \left(\frac{1}{c^2}\right)U(x)T''(t) \tag{8}$$

Divide through by U(x)T(t).

$$\frac{U''(x)}{U(x)} = \left(\frac{1}{c^2}\right)\frac{T''(t)}{T(t)} \tag{9}$$

$$c^2\,\frac{U''(x)}{U(x)} = \frac{T''(t)}{T(t)} \tag{10}$$

Each side of equation (10) must equal a constant. Let $\omega$ be a constant.

$$c^2\,\frac{U''(x)}{U(x)} = \frac{T''(t)}{T(t)} = -\omega^2 \tag{11}$$

The time equation is

$$\frac{T''(t)}{T(t)} = -\omega^2 \tag{12}$$

$$T''(t) = -\omega^2 T(t) \tag{13}$$

$$T''(t) + \omega^2 T(t) = 0 \tag{14}$$

Propose a solution

$$T(t) = a \sin(\omega t) + b \cos(\omega t) \tag{15}$$

$$T'(t) = a\,\omega \cos(\omega t) - b\,\omega \sin(\omega t) \tag{16}$$

$$T''(t) = -a\,\omega^2 \sin(\omega t) - b\,\omega^2 \cos(\omega t) \tag{17}$$

Verify the proposed solution. Substitute into equation (14).

$$-a\,\omega^2 \sin(\omega t) - b\,\omega^2 \cos(\omega t) + \omega^2 \left[\sin(\omega t) + \omega^2 \cos(\omega t)\right] = 0 \tag{18}$$

$$0 = 0 \tag{19}$$

Equation (15) is thus verified as a solution.

There is not a unique $\omega$, however, in equation (11). This is demonstrated later in the derivation. Thus a subscript n must be added as follows.

$$T_n(t) = a_n \sin(\omega_n t) + b_n \cos(\omega_n t) \tag{20}$$

The spatial equation is

$$c^2 \frac{U''(x)}{U(x)} = -\omega^2 \tag{21}$$

$$c^2 U''(x) = -\omega^2 U(x) \tag{22}$$

$$c^2 U''(x) + \omega^2 U(x) = 0 \tag{23}$$

$$U''(x) + \frac{\omega^2}{c^2} U(x) = 0 \quad (24)$$

Equation (24) is similar to equation (14). Thus, a solution can be found by inspection.

$$U(x) = d \sin\left(\frac{\omega x}{c}\right) + e \cos\left(\frac{\omega x}{c}\right) \quad (25)$$

The slope equation is

$$U'(x) = \left[\frac{\omega}{c}\right]\left[d \cos\left(\frac{\omega x}{c}\right) - e \sin\left(\frac{\omega x}{c}\right)\right] \quad (26)$$

Now consider three boundary condition cases as shown in the following appendices.

REFERENCE


1. L. Meirovitch, Analytical Methods in Vibrations, Macmillan, New York, 1967.

## APPENDIX A

Case I. Fixed-Fixed

The left boundary condition is

$$u(0,t) = 0 \qquad \text{(zero displacement)} \tag{A-1}$$

$$U(0)T(t) = 0 \tag{A-2}$$

$$U(0) = 0 \tag{A-3}$$

The right boundary condition is

$$u(L,t) = 0 \qquad \text{(zero displacement)} \tag{A-4}$$

$$U(L)T(t) = 0 \tag{A-5}$$

$$U(L) = 0 \tag{A-6}$$

Substitute equation (A-3) into (25).

$$e = 0 \tag{A-7}$$

Thus, the displacement equation becomes

$$U(x) = d \sin\left(\frac{\omega x}{c}\right) \tag{A-8}$$

Substitute equation (A-6) into (A-8).

$$d \sin\left(\frac{\omega L}{c}\right) = 0 \tag{A-9}$$

The constant d must be non-zero for a non-trivial solution. Thus,

$$\frac{\omega_n L}{c} = n\pi, \quad n = 1, 2, 3, \ldots \tag{A-10}$$

The $\omega$ term is given a subscript n because there are multiple roots.

$$\omega_n = n\pi \frac{c}{L}, \quad n = 1, 2, 3, \ldots \tag{A-11}$$

The displacement function the fixed-fixed rod is

$$U_n(x) = d_n \sin\left(\frac{\omega_n x}{c}\right) \tag{A-12}$$

$$U_n(x) = d_n \sin\left(\frac{n\pi x}{L}\right) \tag{A-13}$$

Substitute the natural frequency term into the time equation.

$$T_n(t) = a_n \sin\left(\frac{n\pi c t}{L}\right) + b_n \cos\left(\frac{n\pi c t}{L}\right) \tag{A-14}$$

The displacement function is thus

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \left[ d_n \sin\left(\frac{n\pi x}{L}\right)\right]\left[ a_n \sin\left(\frac{n\pi c t}{L}\right) + b_n \cos\left(\frac{n\pi c t}{L}\right)\right]\right\} \tag{A-15}$$

The coefficients can be simplified as follows

$$A_n = d_n a_n \tag{A-16}$$

$$B_n = d_n b_n \tag{A-17}$$

By substitution, the displacement equation is

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \left[ \sin\left(\frac{n\pi x}{L}\right)\right]\left[ A_n \sin\left(\frac{n\pi c t}{L}\right) + B_n \cos\left(\frac{n\pi c t}{L}\right)\right]\right\} \tag{A-18}$$

APPENDIX B

Case II.  Fixed-Free

The left boundary conditions is

$$u(0,t) = 0 \qquad \text{(zero displacement)} \tag{B-1}$$

$$U(0)T(t) = 0 \tag{B-2}$$

$$U(0) = 0 \tag{B-3}$$

The right boundary condition is

$$\left.\frac{\partial}{\partial x} u(x,t)\right|_{x=L} = 0 \qquad \text{(zero stress)} \tag{B-4}$$

$$U'(L)T(t) = 0 \tag{B-5}$$

$$U'(L) = 0 \tag{B-6}$$

Substitute equation (B-3) into (25).

$$e = 0 \tag{B-7}$$

Thus, the displacement equation becomes

$$U(x) = d\sin\left(\frac{\omega x}{c}\right) \tag{B-8}$$

$$U'(x) = \left[\frac{\omega}{c}\right]\left[d\cos\left(\frac{\omega x}{c}\right)\right] \tag{B-9}$$

Substitute equation (B-6) into equation (B-9).

$$d\cos\left(\frac{\omega L}{c}\right) = 0 \qquad \text{(B-10)}$$

The constant d must be non-zero for a non-trivial solution. Thus,

$$\frac{\omega_n L}{c} = \left(\frac{2n-1}{2}\right)\pi, \quad n = 1, 2, 3, \ldots \qquad \text{(B-11)}$$

The $\omega$ term is given a subscript n because there are multiple roots.

$$\omega_n = \left(\frac{2n-1}{2}\right)\pi\frac{c}{L}, \quad n = 1, 2, 3, \ldots \qquad \text{(B-12)}$$

The displacement function for the fixed-free rod is

$$U_n(x) = d_n \sin\left(\frac{\omega_n x}{c}\right) \qquad \text{(B-13)}$$

$$U_n(x) = d_n \sin\left(\frac{(2n-1)\pi\ x}{2L}\right) \qquad \text{(B-14)}$$

Substitute the natural frequency term into the time equation.

$$T_n(t) = a_n \sin\left(\frac{(2n-1)\pi c t}{2L}\right) + b_n \cos\left(\frac{(2n-1)\pi c t}{2L}\right) \qquad \text{(B-15)}$$

The displacement function is thus

$$u(x,t) = \sum_{n=1}^{\infty}\left\{\left[d_n \sin\left(\frac{(2n-1)\pi x}{2L}\right)\right]\left[a_n \sin\left(\frac{(2n-1)\pi c t}{2L}\right) + b_n \cos\left(\frac{(2n-1)\pi c t}{2L}\right)\right]\right\}$$

(B-16)

Simplify the coefficients.

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \left[ \sin\left( \frac{(2n-1)\,\pi\, x}{2L} \right) \right] \left[ A_n \sin\left( \frac{(2n-1)\,\pi\, c\, t}{2L} \right) + B_n \cos\left( \frac{(2n-1)\,\pi\, c\, t}{2L} \right) \right] \right\} \tag{B-17}$$

Now determine the effective mass of the rod for the fundamental mode. The stiffness k at free end of the fixed-free longitudinal rod is

$$k = \frac{EA}{L} \tag{B-18}$$

The formula for the fundamental frequency of a single-degree-of-freedom system is

$$\omega_1 = \sqrt{\frac{k}{m}} \tag{B-19}$$

Solve for the mass m.

$$m = \frac{k}{\omega_1{}^2} \tag{B-20}$$

Substitute the stiffness term from equation (B-18).

$$m = \frac{EA}{\omega_1{}^2 L} \tag{B-21}$$

Add a subscript e to denote that the mass is the effective mass.

$$m_e = \frac{EA}{\omega_1{}^2 L} \tag{B-22}$$

Calculate the fundamental frequency from equation (B-12).

$$\omega_1 = \left(\frac{1}{2}\right) \pi \frac{c}{L} \tag{B-23}$$

$$\omega_1 = \left(\frac{1}{2}\right) \pi \frac{1}{L} \sqrt{\frac{E}{\rho}} \tag{B-24}$$

$$\omega_1^2 = \left(\frac{\pi^2}{4L^2}\right)\frac{E}{\rho} \tag{B-25}$$

Substitute the frequency term from equation (B-25) into (B-22).

$$m_e = \frac{4L^2 \rho EA}{E\pi^2 L} \tag{B-26}$$

$$m_e = \frac{4L\rho A}{\pi^2} \tag{B-27}$$

Let M be the mass of the rod. The effective mass is

$$m_e = \frac{4}{\pi^2} M \tag{B-28}$$

APPENDIX C

Case III. Free-Free

The left boundary conditions is

$$\frac{\partial}{\partial x} u(x,t)\Big|_{x=0} = 0 \qquad \text{(zero stress)} \tag{C-1}$$

$$U'(0)T(t) = 0 \tag{C-2}$$

$$U'(0) = 0 \tag{C-3}$$

The right boundary condition is

$$\frac{\partial}{\partial x} u(x,t)\Big|_{x=L} = 0 \qquad \text{(zero stress)} \tag{C-4}$$

$$U'(L)T(t) = 0 \tag{C-5}$$

$$U'(L) = 0 \tag{C-6}$$

Apply equation (C-3) to (25).

$$d = 0 \tag{C-7}$$

Thus

$$U(x) = \cos\left(\frac{\omega x}{c}\right) \tag{C-8}$$

The slope equation is

$$U'(x) = -\left[\frac{\omega}{c}\right]\left[e \sin\left(\frac{\omega x}{c}\right)\right] \tag{C-9}$$

Substitute equation (C-6) into (C-9).

$$e \sin\left(\frac{\omega L}{c}\right) = 0 \tag{C-10}$$

The constant e must be non-zero for a non-trivial solution. Thus,

$$\frac{\omega_n L}{c} = n\pi, \quad n = 1, 2, 3, \ldots \tag{C-11}$$

The $\omega$ term is given a subscript n because there are multiple roots.

$$\omega_n = n\pi \frac{c}{L}, \quad n = 1, 2, 3, \ldots \tag{C-12}$$

The displacement function for the free-free rod is

$$U_n(x) = e_n \cos\left(\frac{\omega_n x}{c}\right) \tag{C-13}$$

$$U_n(x) = e_n \cos\left(\frac{n\pi x}{L}\right) \tag{C-14}$$

Substitute the natural frequency term into the time equation.

$$T_n(t) = a_n \sin\left(\frac{n\pi c t}{L}\right) + b_n \cos\left(\frac{n\pi c t}{L}\right) \tag{C-15}$$

The displacement function is thus

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \left[ e_n \cos\left(\frac{n\pi x}{L}\right) \right] \left[ a_n \sin\left(\frac{n\pi c t}{L}\right) + b_n \cos\left(\frac{n\pi c t}{L}\right) \right] \right\} \tag{C-16}$$

Simplify the coefficients.

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \left[ \cos\left(\frac{n\pi x}{L}\right) \right] \left[ A_n \sin\left(\frac{n\pi c t}{L}\right) + B_n \cos\left(\frac{n\pi c t}{L}\right) \right] \right\} \tag{C-17}$$

## APPENDIX D

Fixed-Free Rod Subjected to Initial Displacement and Initial Velocity

Recall

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \sin\left(\frac{\omega_n x}{c}\right) \left[A_n \sin(\omega_n t) + B_n \cos(\omega_n t)\right] \right\} \tag{D-1}$$

Let

$$u(x,0) = f(x) \tag{D-2}$$

$$\dot{u}(x,0) = g(x) \tag{D-3}$$

$$f(x) = \sum_{n=1}^{\infty} \left\{ B_n \sin\left(\frac{\omega_n x}{c}\right) \right\} \tag{D-4}$$

Premultiply by $\sin\left(\frac{\omega_m x}{c}\right)$ and integrate.

$$\int_0^L f(x) \sin\left(\frac{\omega_m x}{c}\right) dx = \sum_{n=1}^{\infty} \left\{ B_n \int_0^L \sin\left(\frac{\omega_m x}{c}\right) \sin\left(\frac{\omega_n x}{c}\right) dx \right\} \tag{D-5}$$

$$\omega_m = \left(\frac{2m-1}{2}\right) \pi \frac{c}{L}, \quad m = 1,2,3,\ldots \tag{D-6}$$

For $m \neq n$,

The integral on the right hand side of (D-5) goes to zero. The steps are omitted for brevity.

For $m = n$,

$$\int_0^L f(x)\sin\left(\frac{\omega_m x}{c}\right)dx = B_m \int_0^L \sin^2\left(\frac{\omega_m x}{c}\right)dx \tag{D-7}$$

$$\int_0^L f(x)\sin\left(\frac{\omega_m x}{c}\right)dx = \frac{1}{2}B_m \int_0^L \left[1-\cos\left(\frac{2\omega_m x}{c}\right)\right]dx \tag{D-8}$$

$$\int_0^L f(x)\sin\left(\frac{\omega_m x}{c}\right)dx = \frac{1}{2}B_m \left[x - \left(\frac{c}{2\omega_m}\right)\sin\left(\frac{2\omega_m x}{c}\right)\right]\Bigg|_0^L \tag{D-9}$$

$$\int_0^L f(x)\sin\left(\frac{\omega_m x}{c}\right)dx = \frac{1}{2}B_m \left[L - \left(\frac{c}{2\omega_m}\right)\sin\left(\frac{2\omega_m L}{c}\right)\right] \tag{D-10}$$

$$\omega_m = \left(\frac{2m-1}{2}\right)\pi\frac{c}{L}, \quad m = 1, 2, 3, \ldots \tag{D-11}$$

$$\int_0^L f(x)\sin\left(\frac{\omega_m x}{c}\right)dx = \frac{1}{2}B_m L \tag{D-12}$$

$$B_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\omega_m x}{c}\right)dx \tag{D-13}$$

$$\dot{u}(x,t) = \sum_{n=1}^{\infty} \left\{ \omega_n \sin\left(\frac{\omega_n x}{c}\right) \left[ A_n \cos(\omega_n t) - B_n \sin(\omega_n t) \right] \right\} \quad \text{(D-14)}$$

$$g(x) = \sum_{n=1}^{\infty} \left\{ \omega_n \sin\left(\frac{\omega_n x}{c}\right) \left[ A_n \right] \right\} \quad \text{(D-15)}$$

$$\int_0^L g(x) \sin\left(\frac{\omega_m x}{c}\right) dx = \omega_m \sum_{n=1}^{\infty} \left\{ A_n \int_0^L \sin\left(\frac{\omega_m x}{c}\right) \sin\left(\frac{\omega_n x}{c}\right) dx \right\} \quad \text{(D-16)}$$

Equation (D-16) is similar to (D-5).

$$A_m = \frac{2}{\omega_m L} \int_0^L f(x) \sin\left(\frac{\omega_m x}{c}\right) dx \quad \text{(D-17)}$$

Again,

$$\omega_m = \left(\frac{2m-1}{2}\right) \pi \frac{c}{L}, \quad m = 1, 2, 3, \ldots \quad \text{(D-11)}$$