THE RAYLEIGH DISTRIBUTION
Revision A

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December 3, 2012

Introduction

Consider a lightly damped, single-degree-of-freedom system subjected to broadband random excitation. The system will tend to behave as a bandpass filter. The bandpass filter center frequency will occur at or near the system's natural frequency.

The system response will thus tend to be narrowband random. The probability distribution for its instantaneous values will tend to follow a Normal distribution, which is the same distribution corresponding to a broadband random signal.

The absolute values of the system's response peaks, however, will have a Rayleigh distribution.

Derivation

From Reference 1, the probability density function $n(A;\mu,\sigma)$ of a Rayleigh distribution is given by

$$n(A,\sigma) = \frac{A}{\sigma^2} \exp\left\{ -\frac{1}{2}\left[\frac{A}{\sigma}\right]^2 \right\}, \quad A > 0 \tag{1}$$

where

A is the absolute value of the amplitude,

σ is the standard deviation.

The Rayleigh distribution curve has the shape shown in Figure 1.

For this distribution, the probability P that the absolute amplitude A has a value for $A>\lambda\sigma$ is obtained by integrating the area under the probability density curve.

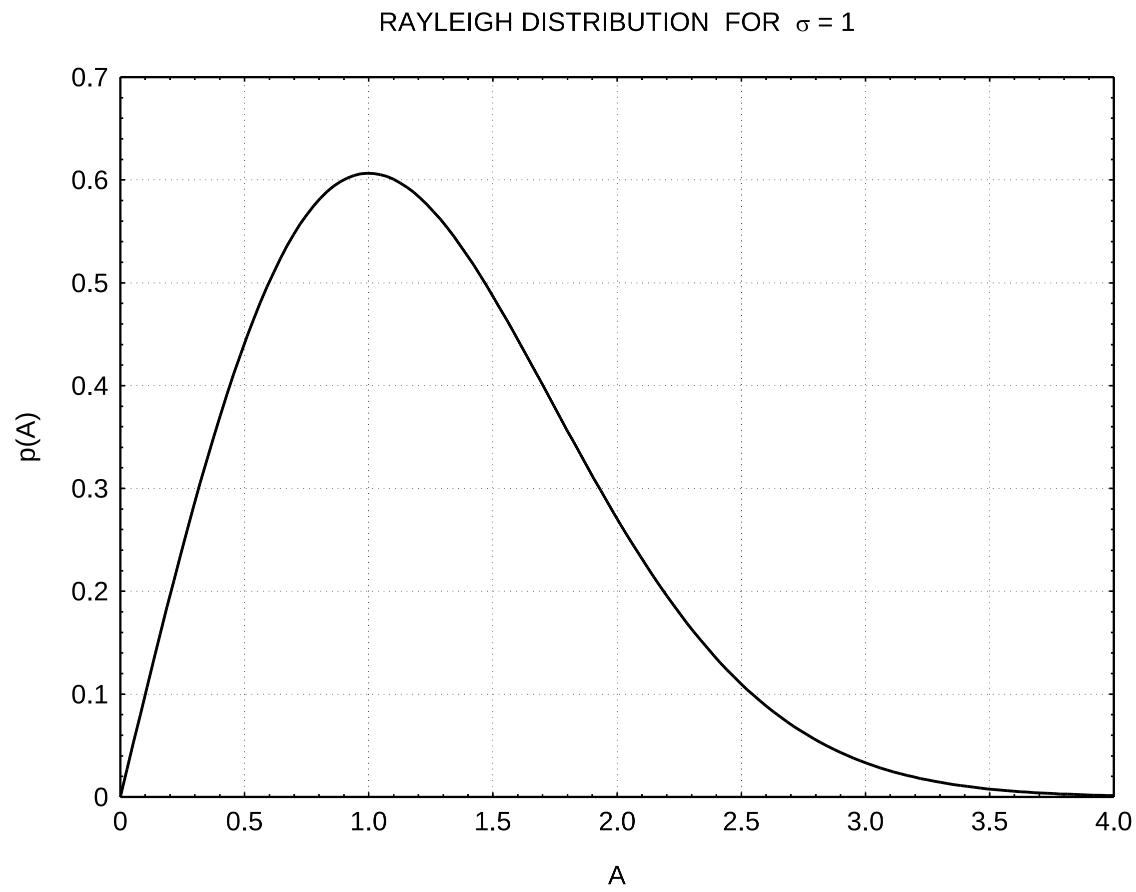

Figure 1.

$$P(\lambda\sigma < A < \infty) = \int_{\lambda\sigma}^{\infty} n(A,\sigma)\,dA\,, \quad A > 0 \tag{2}$$

$$P(\lambda\sigma < A < \infty) = \frac{1}{\sigma^2}\int_{x_1}^{x_2} A\exp\left\{-\frac{1}{2}\left[\frac{A}{\sigma}\right]^2\right\}dA \tag{3}$$

Let

$$Y = \frac{1}{2}\left[\frac{A}{\sigma}\right]^2 \tag{4}$$

$$dY = \left[\frac{A}{\sigma^2}\right] dA \tag{5}$$

Substitute into the integral.

$$P\left(y_1 < Y < y_2\right) = \int_{y_1}^{y_2} \exp\{-Y\} dY \tag{6}$$

$$P\left(y_1 < Y < y_2\right) = -\exp\{-Y\}\Big|_{y_1}^{y_2} \tag{7}$$

$$P\left(\lambda\sigma < A < \infty\right) = -\exp\left\{-\frac{1}{2}\left[\frac{A}{\sigma}\right]^2\right\}\Bigg|_{\lambda\sigma}^{\infty} \tag{8}$$

$$P\left(\lambda\sigma < A < \infty\right) = \exp\left\{-\frac{1}{2}\left[\frac{\lambda\sigma}{\sigma}\right]^2\right\} \tag{9}$$

$$P\left(\lambda\sigma < A < \infty\right) = \exp\left\{-\frac{1}{2}\lambda^2\right\} \tag{10}$$

Sample probability values are given in Table 1, as calculated from equation (10).

Table 1. Rayleigh Distribution Probability	
λ	Prob [A > $\lambda\sigma$]
0.5	88.25 %
1.0	60.65 %
1.5	32.47 %
2.0	13.53 %
2.5	4.39 %
3.0	1.11 %
3.5	0.22 %
4.0	0.034 %
4.5	4.0e-03 %
5.0	3.7e-04 %
5.5	2.7e-05 %
6.0	1.5e-06 %

Example

Consider a band-limited, white noise, random acceleration signal with a standard deviation of 1 G and duration of 300 seconds. The signal was lowpass filtered at 500 Hz.

Apply the signal in Figure 2 as a base excitation function to a single-degree-of-freedom system with a natural frequency of 100 Hz and a damping ratio of 5%.

The response is calculated using the Smallwood, ramp invariant digital recursive filtering relationship.

0.4-second segments of the input and response time histories are shown in Figures 2 and 3, respectively.

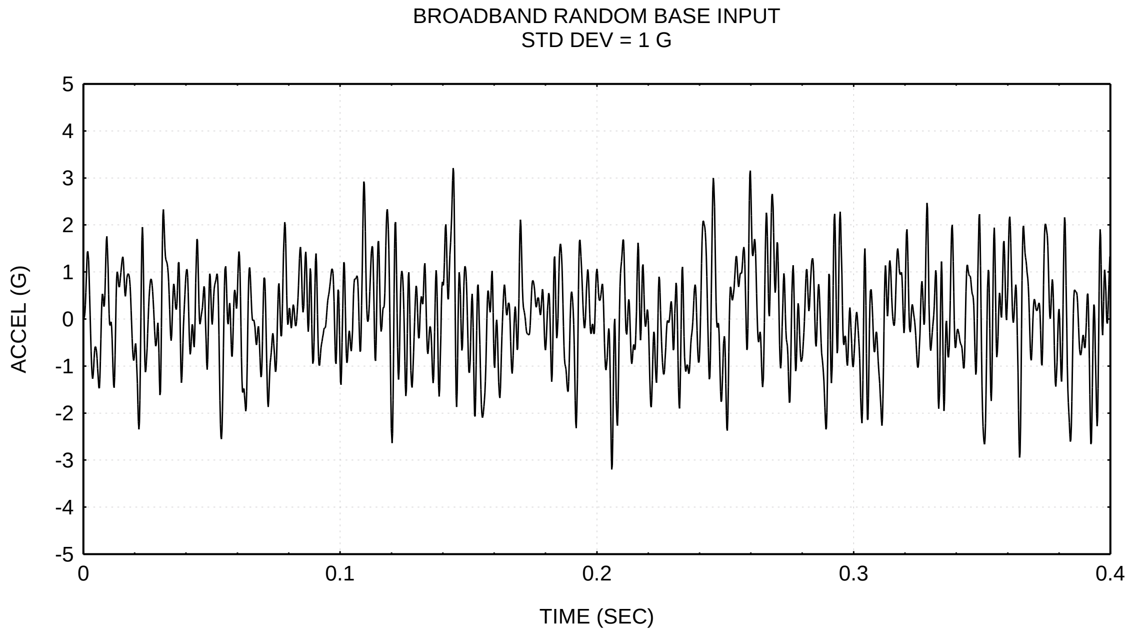

Figure 2.

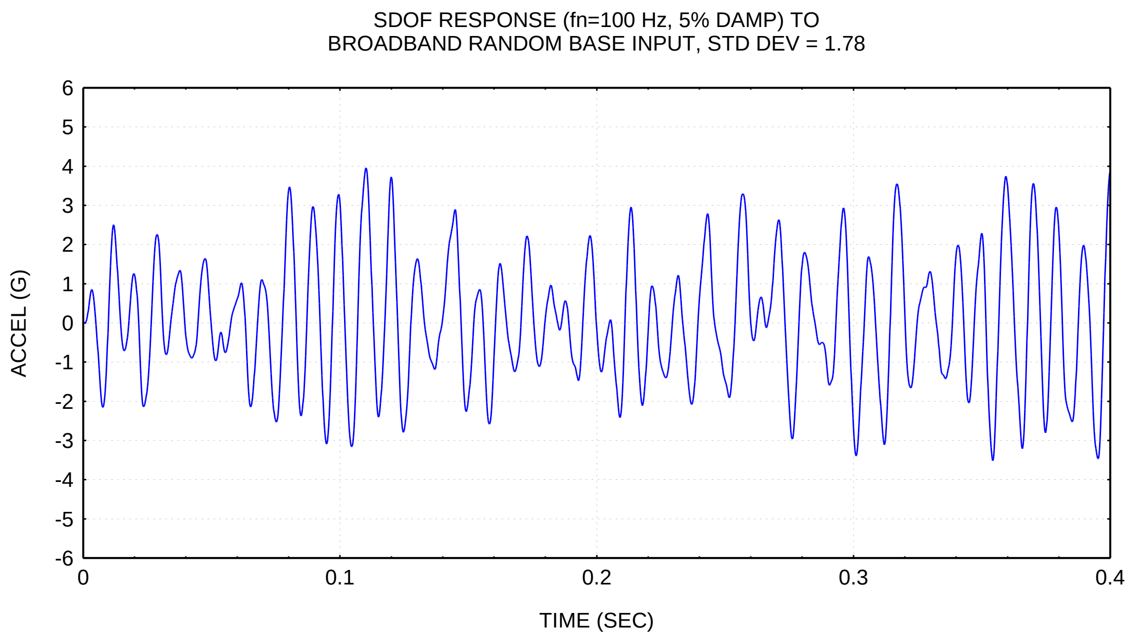

Figure 3.

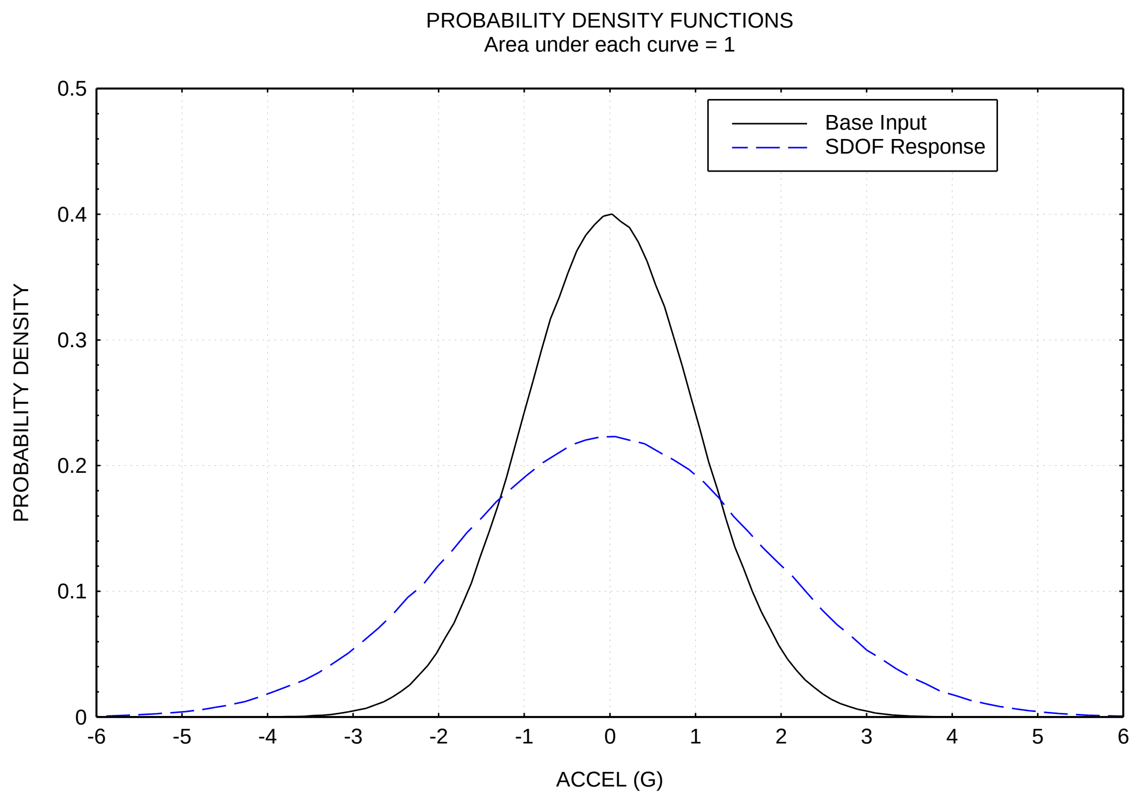

Figure 4.

The probability density functions for each of the two time histories are given in Figure 4. Note that the functions were calculated for the full 300 second duration.

Each probability density function has a Gaussian, or Normal, distribution. The response curve is wider since it has a higher standard deviation value.

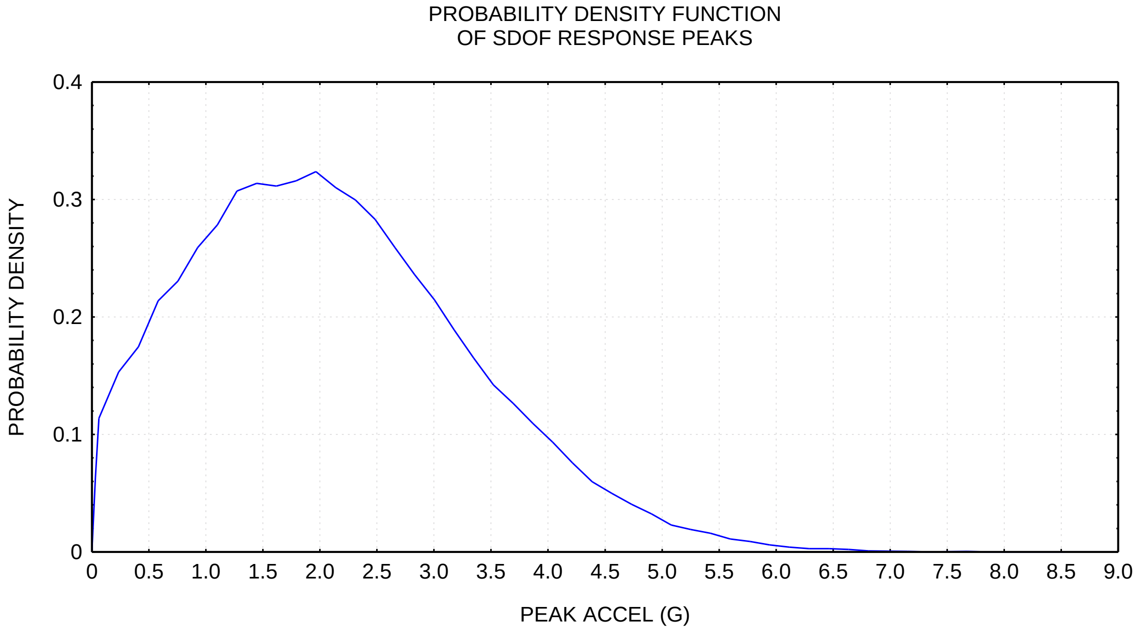

Figure 5.

The probability density function of the absolute response peaks is given in Figure 5. The function was calculated for the full 300 second duration. Its shape is approximately that of a Rayleigh distribution.

The area under the curve is 1.

Absolute Response Statistics

Both the input and response time history had a sample rate of 5000 samples per second. The total number of points for each was thus 1,500,000 for the 300 second duration.

The response time history had a standard deviation σ = 1.78 G. The three sigma value was thus 3σ = 5.34 G.

The response time history had 64,052 peaks. There were 633 peaks above 3σ. Thus, 0.99% of the peaks were above 3σ.

The theoretical percentage of peaks above 3σ is 1.11%, per the Rayleigh distribution.

<u>Maximum Peak Response</u>

The peak response value was 8.68 G, which is 4.88σ.

$$[\ 8.68 \text{ G peak}/\ 1.78 \text{ G}\ ] = 4.88\sigma$$

The probability of 4.88σ is 6.743-04% per the Rayleigh distribution, equation (10).

Given 64,052 peaks, 0.43 peaks should have been above 4.88σ.

The maximum expected peak can be calculated from the formula in Appendix A.

$$\begin{aligned} c_n &= \sqrt{2\ln(fn\,T)} \\ &= \sqrt{2\ln(100\text{Hz})(300\sec)} \\ &= 4.541 \end{aligned} \tag{11}$$

$$\begin{aligned} \max_n &= c_n + \frac{0.5772}{c_n} \\ &= 4.541 + \frac{0.5772}{4.541} \\ &= 4.67 \end{aligned} \tag{12}$$

The maximum expected peak was thus 4.67σ or 8.29 G.

The observed peak of 8.68 G was 5% higher than the expected value.

Rice Analysis, Zero Crossing Ratio

The response time history has the following characteristic.

zero crossings/peaks = 0.93

Both the number of zero crossings and the number of peaks included both positive and negative values.

Note that the ratio would equal 1 for a pure sine function. The narrowband response of an SDOF system approaches 1.

In comparison, the band-limited white noise base input had a ratio of 0.83.

Rice Analysis, Characteristic Frequency

The characteristic frequency N_o was calculated using equation (B-2) via a Matlab script.

The response time history frequency was N_o = 99.5 Hz.

This frequency is approximately equal to the natural frequency of 100 Hz.

The characteristic frequency is also the number of positive-slope zero crossings per unit time.

References

1. W. Thomson, Theory of Vibration with Applications, Second Edition, Prentice-Hall, New Jersey, 1981.

2. David O. Smallwood, An Improved Recursive Formula for Calculating Shock Response Spectra, Shock and Vibration Bulletin, No. 51, May 1981.

3. K. Ahlin, Comparison of Test Specifications and Measured Field Data, Sound & Vibration, September 2006.

4. N.C. Nigam, Introduction to Random Vibration, Massachusetts Institute of Technology, 1993.

5. O.S. Rice, Mathematical Analysis of Random Noise, Dover, New York, 1954.

APPENDIX A

Maximum Expected Peak

Consider a single-degree-of-freedom system with the index n subjected to random vibration. The maximum response $\max_n$ can be estimated by the following equations.

$$c_n = \sqrt{2 \ln (fn\, T)} \tag{A-1}$$

$$C_n = c_n + \frac{0.5772}{c_n} \tag{A-2}$$

$$\max_n = C_n \, \sigma_n \tag{A-3}$$

where

fn	is the natural frequency
T	is the duration
ln	is the natural logarithm function
σ_n	is the standard deviation of the oscillator response

The equations in this appendix are taken from References 3 and 4. They are derived from the Gaussian and Rayleigh distributions.

APPENDIX B

<u>Rice's Equations</u>

The following formulas are taken from Reference 5.

The number of crossings of a given level y per unit time and with positive slope N(y) is

$$N(y) = N_o \exp\left[-\frac{1}{2}\left(\frac{y}{\sigma_y}\right)^2 \right] \tag{B-1}$$

where

N_o is the number of zero crossings per unit time

σ_y is the standard deviation

Equation (B-1) is also the density function of the Rayleigh distribution.

The Rice characteristic frequency N_o is

$$N_o = \frac{1}{2\pi}\frac{\sigma_{dy/dt}}{\sigma_y} \tag{B-2}$$

where

$\sigma_{dy/dt}$ is the standard deviation of the derivative of the time history

σ_y is the standard deviation of the time history

Note that each standard deviation may be calculated in either the time or frequency domain.

The characteristic frequency is also the number of positive-slope zero crossings per unit time.