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MODAL DENSITY OF THIN CIRCULAR CYLINDERS

by David K. Miller and Franklin D. Hart

Prepared by
NORTH CAROLINA STATE UNIVERSITY
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SUMMARY

A combined analytical and experimental study is made of the modal density of a thin cylindrical shell. Previous analytical work is discussed and an integral form solution is presented and evaluated numerically. Having cognizance of the experimental results, it is concluded that the integral form solution gives an accurate method for computing the cumulative number of resonant modes and the modal density of a thin cylindrical shell.

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INTRODUCTION

The concept of modal density arises from the fact that any continuous structure possesses an infinite number of natural modes of vibration. For this reason it is sometimes helpful when dealing with structures to introduce the concept of modal density. The modal density of a structure is essentially the density of the modes of vibration with respect to frequency. It is an indication of the spacing of the natural modes in the frequency domain.

When dealing with structures excited in a very complex or random fashion it is often useful to resort to statistical methods to determine information concerning the response of the structure to such loading. In order to apply this type of analysis it is found that the modal density of the structure in question must be known. Hence, in order to apply a statistical type of analysis to a structural response problem, it is necessary to know the modal density of the basic structural elements such as beams, plates, and shells. Since the cylindrical shell is a fairly common and basic structural component, especially in the spacecraft industry, it is logical that this should be a structural shape of some interest.

In the literature it is generally conceded that for vibrational modes with frequencies well above the so called ring frequency (that frequency at which the wave length in the material is equal to the circumference of the cylindrical structure) that the modal density of the cylindrical shell is equal to one-half the modal density of a flat plate with the same surface area. However below the ring frequency there is some disagreement in the literature as to what the proper expression for the modal density should be. It is therefore the purpose of this paper to obtain an expression for the modal density in this frequency range, and to present experimental results to verify the conclusions which have been reached.

The problem of the modal density of thin cylindrical shells is preceded by a discussion of the concept of modal density. One method which may be used in determining this property is also discussed in detail with several examples to illustrate the technique. Finally some mention is made concerning the application of the property of modal density with respect to the statistical energy method of analyzing complex vibrational problems and in relation to room acoustics.

The cylindrical shell problem is dealt with by presenting in detail three derivations for the number of resonant modes and the modal density. The first two derivations have already appeared in the literature. They are evaluated with respect to their assumptions and their differences are noted. The third derivation, which is essentially a modification of one of the first two, is then presented. The final derivation yields integral expressions for both the number of resonant modes and the modal density. These integral expressions have been evaluated numerically and are tabulated and plotted in dimensionless form.

Finally the experimental program which was conducted is discussed in detail. The results of this program have also been tabulated and plotted in dimensionless form and compared with the three analytical approaches presented.

The results of the investigation are then discussed in detail and conclusions are drawn as to the correct expression for the modal density of a thin cylindrical shell. Some remarks have also been made concerning the extent to which the concept of modal density is a useful and valid one, especially in the low frequency ranges.

REVIEW OF LITERATURE

The problem of determining the modal density of any given structure is equivalent to ascertaining the distribution of eigenvalues of large order corresponding to high mode numbers. A general discussion of the asymptotic distribution of eigenvalues for various classes of differential equations is given by Courant and Hilbert (1953). Expressions for the number of eigenvalues up to a given bound are given for differential equations with one, two, and three independent space variables. Although the treatment of the subject by Courant and Hilbert is approached from a basic mathematical point of view, the results have direct physical interpretation. It is pointed out that boundary conditions have no effect on the asymptotic distribution of the eigenvalues.

Bolotin (1962) has also given considerable attention to the asymptotic method in his studies of eigenvalue determination. In (1962) Bolotin presented a discussion of the asymptotic behavior of the eigenvalues for a generalized rectangular region of arbitrary dimension. He applied this technique to the problem of vibration of plates and shells, where the number of eigenvalues correspond to the number of natural frequencies of vibration. Correction factors were also introduced to extend the work of Courant and Hilbert (1953) to low mode numbers where boundary conditions must sometimes be considered. Bolotin (1960) presented a detailed discussion of the effect of edge conditions on the vibrational modes of elastic shells. At this point, he clarified the type of edge conditions one would expect to encounter, as well as the types of shells and mode numbers where the effects of the boundary conditions would be important.

In (1963) Bolotin presented a general treatment of the eigenvalue density problem for a general thin elastic shell of revolution. Bolotin again used the asymptotic method discussed by Courant and Hilbert (1953) in his work and obtained expressions for the number of natural frequencies and the modal density of a general elastic shell of revolution as elliptic integrals. The results of this work were also extended to the specific cases of the spherical shell and the circular cylindrical shell. Bolotin (1965) presents a discussion which is essentially an extension of his previous work in which he discusses the concentration points of natural modes, as well as the effects of shear and rotary inertia. These factors had previously not been considered in his derivation of the modal density expression for a thin elastic shell.

Without apparent knowledge of Bolotin's work, Heckl (1962) developed an expression for the natural frequencies of a cylindrical shell using impedance methods. He then represented the number of natural modes by a finite sum over all possible modes of vibration possible up to some upper frequency. He then replaced the summation by an integral and obtained an approximate expression for the modal density of a thin cylindrical shell. Heckl also presented some experimental findings in his report.

It may be noted that the results of Heckl (1962) are more frequently discussed in the literature when the modal density of a cylindrical shell is presented. The reason for this is probably due to the simplicity of Heckl's results and to the fact that they are in closed form. In addition, Heckl's results have been more readily available, since the work of Bolotin (1963) has only recently been translated from the Russian.

Another recent and fairly significant work in this field is that of Smith and Lyon (1965). This work treats the entire field structural vibration and specifically with regard to excitation via a sould field. The concept of modal density is introduced and its application with regard to structural vibration is discussed in detail.

The difference between the results presented by Bolotin (1963) and those presented by Heckl (1962) is the major reason for this paper. Both theories are seen to agree for frequencies above the ring frequency; however, below the ring frequency there is sufficient difference in the results presented to warrant further investigation of the matter.

THE CONCEPT OF MODAL DENSITY

General Discussion

It is well known that any structure such as a beam, plate, or shell has an infinite number of resonant frequencies at which it may vibrate, with each frequency corresponding to each of the principal modes. In dealing with dynamic response problems wherein the input forcing quantity has a broad spectral content, many modes will participate in the overall motion of the vibrating system. In such cases, it is sometimes useful to introduce the concept of modal density. For a given structure, the modal density is defined as the asymptotic expression for the density of the frequency distribution obtainable from the frequency equation of the structure. Thus, it is the continuous function obtained by successively dividing the number of resonant frequencies contained in a frequency interval $\Delta\omega$ by the interval width, $\Delta\omega$. If $\Delta N(\omega)$ is the total number of resonances in the frequency band $\Delta\omega$, then the modal density at the center band frequency ω_c in the interval $\Delta\omega$ may be written as,

$$n(\omega_c) = \frac{\Delta N(\omega)}{\Delta\omega} \quad . \tag{1}$$

It may be noted that the inverse of the modal density gives an indication of the spacing of the resonant frequencies in the frequency domain. Hence, a high modal density would indicate many resonant frequencies in a short frequency band, as well as the fact that the distance between the resonances is small in the frequency domain.

The determination of the modal density is essentially a mathematical problem. It involves the determination of the frequency equation for the structure under consideration from the appropriate equations of motion and then the summation of the resonant frequencies over all possible modes of vibration. This yields an expression for the number of resonant modes in terms of frequency. Differentiation of this expression with respect to frequency will then yield the expression for the modal density also in terms of frequency. Hence the problem is really the determination of the expression for the number of resonant frequencies up to an arbitrary frequency once the frequency equation for the structure in question has been determined. Up to and beyond this point the analysis, although complex, is fairly conventional and straightforward.

Method for Determining the Modal Density

In order to obtain the number of natural frequencies or the number of eigenvalues less than some arbitrary frequency, the frequency equation for the structure in question is first expressed in terms of the wave numbers. The wave numbers are essentially the eigenvalues which arise from the solution of the differential equations of motion to obtain the frequency equation. Bolotin (1963) expresses the frequency equation for a generalized rectangular region in terms of the wave numbers k_1 and k_2, Figure 1. The quantities k_1 and k_2 were determined by the solution for the eigenfunctions and are given by,

$$k_1a_1 = m\pi + 0(1) \qquad k_2a_2 = n\pi + 0(1) \qquad m,n = 1,2,3,... \tag{2}$$

where a_1 and a_2 are the principal dimensions of the surface. These equations are analogous to the expressions obtained by Courant and Hilbert (1953) for the case of the flat plate and thin membrane with the exception of the order one expression which is added to each of the expressions. This term is a correction factor to reflect the effect of the boundary conditions on the wave number expressions. Once the wave numbers have been defined, the concept of a k-space is introduced. A rectangular domain is set up with the coordinates k_1 and k_2 and divided into increments of size Δk_1 by Δk_2, Figure 2. From the expressions above it may be seen that for the generalized rectangular region the increments would be of size $\frac{\pi}{a_1}$ by $\frac{\pi}{a_2}$. Hence, two families of curves corresponding to the equations for k_1 and k_2 have been constructed and the intersections of the curves represent values of k_1 and k_2 which will satisfy the frequency equation.

Now according to Courant and Hilbert (1953) the number of eigenvalues or natural frequencies may be found by taking the surface integral over the k-space and dividing by the dimensions of one element of the k-space.

$$N(\omega) \approx \frac{1}{\Delta k_1 \Delta k_2} \iint_s dk_1 dk_2 \ . \qquad (3)$$

The limits on the surface integral are such that the solution to the frequency equation exists. This is indicated by the curve shown on the k-space in Figure 2. The curve represents the maximum permissible wave numbers for the arbitrary frequency at which the number of resonances is desired. Hence the number of natural frequencies is proportional to the surface integral over the k-space divided by the size

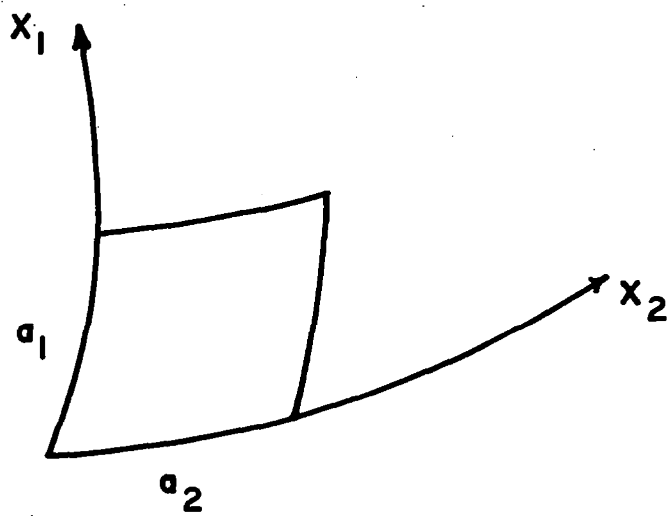

FIGURE 1. GENERALIZED RECTANGULAR REGION

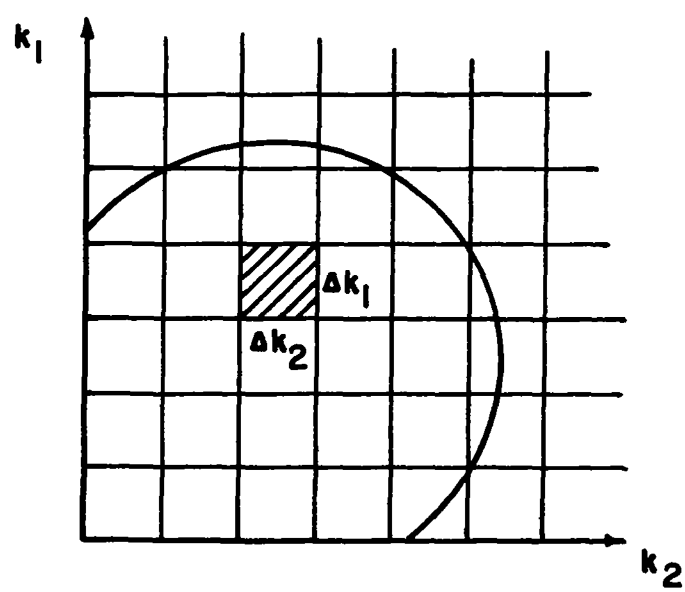

FIGURE 2. k-SPACE FOR A GENERALIZED RECTANGULAR REGION

of an element in the k-space. Looked at another way, since the area of the k-space is equal to the surface integral over the k-space and the area is also equal to the number of intersections times the area of one element, the double integral divided by the area of one element is obviously equal to the number of intersections. Each intersection represents a solution to the frequency equation and therefore the number of intersections is the number of natural frequencies.

Naturally this idea is not valid for low mode numbers if the elements of the k-space are not all the same size, but this still does not pose any difficulty at fairly high mode numbers. For the lower mode numbers, however, the mode shapes are affected to a larger extent by the edge conditions of the structure in question. Therefore, the concept of the k-space is not really valid for wave numbers where the edge conditions are dominant in the determination of the mode shapes. Bolotin (1960) has examined this problem in some detail and found that for a plate or a spherical shell that the edge effects never dominate and that for a cylindrical shell the edge effects only dominate for fairly small wave numbers. Hence, although it is usually not important, it should be kept in mind that the above method for determining the eigenvalues is not always valid and may lead to problems at low wave numbers.

The problem has now been reduced to one of simply putting in the appropriate values for the wave numbers and evaluating the double integral over the k-space where the frequency equation is satisfied. As was stated above, once the expression for the number of eigenvalues or natural frequencies has been obtained it is a fairly simple matter to differentiate the expression with respect to frequency to obtain the expression for the modal density. It may be noted that the above expression is given for wave propagation in two dimensional space. For the one or three dimensional cases the integral is either a single or a triple integral and the element size is adjusted accordingly.

Example Modal Density Calculations

Simply Supported Beam

In order to clarify the procedure to some extent it may be of some value to look at some simple cases. First of all the case of the simply supported beam will be examined, Figure 3a. The frequency equation for this case may be written as simply, Smith and Lyon (1965),

$$\omega = \frac{m^2 \pi^2}{\ell^2} K C_L \qquad m = 1,2,\ldots , \tag{4}$$

where C_L is the longitudinal velocity of wave propagation in the beam material along the beam, ℓ is the length of the beam, and K is the radius of gyration taken in a plane perpendicular to the plane in which bending occurs. The wave number k_1 may now be defined as,

$$k_1 = \frac{m\pi}{\ell} \quad . \tag{5}$$

Hence the change in wave number from one mode to the next is given by the following expression:

$$\Delta k_1 = \frac{\pi}{\ell} \, . \tag{6}$$

Since the waves in the beam are propagated only in a single direction (along the length of the beam) the k-space is one dimensional, Figure 3b, and the equation for the number of resonant frequencies becomes

$$N(\omega) = \frac{1}{\Delta k_1} \int_0^{k_1} dk_1 \tag{7}$$

which leads to

$$N(\omega) = \frac{\ell}{\pi} k_1 \quad . \tag{8}$$

But k_1, from equations (4) and (5), becomes

$$k_1 = \sqrt{\frac{\omega}{KC_L}} \quad . \tag{9}$$

Therefore the expression for the number of resonant frequencies is

$$N(\omega) = \frac{\ell}{\pi} \sqrt{\frac{\omega}{KC_L}} \quad . \tag{10}$$

Differentiating the expression with respect to frequency yields,

$$n(\omega) = \frac{\ell}{2\pi} \frac{1}{\sqrt{\omega KC_L}} \quad . \tag{11}$$

Equation (11) is the expression for the modal density of a simply supported beam. If the thickness of the beam is h, the radius of gyration is given by $h/\sqrt{12}$.

Simply Supported Rectangular Plate

A slightly better example is given by the case of a rectangular plate, with simply supported edges, Figure 4a. The frequency equation for the plate may be expressed in the following form, Smith and Lyon (1965),

$$\omega = \left(\frac{m^2\pi^2}{\ell_1^2} + \frac{n^2\pi^2}{\ell_2^2}\right) KC_L \qquad m,n = 1,2,3,\ldots \tag{12}$$

where ℓ_1 and ℓ_2 are the length and width of the plate and K and C_L are the same as in the case of the beam. Again the wave numbers are first defined as,

$$k_1 = \frac{m\pi}{\ell_1} \qquad k_2 = \frac{n\pi}{\ell_2} \quad . \tag{13}$$

The changes in the two wave numbers are then given by the expressions

$$\Delta k_1 = \frac{\pi}{\ell_1} \qquad \Delta k_2 = \frac{\pi}{\ell_2} \quad . \tag{14}$$

In this case two space variables are involved and therefore a two dimensional k-space is required, Figure 4b. The equation for the number of natural frequencies is written as in equation (3),

$$N(\omega) \approx \frac{1}{\Delta k_1 \Delta k_2} \int_s \int dk_1 dk_2 \ .$$

It is convenient in the case of the plate to integrate over the surface of the k-space using cylindrical coordinates. Therefore $k_1^2 + k_2^2 = r^2$ and equation (3) becomes,

$$N(\omega) = \frac{\ell_1 \ell_2}{\pi^2} \int_0^r \int_0^{\pi/2} r d\theta dr \ . \tag{15}$$

Integrating once with respect to r gives,

$$N(\omega) = \frac{\ell_1 \ell_2}{2\pi^2} \int_0^{\pi/2} r^2 d\theta. \tag{16}$$

Then integrating with respect to θ and substituting for r^2, the following expression for the number of resonant frequencies is obtained.

$$N(\omega) = \frac{\ell_1 \ell_2}{4\pi} \frac{\omega}{KC_L} \quad . \tag{17}$$

Finally equation (17) is differentiated with respect to frequency, ω, to obtain the expression for the modal density of a rectangular plate.

$$n(\omega) = \frac{\ell_1 \ell_2}{4\pi} \frac{1}{KC_L} \quad . \tag{18}$$

If the plate thickness is h, the radius of gyration is $h/\sqrt{12}$ and the expression becomes the same as that given by Heckl (1962).

$$n(\omega) = \frac{\ell_1 \ell_2}{2\pi} \frac{\sqrt{3}}{hC_L} \quad . \tag{19}$$

It is interesting to note that the modal density of a flat plate is a constant for a given plate and thus is independent of frequency.

Clamped Circular Plate

As a final example the case of a flat circular plate will be examined, Figure 5a. For high frequencies, the frequency equation for a plate with either clamped or free edge conditions is given by, Rayleigh (1945),

$$\omega = \frac{\pi^2}{4a^2} KC_L (n + 2m)^2 \qquad m,n = 1,2,3,... \tag{20}$$

where a is the plate radius, K the radius of gyration, and C_L the velocity of wave propagation along the plate. Rewriting the frequency equation as

$$\omega = \left(\frac{\pi n}{2a} + \frac{\pi m}{a}\right)^2 KC_L \ , \tag{21}$$

the wave numbers may be defined in the following manner:

$$k_1 = \frac{\pi n}{2a} , \qquad k_2 = \frac{\pi m}{a} . \tag{22}$$

Hence, the change in the wave numbers from one mode of vibration to the next is given by

$$\Delta k_1 = \frac{\pi}{2a} , \qquad \Delta k_2 = \frac{\pi}{a} . \tag{23}$$

The wave propagation in the circular plate is again two dimensional, as in the case of the rectangular plate, so that the k-space is also two dimensional, Figure 5b. The number of eigenvalues is again given by equation (3). Substituting the values for the change in wave numbers and converting to cylindrical coordinates yields,

$$N(\omega) = \frac{a^2}{\pi^2} \int_{\theta_1}^{\theta_2} r^2 d\theta . \tag{24}$$

Noting that the frequency equation may be written in the form

$$(k_1 + k_2)^2 = \frac{\omega}{KC_L} , \tag{25}$$

and converting the equation to cylindrical coordinates and solving for r^2 yields,

$$r^2 = \frac{\omega}{KC_L} \frac{1}{(\sin\theta + \cos\theta)^2} . \tag{26}$$

Hence, r^2 may be eliminated from the above integral expressions for the number of eigenvalues. The limits on the integral are from 0 to $\pi/2$ since the argument of the integral exists for all values of θ in the first quadrant. Therefore,

$$N(\omega) = \frac{a^2}{\pi^2} \frac{\omega}{KC_L} \int_0^{\pi/2} \frac{d\theta}{(\sin\theta + \cos\theta)^2} , \tag{27}$$

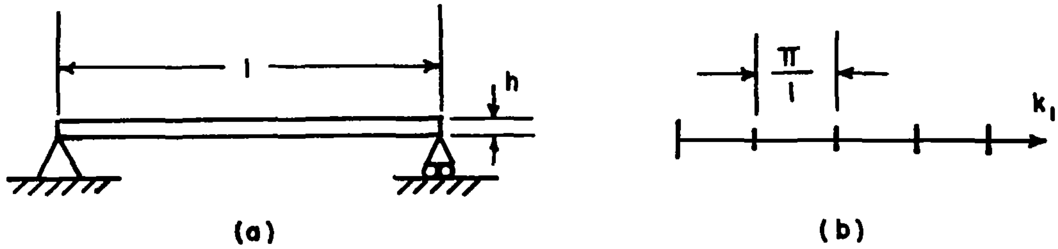

FIGURE 3. SIMPLY SUPPORTED BEAM & k-SPACE

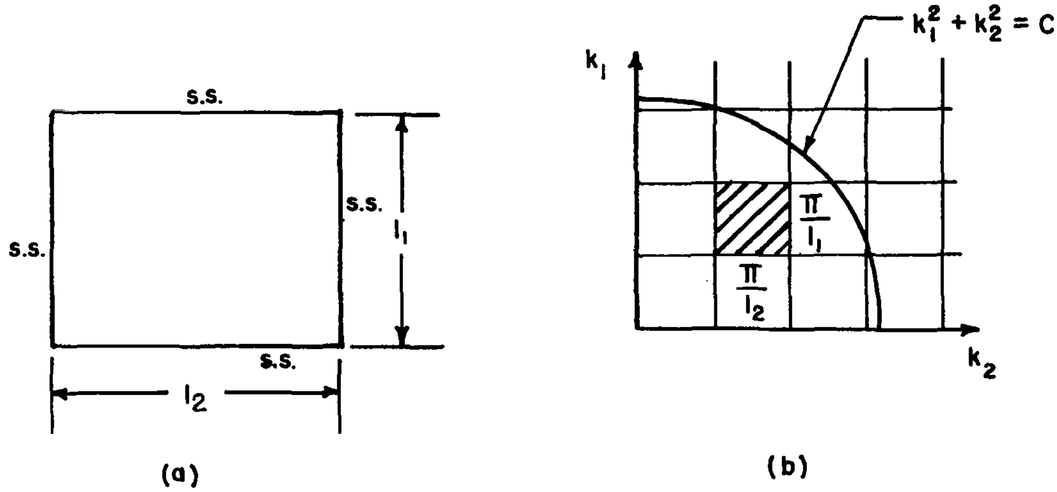

FIGURE 4. SIMPLY SUPPORTED RECTANGULAR PLATE & k-SPACE

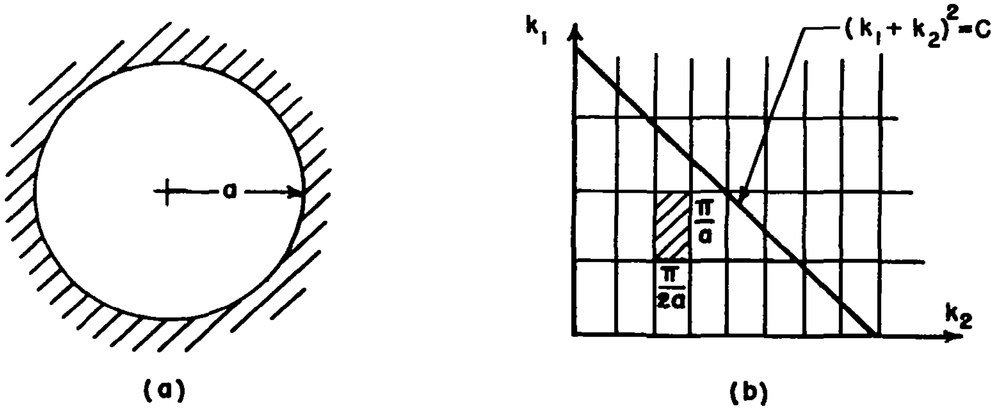

FIGURE 5. CLAMPED CIRCULAR PLATE & k-SPACE

which may be written in the form

$$N(\omega) = \frac{a^2}{2\pi^2} \frac{\omega}{C_L K} \int_0^{\pi} \frac{d\alpha}{1 + \sin\alpha} \quad . \tag{28}$$

Integration yields

$$N(\omega) = \frac{a^2}{\pi^2} \frac{\omega}{KC_L} \quad . \tag{29}$$

Differentiating the above expression with respect to frequency produces the expression for the modal density.

$$n(\omega) = \frac{a^2}{\pi^2 KC_L} \quad . \tag{30}$$

The radius gyration is again $h/\sqrt{12}$ where h is the plate thickness. The final expression is

$$n(\omega) = \frac{\pi a^2 \sqrt{12}}{\pi^3 hC_L} \quad . \tag{31}$$

Again it may be noted, as in the case of the rectangular plate, that the modal density is a function independent of the frequency. Therefore for two plates of equal area, thickness, and of the same material, one circular and the other rectangular, the ratio of the modal densities is found to be

$$\frac{n(\omega)_c}{n(\omega)_r} = \frac{\sqrt{12}\ (\pi a^2)}{\pi^3 hC_L} \Big/ \frac{\ell_1 \ell_2 \sqrt{3}}{2h\pi C_L} = \frac{4}{\pi^2} \quad . \tag{32}$$

Hence, it is evident that for a plate the modal density is independent of the frequency of vibration, but is not independent of the shape or geometry of the plate. Equation (32) implies that at a given frequency (high frequency due to an assumption in the circular plate derivation) that the modal density of the rectangular plate will be approximately two and one-half times that of the circular plate.

Application of Modal Density

Now that the concept of modal density has been defined and some examples given showing how it may be evaluated for simple structures, a brief discussion of the application of the concept will be undertaken as it relates to statistical energy methods and room acoustics.

The statistical energy method is of great value when the input force or excitation to a structure is fairly complex in nature, and especially when the input is random. Fairly broad band random inputs to structures are fairly common in the fields of acoustics and turbulent flow. These types of loading or excitation are extremely difficult and sometimes impossible to handle in a reasonable fashion with classical methods. Hence, it is necessary to resort to a statistical approach. Basically this approach examines the statistical properties of the input function and predicts the statistical properties of the response function of the structure. In this way sufficient information may be obtained for design purposes. In this type of analysis the property of modal density of the structure is one of the parameters of importance. Given sufficient information about the input function and the modal density it is possible to determine the number of resonant frequencies excited by the input and to what extent. This then gives some insight into the response of the structure to the given excitation and some insight into the amount of energy which will be absorbed or transmitted by the structure. A detailed description of this procedure is given by Smith and Lyon (1965).

The concept of modal density also proves to be valuable in the field of room acoustics. In the case of room acoustics it is usually the frequency separation or the inverse of the modal density which is of interest. In this case the density of the eigenvalues in a three dimensional space is required. Hence, if the technique described earlier for finding the number of eigenvalues is used, a triple integral would be involved. It may be shown that the final expression for the modal spacing for a room, Morse and Bolt (1944), is given by,

$$\langle \delta_w \rangle = 2\pi c^3 \;/\; \omega^2 V_o \;, \tag{33}$$

where c is velocity of wave propagation in the air, V is the volume of the room, and ω is the frequency of interest. It may be noted that in room acoustics the irregular transmission of low frequencies is due to the small number of modes which may be excited at the lower frequencies. Hence the acoustical properties of rooms or air spaces are definitely related to the concept of the modal density of the room.

Other uses of the concept of modal density could be given; however, the above examples are sufficient to point out the usefulness of this concept with regard to engineering applications.

THEORETICAL DEVELOPMENT FOR SHELLS OF REVOLUTION

Introduction

In this section three different derivations of the modal density expression for the thin cylindrical shell will be presented in detail. The first presentation is that of Bolotin (1963) in which the general shell of revolution is discussed and then adapted to the case of the thin cylindrical shell. The second presentation is that of Heckl (1962) in which the expressions are found for the cylindrical shell alone. The final presentation is essentially a modification of Bolotin's work for the specific case of the cylindrical shell.

First Presentation

In his general derivation for thin shells of revolution, Bolotin (1963) examined the case of shells with two principal radii of curvature, neglecteding the effects of tangential and inertial forces. The presentation is also restricted to shells which are simply supported at their edges. However, it is mentioned that the effects of boundary conditions on the vibrational modes are limited, and that only the first few modes of vibration are affected significantly. Hence the edge conditions are of little significance in the modal density expression development. In his work, Bolotin used the method described by Courant and Hilbert (1953) which has already been discussed to determine the number of resonant modes. Therefore the number of resonant modes in the shell is given by equation (3)

$$N(\omega) \simeq \frac{1}{\Delta k_1 \Delta k_2} \int_s\int dk_1 dk_2$$

where k_1 and k_2 are the wave numbers. In this case Δk_1 and Δk_2 are given by π/a_1 and π/a_2 respectively where a_1 and a_2 are the principal dimensions of the shell surface. The surface integral is then put into cylindrical coordinates to obtain the following expression:

$$N(\omega) = \frac{a_1 a_2}{\pi^2} \int\int_S r dr d\theta = \frac{a_1 a_2}{2\pi^2} \int_{\theta_1}^{\theta_2} r^2 d\theta \ . \tag{34}$$

The value of r which appears in equation (34) may be found from the frequency equation for the general shell of revolution. The frequency equation was obtained by solving the following differential equations for the shell,

$$\begin{aligned} D\Delta\Delta w - \left(\frac{1}{R_2}\frac{\partial^2\phi}{\partial x_1^2} + \frac{1}{R_1}\frac{\partial^2\phi}{\partial x_2^2}\right) - \rho h\omega^2 &= 0 \\ \frac{1}{Eh}\Delta\Delta\phi + \frac{1}{R_2}\frac{\partial^2 w}{\partial x_1^2} + \frac{1}{R_1}\frac{\partial^2 w}{\partial x_2^2} &= 0 \end{aligned} \tag{35}$$

where x_1 and x_2 are the general curvilinear coordinates; R_1 and R_2 are the principal radii of curvature, D is the plate stiffness $\left(D = \frac{Eh^3}{12(1-\mu)^2} \simeq \frac{Eh^3}{12}\right)$, ρ is the density, h the thickness of the shell, E the modulus of elasticity, w the normal deflection, ϕ the stress function for the middle surface, and ω the frequency of vibration. The solution of these equations results in the following frequency equation,

$$\omega^2 = \frac{D}{\rho h}\left[(k_1^2 + k_2^2) + \frac{Eh}{DR_1^2}\frac{(k_1^2\chi + k_2^2)}{(k_1^2 + k_2^2)}\right]; \quad \chi = \frac{R_1}{R_2} \ . \tag{36}$$

This expression is converted to cylindrical coordinates as before and solved for $r^2 = k_1^2 + k_2^2$. Hence the following expression is obtained.

$$r^2 = [\omega^2 - \Omega_R^2 (\mathcal{X} \cos^2\theta + \sin^2\theta)^2]^{1/2} (\frac{\rho h}{D})^{1/2} . \qquad (37)$$

The expression for the number of resonant modes from equation (3) becomes

$$N(\omega) = \frac{a_1 a_2}{2\pi^2} (\frac{\rho h}{D})^{1/2} \int_{\theta_1(\omega)}^{\theta_2(\omega)} [\omega^2 - \Omega_R^2 (\mathcal{X} \cos^2\theta + \sin^2\theta)^2]^{1/2} d\theta \qquad (38)$$

where Ω_R is equal to $\frac{1}{R_1} (\frac{E}{\rho})^{1/2}$. Equation (38) is an integral expression for the number of resonant frequencies up to a bounding frequency for a thin shell of revolution. The limits on the integral are specified as taken over the portion of the first quadrant where the argument of the integral remains real. In other words negative square roots are not allowed.

Differentiation of expression (38) with respect to frequency immediately yields the expression for the modal density. It may be noted that when differentiation is taken under the integral using Leibnitz's rule, the extra terms which normally appear are always zero due to the way in which the limits on the integral are specified. The result is

$$n(\omega) = \frac{a_1 a_2}{2\pi^2} (\frac{\rho h}{D})^{1/2} \omega \int_{\theta_1(\omega)}^{\theta_2(\omega)} \frac{d\theta}{[\omega^2 - \Omega_R^2 (\mathcal{X} \cos^2\theta + \sin^2\theta)^2]^{1/2}} . \qquad (39)$$

Finally Bolotin writes equations (38) and (39) in the following form.

$$N(\omega) = \frac{a_1 a_2}{4\pi} \left(\frac{\rho h}{D}\right)^{1/2} \omega\, H\left(\frac{1}{\nu}, \chi\right); \quad \frac{1}{\nu} = \frac{\Omega_R}{\omega}$$

$$n(\omega) = \frac{a_1 a_2}{4\pi} \left(\frac{\rho h}{D}\right)^{1/2} H_1\left(\frac{1}{\nu}, \chi\right) \tag{40}$$

where

$$H\left(\frac{1}{\nu}, \chi\right) = \frac{2}{\pi} \int_{\theta_1}^{\theta_2} \left[1 - \frac{1}{\nu^2} (\chi \cos^2\theta + \sin^2\theta)^2\right]^{1/2} d\theta$$

$$H_1\left(\frac{1}{\nu}, \chi\right) = \frac{2}{\pi} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\left[1 - \frac{1}{\nu^2} (\chi \cos^2\theta + \sin^2\theta)^2\right]^{1/2}} \, . \tag{41}$$

The expressions for both the number of resonant frequencies and the modal density of a thin shell of revolution are now expressed as integrals. Bolotin goes on to show that the expression H_1 may be expressed as an eliptic integral for various ranges of $1/\nu$.

The general expressions (40) and (41) may now be applied to the specific case of the cylindrical shell. For the cylinder, χ is equal to zero, and a_1 and a_2 are the dimensions of the shell surface. Thus, the quantities a_1 and a_2 are equivalent to the length of the cylinder, ℓ, and one-half the circumference of the cylinder, πa. The reason that only half the circumference of the cylinder is used is to take into account that the cylinder is a closed surface and that the vibrational modes are limited to one-half by this fact. The following expressions result immediately from (40) and (41).

$$N(\omega) = \frac{a\ell}{4} \left(\frac{\rho h}{D}\right)^{1/2} \omega\, H\left(\frac{1}{\nu}, 0\right)$$

$$n(\omega) = \frac{a\ell}{4} \left(\frac{\rho h}{D}\right)^{1/2} H_1\left(\frac{1}{\nu}, 0\right) \, . \tag{42}$$

The expression H_1 may be used as in equation (41) or it may be used in the following elliptic integral form:

$$\text{for } \nu > 1: \quad H_1(\tfrac{1}{\nu},0) = \frac{2}{\pi\sqrt{1+1/\nu}} \, K_\varepsilon\left(\sqrt{\frac{2/\nu}{1+1/\nu}}\right) ,$$

$$\text{and} \quad \text{for } \nu < 1: \quad H_1(\tfrac{1}{\nu},0) = \frac{\sqrt{2}}{\pi\sqrt{1/\nu}} \, K_\varepsilon\left(\sqrt{\frac{1+1/\nu}{2/\nu}}\right) , \tag{43}$$

where K_ε represents the complete elliptic integral of the first kind. Equation (43) is Bolotin's result for a cylinder of radius a and length ℓ.

Second Presentation

The analysis presented in this section is due to Heckl (1962). The results obtained from this analysis are the ones most commonly quoted in the literature at the present when the modal density of a cylindrical shell is mentioned.

Heckl's derivation is for the case of a thin infinitely long cylindrical shell. In his analysis Heckl started with the following shell equations.

$$\begin{aligned} &\alpha v + n_o v_t + \mu k a v_a = i\nu^2 p_o/\omega\rho h \\ &n_o v + [n_o^2 + \nu^2 + \tfrac{1}{2}(1-\mu)k^2a^2]v_t + \tfrac{1}{2}(1+\mu)n_o k a v_a = 0 \\ &\mu k a v + \tfrac{1}{2}(1+\mu)n_o k a v_t + [k^2a^2 + \tfrac{1}{2}(1-\mu)n_o^2 - \nu^2]v_a = 0 \end{aligned} \tag{44}$$

where v, v_t, and v_a are the radial, tangential, and axial components of velocity amplitude, n_o is the half number of modes in the circumferential direction, k is the wave number in the axial direction,

μ is Poisson's ratio, h is the shell thickness, ω is the exciting frequency, p_o is the amplitude of excitation, a is the cylinder radius, and ν is the dimensionless frequency of vibration given by

$$\nu = \omega a/C_L \ ,$$

where C_L is the velocity of wave propagation in the shell material. It may also be noted that α used by Heckl in this work is given by the following expression.

$$\alpha = 1-\nu^2 + \{(n_o^2+k^2a^2)^2 - \frac{1}{2}[n_o^2(4-\mu)-2-\mu](1-\mu)^{-1}\}h^2/12a^2 \ . \tag{46}$$

Equation (44) was subsequently solved for the impedance of the cylindrical shell and then the frequency equation of the cylinder was determined by letting the impedance go to zero. This resulted in the following frequency equation.

$$\nu^2 = (1-\mu^2)(m\pi a/\ell)^4 x[(m\pi a/\ell)^2+n_o^2]^{-2} + \{[(m\pi a/\ell)^2+n_o^2]^2 - \frac{1}{2}[n_o^2(4-\mu)-2-\mu](1-\mu)^{-1}\}\frac{h^2}{12a^2} \ . \tag{47}$$

Finally by neglecting several of the terms in the frequency equation Heckl arrived at the following approximate frequency expression.

$$\nu^2 \simeq [\sigma^2(n_o^2+\sigma^2)^{-1} + \beta(n_o^2+\sigma^2)]^2; \quad \sigma = \frac{m\pi a}{\ell} \qquad \beta = h/2\sqrt{3}\ a \ . \tag{48}$$

It may be shown that the terms which Heckl neglected have little effect on the frequency expression for frequencies above the ring frequency ($\nu = 1$). However, below the ring frequency the effect may be as much as forty percent of the actual value. Plots of the exact expression and the approximate expression have been made for various values of m and n, and the differences may be noted in Figure 6. Hence it is expected that the values obtained for both the number of resonant frequencies and the modal density of the cylinder will be somewhat on the conservative side since the frequency equation used by Heckl predicts frequencies somewhat higher than is actually the case.

Finally solving for σ and then summing over all possible values of n, Heckl was able to obtain the following expressions for the number of resonant frequencies and the modal density.

$$N(\omega) = \frac{\ell}{\pi a} \sum_{n_o=0,1}^{n_o=(\nu/\beta)^{1/2}} \sigma \; ; \quad n(\omega) = \frac{\ell}{\pi a} \int_{0,1}^{(\nu/\beta)^{1/2}} \frac{\partial\sigma}{\partial\nu}\, d\nu \tag{49}$$

where the lower limit is 1 for $\nu < 1$, and 0 for $\nu > 1$. Further simplification leads to the following approximate expressions.

$\nu > 1$

$$N(\omega) = \ell\nu/4a\beta = \sqrt{3}\ \ell a\omega/2C_L h$$

$$n(\omega) = \ell/4a\beta = \sqrt{3}\ \ell/2h$$

(50)

$\nu < 1$

$$N(\omega) = \{\frac{1}{2}(2\nu-1)[\frac{1}{2}\pi + \text{arc sin }(2\nu-1) + (\nu-\nu^2)^{1/2}\}\ \ell/4a\beta \approx \frac{3\ell\nu^{3/2}}{8\pi a\beta}$$

$$n(\omega) = [\frac{1}{2}\pi + \text{arc sin }(2\nu-1)]\ \ell/4\pi a\beta\ .$$

These are the final expressions obtained by Heckl for the number of resonant modes and the modal density of a thin cylindrical shell.

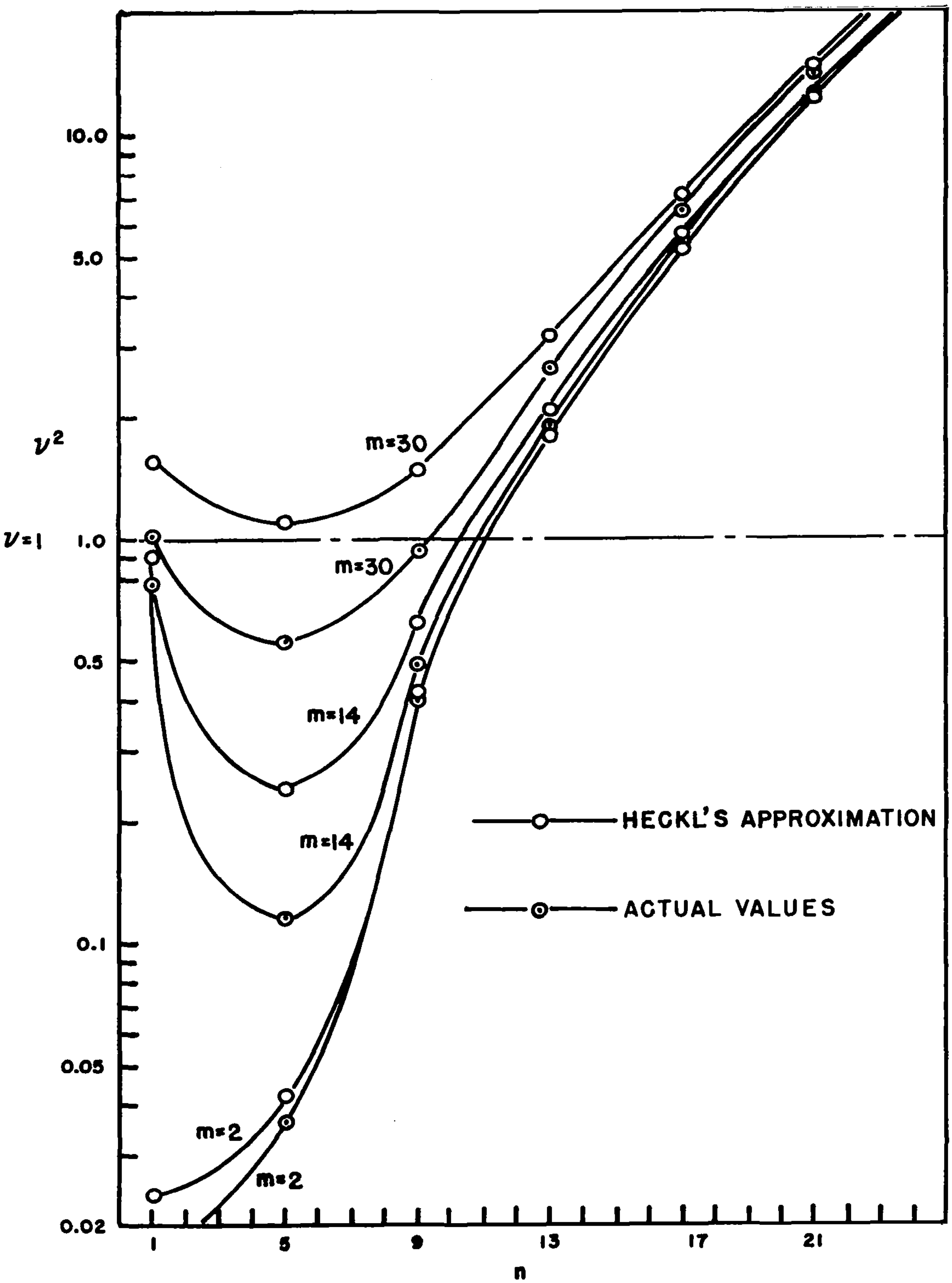

FIGURE 6. HECKL'S APPROXIMATE FREQUENCY EQUATION AND ACTUAL EQUATION VERSUS n FOR VARIOUS VALUES OF m

Third Presentation

The third and final analysis is essentially a modification of the work done by Bolotin (1962). Going back to equation (38) for the number of natural frequencies for a shell of revolution

$$N(\omega) = \frac{a_1 a_2}{2\pi^2} \left(\frac{\rho h}{D}\right)^{1/2} \int_{\theta_1}^{\theta_2} [\omega^2 - \Omega_R^2(\cos^2\theta \mathcal{X} + \sin^2\theta)^2]^{1/2} d\theta \ .$$

At this point the restriction that the shell of revolution is a cylinder is introduced and thus implies that $a_1 = \ell$, $a_2 = a\pi$, and $\mathcal{X} = 0$. The notation is also converted to that used by Heckl, so that

$$N(\omega) = \frac{\ell}{2\pi\beta a} \int_{\theta_1}^{\theta_2} \nu[1 - \frac{1}{\nu^2} \sin^4\theta]^{1/2} d\theta \ , \tag{51}$$

where again the limits on the integral are such that the term in the brackets remains positive in the first quadrant. Hence, the following integral expression for the number of resonant frequencies is obtained.

$$N(\omega) = \frac{\ell\sqrt{3}}{\pi h} \int_0^{\sin^{-1}\sqrt{\nu}} [\nu^2 - \sin^4\theta]^{1/2} d\theta \ . \tag{52}$$

The upper limit on the integration holds for $\nu < 1$. For ν greater than or equal to one, the upper limit $\pi/2$ is used. Differentiating equation (52) with respect to ν, the modal density expression is obtained.

$$n(\omega) = \frac{\ell\sqrt{3}}{2h} \cdot \frac{2}{\pi} \int_0^{\sin^{-1}\sqrt{\nu}} \frac{d\theta}{[1 - \frac{1}{\nu^2} \sin^4\theta]^{1/2}} \ . \tag{53}$$

Again the upper limit must be $\pi/2$ when ν is equal to or greater than one.

Expressions (52) and (53) may be evaluated numerically by means of Simpson's Rule using a digital computer. This has been done and values have been obtained for each of the one-third octave bands. Along with the above calculations, an approximate modal density expression has also been calculated by subtracting the successive N values and then dividing the result by the change in ν from one station to the next. This was done because this is essentially the procedure which was used in calculating the modal density experimentally. The results of the computations appear in Table 1.

Summary

The results of the three presentations for the number of natural frequencies and the modal density have now been obtained. As a final step the notation used by Bolotin will be converted to that of Heckl so that the notation will be consistent for all three cases. This has been done and the principal results of all three derivations appear in Table 2. In order to compare these three sets of results numerical evaluation is carried out and displayed graphically in Figures (7) and (8). Figure 7 shows the number of resonant modes in dimensionless form versus the dimensionless frequency and Figure 8 shows the modal density in dimensionless form versus the dimensionless frequency for all three cases.

It may be noted after examining Figures 7 and 8 that all three presentations converge for frequencies above the ring frequency ($\nu = 1$) as was expected. However, below the ring frequency the differences are quite noticeable. The results of Bolotin and the modified Bolotin

Table 1. Tabulation of dimensionless number of natural frequencies and dimensionless modal density for a cylindrical shell using the modified Bolotin analysis

Center[a/] ν	Upper ν	$N.\pi h/\sqrt{3}\ 1$[b/]	$n.2h/\sqrt{3}\ 1$[c/] approx.	$n.2h/\sqrt{3}\ 1$[b/]
0.00725	0.00812	0.00064	-	0.0751
0.00912	0.0102	0.00090	0.0800	0.0844
0.0115	0.0129	0.00128	0.0899	0.0947
0.0145	0.0162	0.00181	0.101	0.106
0.0182	0.0204	0.00255	0.113	0.119
0.0229	0.0257	0.00361	0.127	0.134
0.0288	0.0323	0.00510	0.143	0.151
0.0363	0.0407	0.00721	0.161	0.169
0.0457	0.0512	0.0102	0.181	0.191
0.0576	0.0645	0.0144	0.203	0.215
0.0725	0.0812	0.0204	0.229	0.242
0.0912	0.102	0.0289	0.258	0.273
0.115	0.129	0.0411	0.291	0.308
0.145	0.162	0.0583	0.329	0.349
0.182	0.204	0.0829	0.373	0.396
0.229	0.257	0.118	0.424	0.451
0.288	0.323	0.169	0.485	0.517
0.363	0.407	0.242	0.557	0.597
0.457	0.512	0.349	0.647	0.698
0.576	0.645	0.509	0.765	0.834

Table continued

Table 1 (continued)

Center[a/] ν	Upper ν	$N \cdot \pi h/\sqrt{3}\ 1$[b/]	$n \cdot 2h/\sqrt{3}\ 1$[c/] approx.	$n \cdot 2h/\sqrt{3}\ 1$[b/]
0.725	0.812	0.755	0.939	1.056
-	1.000	1.1478	-	-
0.912	1.02	1.21	1.38	1.656
1.15	1.29	1.76	1.31	1.181
1.45	1.62	2.35	1.13	1.092
1.82	2.04	3.05	1.07	1.052
2.29	2.57	3.91	1.04	1.031
2.88	3.23	4.98	1.02	1.019
3.63	4.07	6.32	1.01	1.012
4.57	5.12	7.98	1.01	1.007
5.76	6.45	10.08	1.00	1.005
7.25	8.12	12.71	1.00	1.002

a/ Note: the dimensionless frequency values chosen for this tabulation correspond to the one-third octave band frequencies for the cylinder used in the experimental work. This would not necessarily be the case for any other cylinder.

b/ Plotted at upper ν.

c/ Plotted at center ν.

Table 2. Summary of analytical results for the number of natural frequencies and modal density of a cylindrical shell

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Bolotin's Results:

$$N(\omega) = \frac{\ell\sqrt{3}}{\pi h}\,\nu H(\frac{1}{\nu}\,,\,0)$$

$$n(\omega) = \frac{\ell\sqrt{3}}{2h}\,H_1(\frac{1}{\nu}\,,\,0)$$

$\nu > 1$ $$H_1(\frac{1}{\nu}\,,\,0) = \frac{2}{\pi}\sqrt{\frac{\nu}{\nu+1}}\,K_\varepsilon(\sqrt{\frac{2}{\nu+1}})$$

$\nu < 1$ $$H_1(\frac{1}{\nu}\,,\,0) = \frac{\sqrt{2\nu}}{\pi}\,K_\varepsilon(\sqrt{\frac{\nu+1}{2}})$$

Heckl's Results:

$\nu > 1$ $$N(\omega) = \frac{\ell\sqrt{3}}{\pi h}\cdot\frac{\pi}{2}\,\nu$$

$$n(\omega) = \frac{\ell\sqrt{3}}{2h}$$

$\nu < 1$ $$N(\omega) = \frac{\ell\sqrt{3}}{\pi h}\cdot\frac{3}{4}\,\nu^{3/2}$$

$$n(\omega) = \frac{\ell\sqrt{3}}{2h}\cdot\frac{1}{\pi}\,[\frac{1}{2}\pi + \text{arc sin }(2\nu-1)]$$

Modified Bolotin Results:

$$N(\omega) = \frac{\ell\sqrt{3}}{\pi h}\int_0^{\theta_2}[\nu^2 - \sin^4\theta]^{1/2}\,d\theta$$

$$n(\omega) = \frac{\ell\sqrt{3}}{2h}\cdot\frac{2}{\pi}\int_0^{\theta_2}[1 - \frac{1}{\nu^2}\sin^4\theta]^{-1/2}\,d\theta$$

$\nu > 1 \quad \theta_2 = \frac{\pi}{2}\,;$ $\qquad \nu < 1 \quad \theta_2 = \sin^{-1}\sqrt{\nu}$

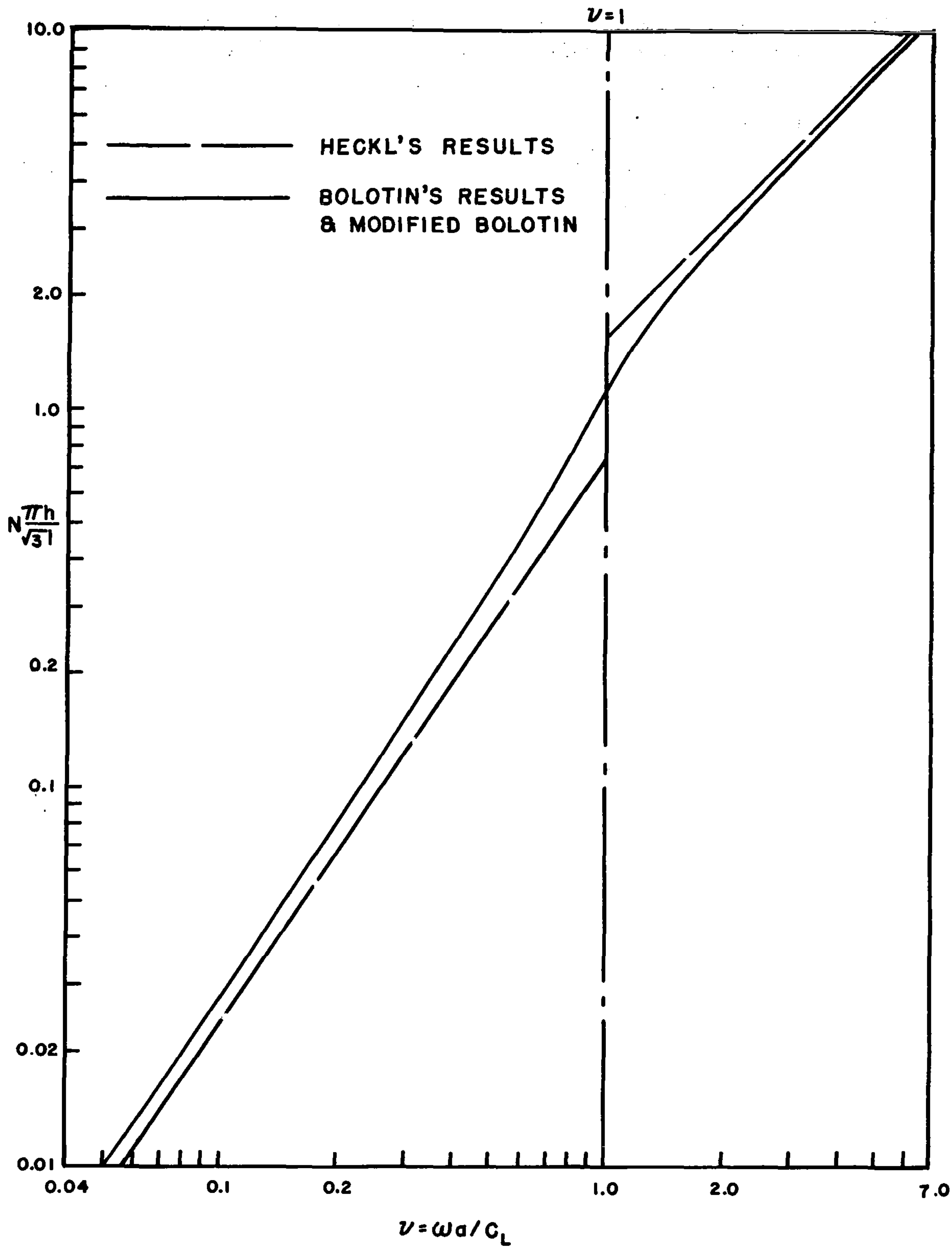

FIGURE 7. ANALYTICAL CURVES FOR NUMBER OF MODES VERSUS FREQUENCY

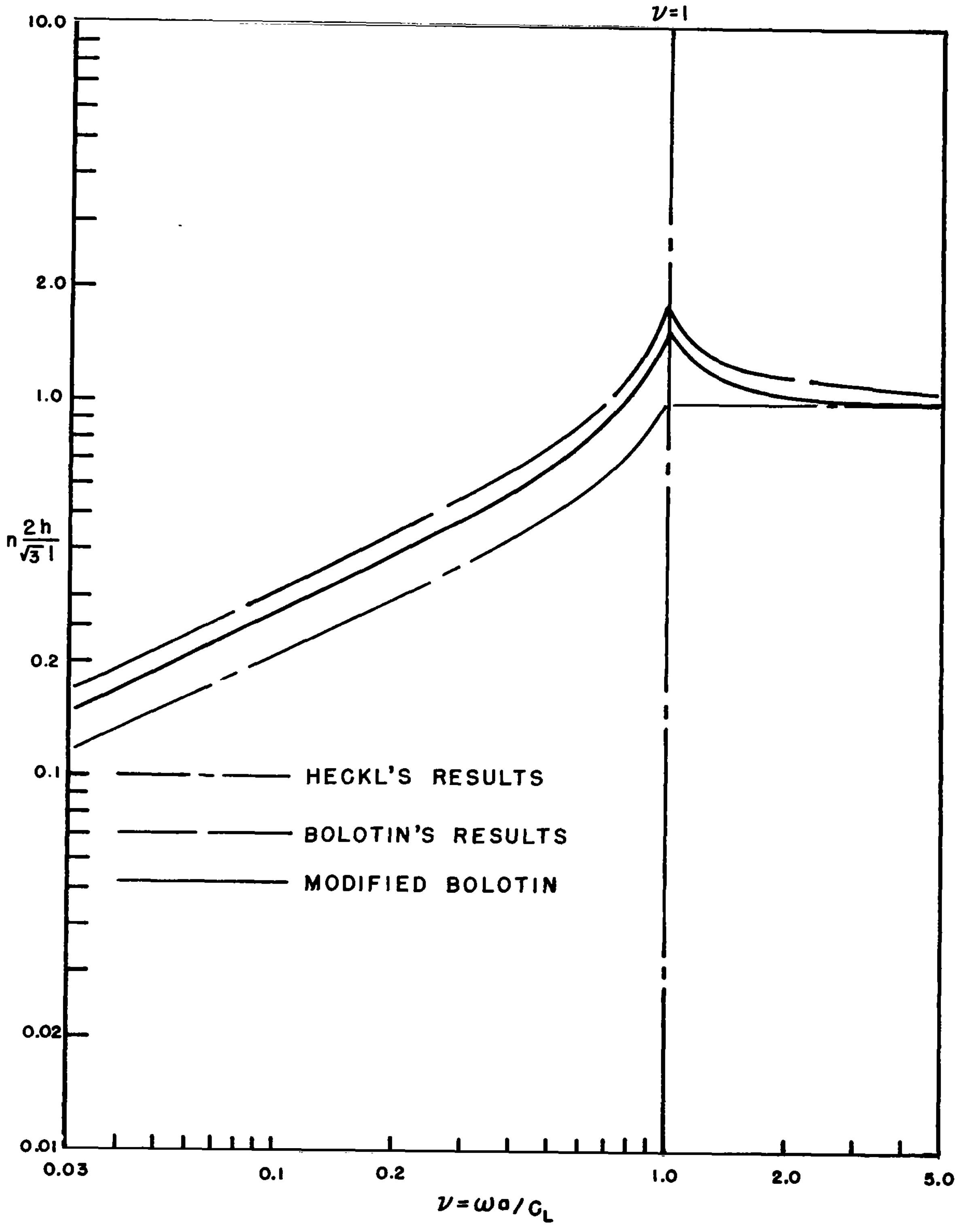

FIGURE 8. ANALYTICAL CURVES FOR MODAL DENSITY VERSUS FREQUENCY

analysis are seen to be the same for the number of resonant frequencies and only slightly different for the modal density. The reason for this slight difference in the modal density curves is due to an approximation made by Bolotin in order to express the modal density in elliptic integral form. It may also be noted that the results of Heckl are lower than the other results for the number of resonant modes and the number of natural frequencies below the ring frequency. Since it was shown earlier that Heckl's results would be on the conservative side it is reasonable to suspect that this is the reason for the difference.

EXPERIMENTAL PROGRAM

Objective

In conjunction with the theoretical work presented, an experimental program was conducted in order to provide a comparison with the analytical work. The main objective of this program was to physically count the resonant frequencies of a thin cylindrical shell, and in this manner to experimentally determine the modal density of the shell. Finally, by non-dimensionalizing the results a comparison could be made between the analytical results and the experimental values.

Experimental Setup

The cylindrical shell which was used in this work was chosen so that it would have a ring frequency well within the range of the equipment being used in the investigation. It was also chosen to have sufficient separation between modes so that the counting of the resonant frequencies would be possible with some degree of accuracy. The dimensions and specifications of the actual cylinder used in this investigation are given in Table 3. The arrangement of the instrumentation used in this work, which is shown in Figure 9, consisted first of an oscillator with a continuous sweep range of twenty to twenty thousand Hz. The signal from the oscillator was fed to a power amplifier and then to an electromagnetic shaker. The shaker in turn excited the test cylinder. An accelerometer was mounted on the freely suspended cylinder as a pickup, and its output was fed through a preamplifier and a signal amplifier to a graphic level recorder. The recorder was also

Table 3. Experimental cylinder specifications and nondimensionalizing conversion factors for experimental data

Cylinder Data

Material	Stainless Steel 304
Diameter (2a)	4.5 in.
Thickness (h)	0.0625 in.
Length (1)	36.0 in.
Young's Modulus (E)	27.6×10^6 lbf/in.^{2}
Density (ρ)	0.280 lbm/in.^{3}
Velocity of Wave Propagation (C_L)	1.95×10^5 in./sec
Ring Frequency	1.38×10^4 Hz

Conversion Factors

Dimensionless Frequency

$\nu = T_1 f$ frequency in Hz

where $T_1 = 2\pi a/C_L = 0.0000725$

Dimensionless Number of Resonant Frequencies

$N = T_2$ (number of resonances)

where $T_2 = \pi h/\sqrt{3}\, \ell = 0.0031489$

Dimensionless Modal Density

$n = T_3$ (number of resonances)$/\Delta\nu$

where $T_3 = 2h/\sqrt{3}\, \ell = 0.002005$

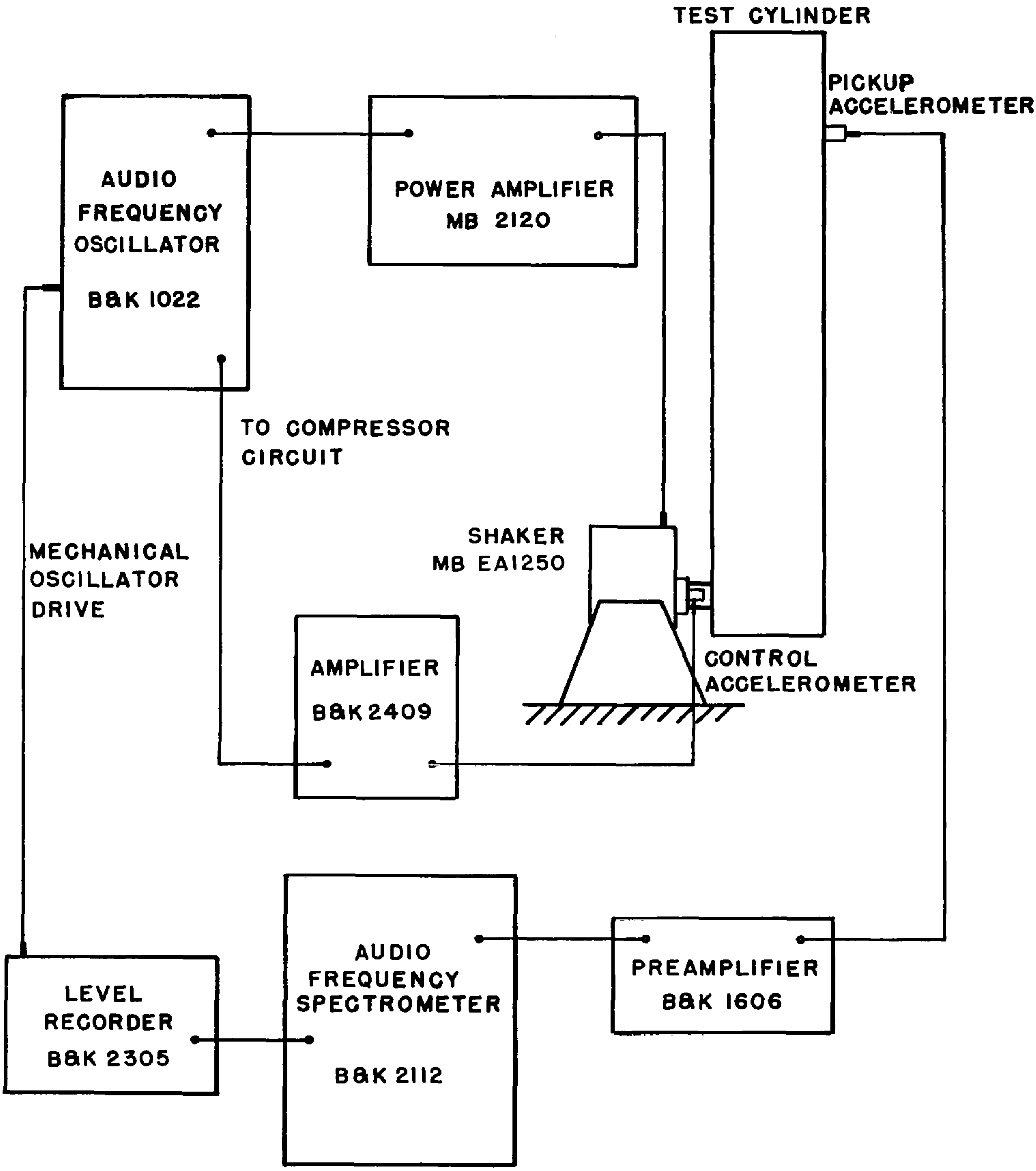

FIGURE 9. ARRANGEMENT OF EXPERIMENTAL INSTRUMENTATION

used to drive the sweep of the oscillator so that the two were synchronized. Another accelerometer pickup was mounted to the shaker head and fed back by means of another signal amplifier to the oscillator feedback circuit. In this way the input to the cylinder was held constant over the desired frequency range. Some care had to be taken, however, in order to insure that the control did not interfere with the output since some mechanical feedback was encountered from the cylinder to the shaker. Several photographs of the experimental apparatus are shown in Figures 10 through 13.

Operation

The pickup accelerometer was mounted to the cylinder by means of a hard wax. This mounting method was used since it offered the best response of several methods tried (in comparison with a rigid mount) and could be used on a curved surface. It also provided a means of mounting the accelerometer without physically altering the cylinder itself. This was desirable since several accelerometer mounting positions were to be used in the experiment. After the accelerometer was mounted, the frequency range was swept slowly and the results recorded on frequency calibrated paper. The entire frequency range over which readings were taken (approximately 200 to 20,000 Hz) was divided into several ranges so that different sweep speeds and paper speeds could be used in order to get adequate separation between the resonant frequencies so that they did not overlap and make counting impossible. Once sufficient data had been collected for several accelerometer locations the data was analyzed.

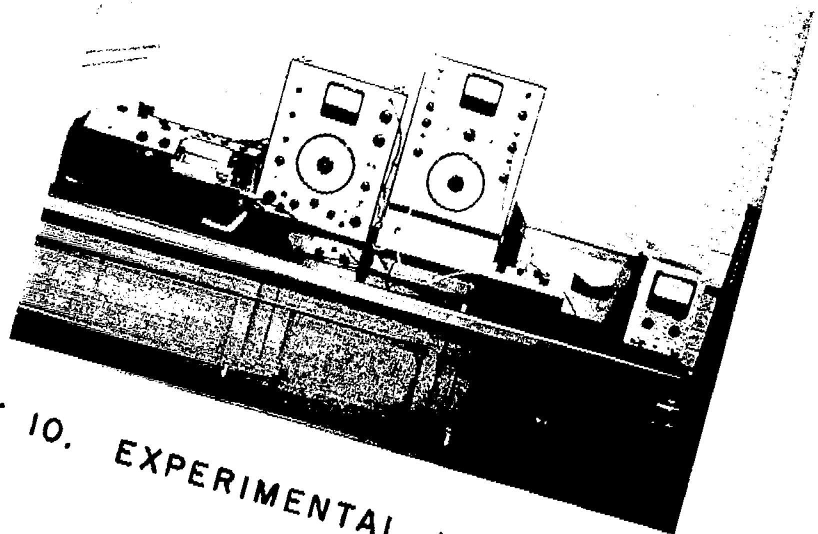

FIGURE 10. EXPERIMENTAL INSTRUMENTATION

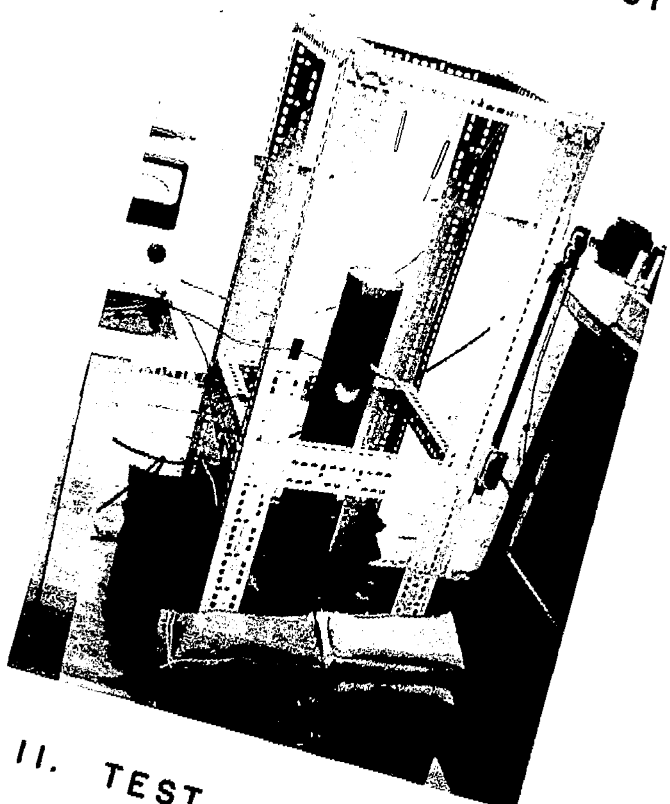

FIGURE 11. TEST CYLINDRICAL SHELL

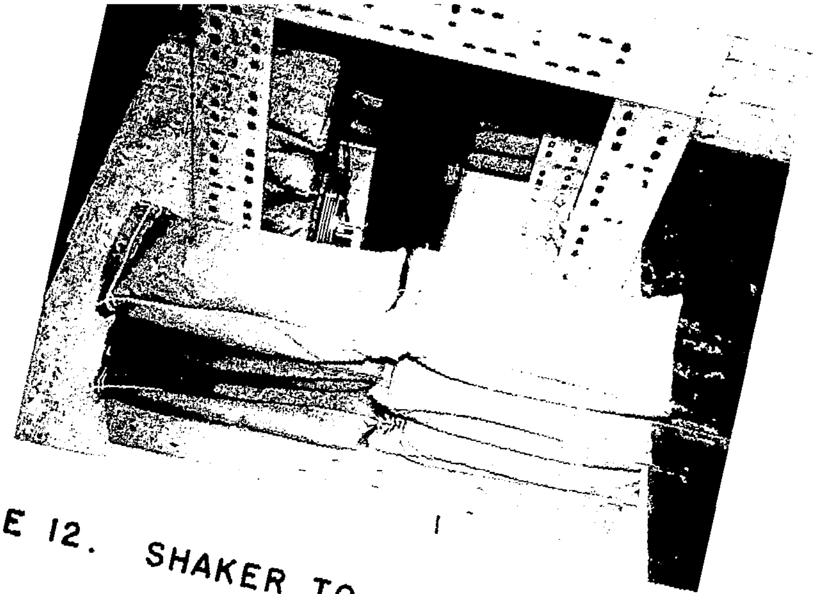

FIGURE 12. SHAKER TO CYLINDER CONNECTION

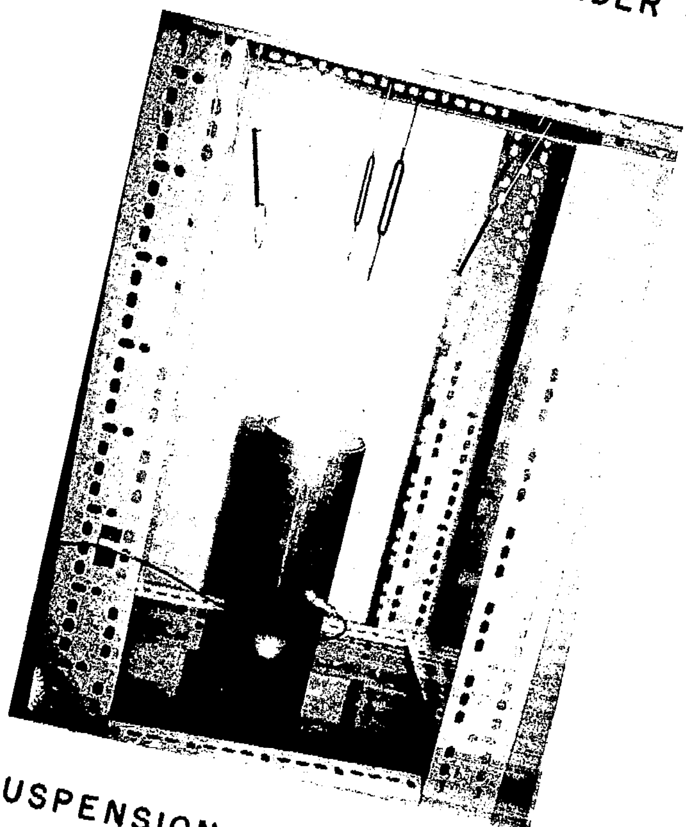

FIGURE 13. SUSPENSION AND PICKUP ACCELEROMETER

Reduction of Data

The recorded data obtained was first broken down into standard one-third octave band frequency ranges, see Table 4. The number of resonant peaks appearing in each of the bands was then counted. In this way, knowing the frequency range and the number of modes, the modal density for each of the one-third octave bands could be determined. The results obtained in this manner were then nondimensionalized using the factors given in Table 3. The results of this reduction are given in Tables 5 through 10, as well as in Figures 14 and 15. Figure 14 shows the data for the number of resonant modes and Figure 15 shows the data for the modal density. In each case the analytical curves for the modified Bolotin approach are shown.

At this point some mention should be made concerning the difficulties encountered in counting the number of resonant frequencies. First of all, since an accelerometer was used as a pickup, only the acceleration at the point of mounting was recorded. Therefore it is obvious that for some of the vibrational modes passed through, the accelerometer could have been either at a node or in the near vicinity of a node. For this reason the magnitude of the peaks recorded varied widely. The regulation circuit helped to keep the mean response in about the same place, however, some of the peaks were very slight while others were quite distinct. This made it difficult at times to distinguish real resonant peaks from stray electrical noise and other effects which were always present. However, this difficulty was predominant only for the lower modes, and became much less of a problem as the number of modes within a given frequency band increased. It was

Table 4. Tabulation of standard one-third octave bands

Center Frequency Hz	Lower Frequency Hz	Upper Frequency Hz
16	14.8	17.8
20	17.8	22.4
25	22.4	28.2
31.5	28.2	35.5
40	35.5	44.7
50	44.7	56.3
63	56.7	70.9
80	70.9	89.2
100	89.2	112
125	112	141
160	141	178
200	178	224
250	224	282
315	282	355
400	355	447
500	447	563
630	563	709
800	709	892
1000	892	1120
1250	1120	1410

Table continued

Table 4 (continued)

Center Frequency Hz	Lower Frequency Hz	Upper Frequency Hz
1600	1410	1780
2000	1780	2240
2500	2240	2820
3150	2820	3550
4000	3550	4470
5000	4470	5630
6300	5630	7090
8000	7090	8920
10000	8920	11200
12500	11200	14100
16000	14100	17800
20000	17800	22400

Table 5. Experimental data for position 1

Center ν	$\Delta\nu$	Upper ν	Number of Modes	Total Number	N[a]	n[b]
0.0576	0.0133	0.0645	2	3	0.0094	0.3015
0.0725	0.0167	0.0812	2	5	0.0157	0.2401
0.0912	0.0210	0.102	3	8	0.0252	0.2854
0.115	0.0265	0.129	3	11	0.0346	0.2269
0.145	0.0333	0.162	7	18	0.0567	0.4215
0.182	0.0419	0.204	9	27	0.0850	0.4307
0.229	0.0528	0.257	11	38	0.1197	0.4177
0.288	0.0665	0.323	21	59	0.1859	0.6332
0.363	0.0837	0.407	28	87	0.2739	0.6707
0.457	0.105	0.512	35	122	0.3842	0.6683
0.576	0.133	0.645	48	170	0.5353	0.7236
0.725	0.167	0.812	72	242	0.7620	0.8644
0.912	0.210	1.02	127	369	1.1619	1.2125
1.15	0.265	1.29	161	530	1.6689	1.2182
1.45	0.333	1.62	193	723	2.2767	1.1621

[a] Number of resonances is plotted at upper ν.

[b] Modal density is plotted at center ν.

Table 6. Experimental data for position 2

Center ν	$\Delta\nu$	Upper ν	Number of Modes	Total Number	N[a]	n[b]
0.0576	0.0133	0.0645	2	3	0.0094	0.3015
0.0725	0.0167	0.0012	3	6	0.0189	0.3602
0.0912	0.0210	0.102	3	9	0.0283	0.2864
0.115	0.0265	0.129	4	13	0.0409	0.3026
0.145	0.0333	0.162	7	20	0.0629	0.4215
0.182	0.0419	0.204	9	29	0.0913	0.4307
0.229	0.0528	0.257	11	40	0.1259	0.4177
0.288	0.0665	0.323	19	59	0.1858	0.5729
0.363	0.0837	0.407	23	82	0.2581	0.5509
0.457	0.105	0.512	35	117	0.3684	0.6683
0.576	0.133	0.645	47	164	0.5164	0.7085
0.725	0.167	0.812	81	245	0.7715	0.9725
0.912	0.210	1.02	116	361	1.1368	1.1075
1.15	0.265	1.29	150	511	1.6091	1.1349
1.45	0.333	1.62	171	682	2.1475	1.0296

[a] Number of resonances is plotted at upper ν.

[b] Modal density is plotted at center ν.

Table 7. Experimental data for position 3

Center ν	$\Delta\nu$	Upper ν	Number of Modes	Total Number	N[a]	n[b]
0.0576	0.0133	0.0645	2	3	0.0094	0.3015
0.0725	0.0167	0.0812	2	5	0.0157	0.2401
0.0912	0.0210	0.102	3	8	0.0252	0.2864
0.115	0.0265	0.129	4	12	0.0378	0.3026
0.145	0.0333	0.162	7	19	0.0598	0.4215
0.182	0.0419	0.204	10	29	0.0913	0.4785
0.229	0.0528	0.257	12	41	0.1291	0.4557
0.288	0.0665	0.323	20	61	0.1921	0.6030
0.363	0.0837	0.407	25	86	0.2708	0.5989
0.457	0.105	0.512	34	120	0.3778	0.6492
0.576	0.133	0.645	48	168	0.5290	0.7236
0.725	0.167	0.812	77	245	0.7715	0.9245
0.912	0.210	1.02	118	363	1.1431	1.1266
1.15	0.265	1.29	154	517	1.6279	1.9652
1.45	0.333	1.62	171	688	2.1664	1.0296

[a] Number of resonances is plotted at upper ν.

[b] Modal density is plotted at center ν.

Table 8. Experimental data for position 4

Center ν	$\Delta\nu$	Upper ν	Number of Modes	Total Number	N[a]	n[b]
0.0576	0.0133	0.0645	3	4	0.01259	0.4523
0.0725	0.0167	0.0812	3	7	0.0220	0.3602
0.0912	0.0210	0.102	3	10	0.0315	0.2864
0.115	0.0265	0.129	4	14	0.0441	0.3026
0.145	0.0333	0.162	7	21	0.0661	0.4215
0.182	0.0419	0.204	9	30	0.0945	0.5427
0.229	0.0528	0.257	12	42	0.1323	0.4557
0.288	0.0665	0.323	18	60	0.1889	0.5427
0.363	0.0837	0.407	26	86	0.2708	0.6228
0.457	0.105	0.512	36	122	0.3842	0.6874
0.576	0.133	0.645	47	169	0.5322	0.7085
0.725	0.167	0.812	78	247	0.7778	0.9365
0.912	0.210	1.02	122	369	1.1619	0.1648
1.15	0.265	1.29	153	522	1.6437	1.1576
1.45	0.333	1.62	168	690	2.1727	1.0115

[a] Number of resonances is plotted at upper ν.

[b] Modal density is plotted at center ν.

Table 9. Experimental data for position 5

Center ν	$\Delta\nu$	Upper ν	Number of Modes	Total Number	N[a]	n[b]
0.0576	0.0133	0.0645	3	4	0.0126	0.4523
0.0725	0.0167	0.0812	2	6	0.0189	0.2401
0.0912	0.0210	0.102	3	9	0.0283	0.2864
0.115	0.0265	0.129	4	13	0.0409	0.3026
0.145	0.0333	0.162	7	20	0.0629	0.4215
0.182	0.0419	0.204	9	29	0.0913	0.4307
0.229	0.0528	0.257	12	41	0.1291	0.4557
0.288	0.0665	0.323	20	61	0.1921	0.6030
0.363	0.0837	0.407	25	86	0.2708	0.5989
0.457	0.105	0.512	33	119	0.3747	0.6301
0.576	0.133	0.645	50	169	0.5322	0.7538
0.725	0.167	0.812	76	245	0.7715	0.9124
0.912	0.210	1.02	115	360	1.1336	1.0979
1.15	0.265	1.29	150	510	1.6059	1.1349
1.45	0.333	1.62	162	672	2.1161	0.9754

[a]Number of resonances is plooted at upper ν.

[b]Modal density is plotted at center ν.

Table 10. Experimental data for position 6

Center ν	$\Delta\nu$	Upper ν	Number of Modes	Total Number	N[a]	n[b]
0.0576	0.0133	0.0645	1	2	0.0063	0.1508
0.0725	0.0167	0.0812	2	4	0.0125	0.2401
0.0912	0.0210	0.102	3	7	0.0220	0.2864
0.115	0.0265	0.129	5	12	0.0378	0.3783
0.145	0.0333	0.162	7	19	0.0598	0.4215
0.182	0.0419	0.204	10	29	0.0913	0.4785
0.229	0.0528	0.257	12	41	0.1291	0.4557
0.288	0.0665	0.323	21	62	0.1952	0.6332
0.363	0.0837	0.407	26	88	0.2771	0.6228
0.457	0.105	0.512	33	121	0.3810	0.6301
0.576	0.133	0.645	49	170	0.5353	0.7387
0.725	0.167	0.812	79	249	0.7841	0.9485
0.912	0.210	1.02	110	359	1.1305	1.0502
1.15	0.265	1.29	145	504	1.5870	1.0971
1.45	0.333	1.62	172	676	2.1287	1.0356

[a] Number of resonances is plotted at upper ν.

[b] Modal density is plotted at center ν.

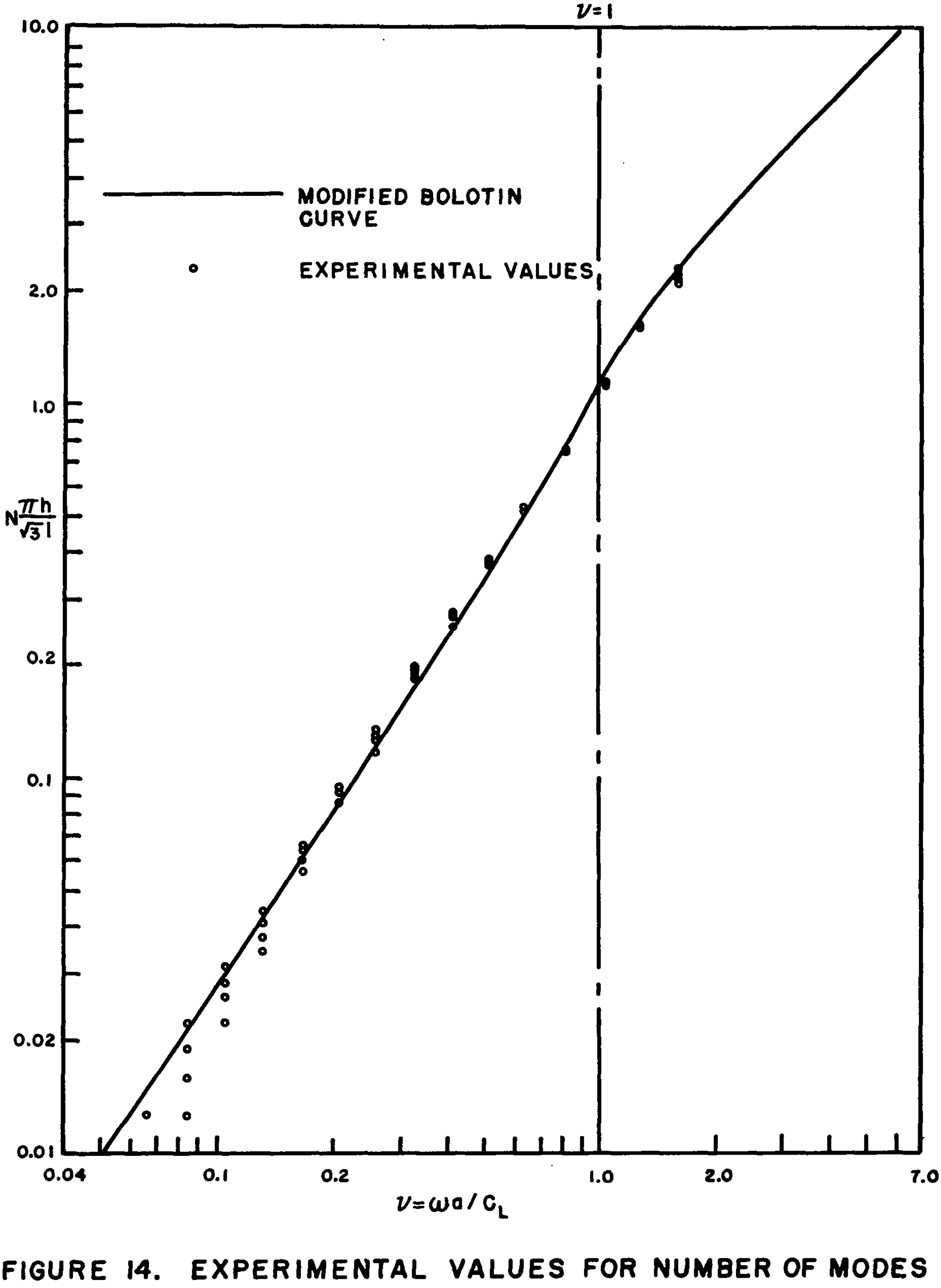

FIGURE 14. EXPERIMENTAL VALUES FOR NUMBER OF MODES VERSUS FREQUENCY

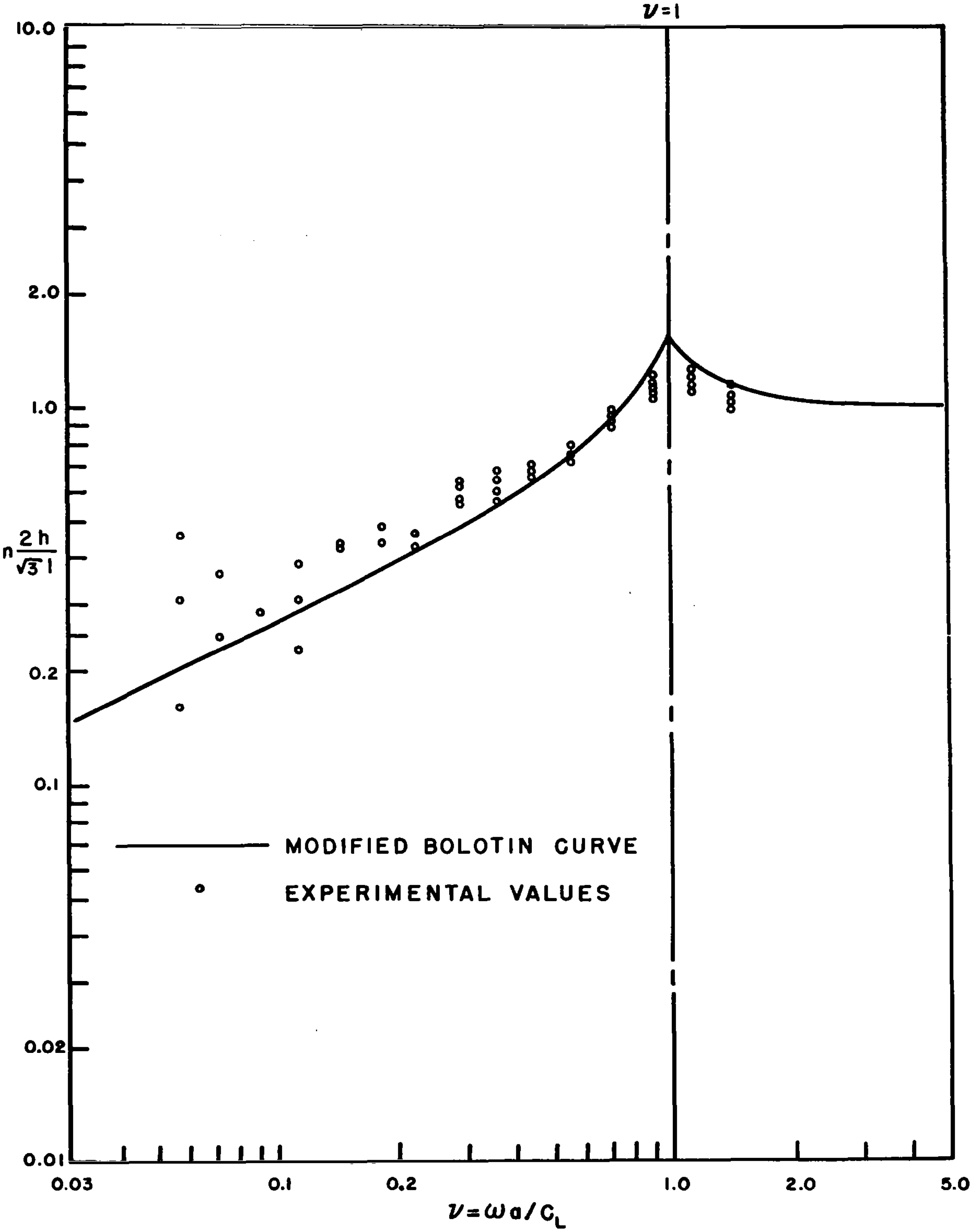

FIGURE 15. EXPERIMENTAL VALUES FOR MODAL DENSITY VERSUS FREQUENCY

also found to be very difficult to count the resonant peaks for the modes near the fundamental mode, and obtain any sort of agreement with theory. This was due to the fact that both the modal density expression and the expression for the number of resonant frequencies are represented by continuous functions, and this is obviously not a very good representation near the fundamental mode of vibration. In this region theory predicts fractional numbers of modes and this is obviously impossible to obtain by physically counting resonant peaks. Hence the reader will note a large degree of scatter in the experimental results for the lower frequencies, as would be expected.

SUMMARY AND CONCLUSIONS

The results of the three analytic procedures for the number of resonant frequencies and the modal density for the cylindrical shell are given in Table 2. Also plots of all three are shown in Figures 7 and 8. It may be seen that Heckl's results are somewhat lower than those obtained by the other two procedures below the ring frequency. The reason for this may be attributed, at least in part, to the approximations made by Heckl in his expression for the natural frequencies of a cylinder. It has been noted already that the terms omitted by Heckl would tend to lead to somewhat conservative results below the ring frequency and this seems to be the case. The results obtained by the other two derivations are seen to be quite close together for the modal density expression and identical for the number of resonant frequencies. The slight difference in the modal density expression may be attributed to two factors. First of all some approximation was necessary in Bolotin's derivation in order to express the modal density as a complete elliptic integral of the first kind, and secondly in the numerical integration of the modal density expression, it was necessary to truncate the interval very slightly (10^{-6}) in order to avoid trouble at the upper limit of integration. The first factor probably tended to increase the modal density slightly below the ring frequency, thus making the results slightly higher than is actually the case. The second factor obviously lowered the modal density expression slightly, thus making the results of the modified Bolotin approach slightly lower than would actually be the case. Hence it is felt that the exact solution to the problem is between the results of the modified Bolotin approach and the results obtained by Bolotin for the modal density.

The results of the experimental work which was performed are shown plotted in Figures 14 and 15 along with the modified Bolotin analytical results for the number of resonant frequencies and the modal density. It may be seen that the experimental results are in excellent agreement with the modified Bolotin curve for the number of resonant frequencies. It has already been pointed out that some scattering of points is to be expected for the lower values in the vicinity of the fundamental mode where these concepts are not really valid. Experimental agreement with the modal density curve is also good, although not quite as good as for the number of resonant modes. For the modal density, the scattering of experimental data is much more pronounced for the lower frequencies as would be expected. However, as the frequency increases and the concept of modal density becomes a better indication of what is really happening, good agreement is obtained between the experimental values and the results of the modified Bolotin approach. Since the modal density is a continuous representation of discrete events, it is not possible to obtain good results at the lower frequencies in the vicinity of the fundamental frequency of the structure.

In conclusion it may be said that either the expression obtained by Bolotin or by the modified Bolotin approach gives values for the number of resonances in a thin cylindrical shell which are in excellent agreement with the experimental values. Hence the modified Bolotin approach gives an accurate integral expression for the number of resonant modes in a cylindrical shell above and below the ring frequency and may be plotted or tabulated in dimensionless form for convenient use. It may be noted also that this expression is continuous at the ring frequency and does not have a discontinuity at this point as did Heckl's approximate expressions.

The conclusions which may be drawn for the modal density expression are not quite as obvious. First of all there is only a slight difference between Bolotin's values and those obtained with the modified Bolotin approach. Secondly the experimental data is such that it is not immediately evident which is the better of the two representations. However, it is felt that the modified Bolotin results are in slightly better agreement with the experimental results and hence should be considered the more accurate of the two. However, there is really not sufficient difference in the two results to warrant further work on this point. The modified Bolotin approach gives an integral expression for the modal density of a thin cylindrical shell which may be plotted or tabulated as a function of frequency in dimensionless form for convenient use by the engineer.

It may also be concluded that the expressions obtained for the number of resonant modes and the modal density are not really valid for frequencies below and in the vicinity of the fundamental mode of the structure. This is due to the fact, which has been mentioned previously, that the continuous functions for the number of resonances and the modal density do not represent what is really happening in this region. In a sense it is statistical information about a small number of events, and therefore is not an adequate representation of the phenomenon.

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APPENDIX. LIST OF SYMBOLS

a	- radius of cylinder or circular plate
a_1 a_2	- principal dimension of shell surface
C_L	- longitudinal wave velocity - $\sqrt{E/\rho}$
c	- sound wave velocity in air
D	- stiffness modulus - $Eh^3/12(1-\mu)^2$
E	- Young's modulus
f	- frequency
h	- thickness of plate or shell
k k_1 k_2	- wave number
ℓ	- length of cylinder or beam
ℓ_1 ℓ_2	- surface dimension of plate
m	- integer value
$N(\omega)$	- number of resonant modes
n	- integer value
n_o	- one-half circumferential modes
$n(\omega)$	- modal density
p_o	- amplitude of excitation
R_1 R_2	- radius of shell curvature
r	- cylindrical coordinate
T_1 T_2 T_3	- conversion factor
V_o	- volume
v	- radial velocity amplitude
v_a	- axial velocity amplitude
v_t	- tangential velocity amplitude

w	-	displacement normal to surface
x_1 x_2	-	generalized coordinate
α	-	stress coefficient
β	-	$h/2 \sqrt{3}\, a$
$< \delta_w >$	-	modal spacing
θ	-	cylindrical coordinate
θ_1 θ_2	-	limits on k-space integral
K	-	radius of gyration
ρ	-	density
μ	-	Poisson's ratio
ν	-	dimensionless frequency - $\omega a/C_L = \omega/\Omega_R$
σ	-	$m\pi a/\ell$
ϕ	-	stress function
χ	-	R_1/R_2
Ω_R	-	$(E/\rho)^{1/2}/R_1$
ω	-	angular frequency

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