RANDOM VIBRATION FATIGUE Revision B

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Introduction

This example is an innovation upon a similar problem in Reference 1.

Consider a power supply mounted on a bracket as shown in Figure 1.

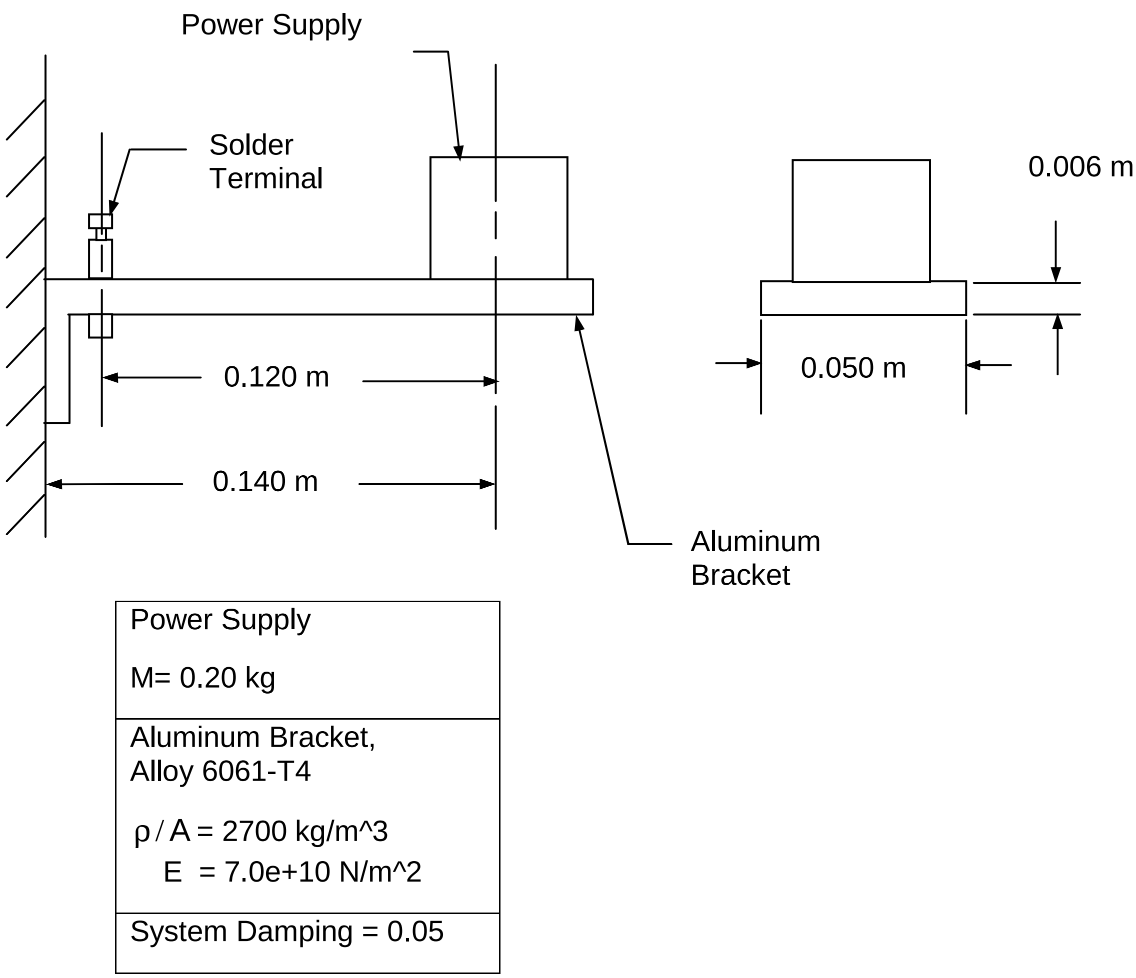

Figure 1.

The area moment of inertia of the beam cross-section I is

$$I = \frac{1}{12} bh^3 \tag{1}$$

$$I = \frac{1}{12} [0.050 \text{ m}][0.006 \text{ m}]^3 \tag{2}$$

$$I = 9.0 \; (10^{-10}) \; \text{m}^4 \tag{3}$$

The stiffness EI is

$$EI = \left[7.0 (10^{+10}) \text{ N/m\^{}2}\right]\left[9.0 \; (10^{-10}) \; \text{m}^4\right] \tag{4}$$

$$EI = 63.0 \quad \text{N m}^2 \tag{5}$$

The mass per length of the beam, excluding the power supply, is

$$\rho = [2700 \text{ kg/m\^{}3}][0.050 \text{ m}][0.006 \text{ m}] \tag{6}$$

$$\rho = 0.810 \; \text{kg/m} \tag{7}$$

The beam mass is

$$\rho L = [0.810 \; \text{kg/m}][0.14 \text{ m}] \tag{8}$$

$$\rho L = 0.113 \text{ kg} \tag{9}$$

Model the system as a single-degree-of-freedom system subjected to base input as shown in Figure 2.

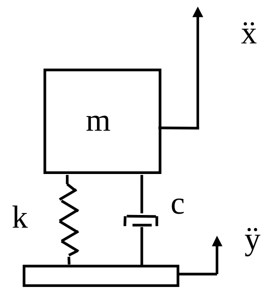

Figure 2.

The natural frequency of the beam, from Reference 2, is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{3EI}{(0.2235\,\rho L + m)L^3}} \tag{10}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{3\left(63.0 \quad N\,m^2\right)}{[0.2235(0.113\ \ kg) + (0.20\ kg)](0.14m\ )^3}} \tag{11}$$

$$f_n = 88.0\ \text{Hz} \tag{12}$$

Again, the damping ratio is $\xi = 0.05$.

Now consider that the bracket assembly is subjected to the random vibration base input level shown in Figure 3 and in Table 1. The duration is 3 minutes.

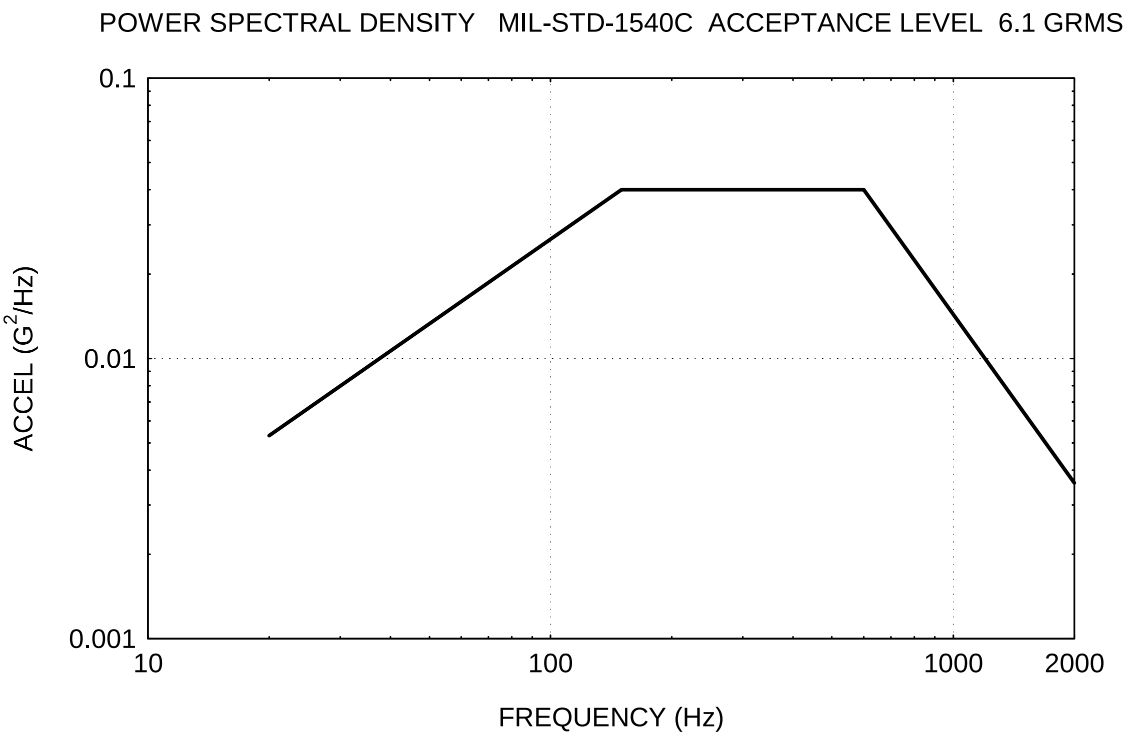

Figure 3.

Table 1. MIL-STD-1540C, PSD, 6.1 GRMS	
Frequency (Hz)	Accel (G^2/Hz)
20	0.0053
150	0.04
600	0.04
2000	0.0036

The acceleration response can be calculated via Miles equation, as shown in Reference 2. The Miles equation, however, makes several restrictive assumptions. For example, the Miles equation assumes a white noise input. In this example, however, the natural frequency falls on a ramp portion of the power spectral density specification.

Thus, a more general approach is used in this example, taken from Reference 3.

The acceleration response $\ddot{x}_{GRMS}$ of the mass at the end of the bracket is

$$\ddot{x}_{GRMS}(f_n, \xi) = \sqrt{\sum_{i=1}^{N} \frac{\left\{1+(2\xi\rho_i)^2\right\}}{\left\{\left[1-\rho_i^{\,2}\right]^2+\left[2\xi\rho_i\right]^2\right\}} \hat{Y}_{APSD}(f_i)\Delta f_i} \quad , \quad \rho_i = f_i / f_n \tag{13}$$

where

ξ is the damping ratio,

f_n is the natural frequency,

$\hat{Y}_{APSD}(f_i)$ is the base input acceleration power spectral density at frequency f_i.

Equation (13) appears cumbersome but is easily implemented via a computer program.

The resulting acceleration is

$$\ddot{x}_{GRMS} = 5.62 \ \text{GRMS} \tag{14}$$

The mean value is zero. Thus the GRMS value is also the 1σ value, where σ is the standard deviation.

$$\ddot{x}\ [1\sigma] = 5.62 \ \text{G} \tag{15}$$

$$\ddot{x}\ [2\sigma] = 11.24 \ \text{G} \tag{16}$$

$$\ddot{x}\ [3\sigma] = 16.86 \ \text{G} \tag{17}$$

The equivalent metric levels are

$$\ddot{x}\ [1\sigma] = 55.1\ \ \mathrm{m/sec^{\wedge}2} \tag{18}$$

$$\ddot{x}\ [2\sigma] = 110.3\ \ \mathrm{m/sec^{\wedge}2} \tag{19}$$

$$\ddot{x}\ [3\sigma] = 165.4\ \ \mathrm{m/sec^{\wedge}2} \tag{20}$$

Consider a free body diagram of the beam as shown in Figure 4.

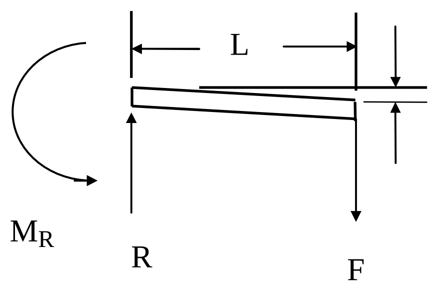

Figure 4.

The reaction moment is

$$M_R = F\,L \tag{21}$$

The force F is equal to the effect mass of the bracket system multiplied by the acceleration level. The effective mass m_e is

$$m_e = (0.2235\,\rho\,L + m) \tag{22}$$

$$m_e = [0.2235\,(0.113\ \ \mathrm{kg}) + (0.20\ \mathrm{kg})] \tag{23}$$

$$m_e = 0.225\ \mathrm{kg} \tag{23}$$

The dynamic force is then calculated as

$$F = m_e \ddot{x} \tag{24}$$

The force levels are

$$F \, [1\sigma] = 12.4 \ \ N \tag{25}$$

$$F \, [2\sigma] = 24.8 \ \ N \tag{26}$$

$$F \, [3\sigma] = 37.2 \ \ N \tag{27}$$

The reaction moments are

$$M_R \, [1\sigma] = 1.74 \ \ Nm \tag{28}$$

$$M_R \, [2\sigma] = 3.48 \ \ Nm \tag{29}$$

$$M_R \, [3\sigma] = 5.22 \ \ Nm \tag{30}$$

The bending moment at the solder terminal is more critical, however, since this location has a thru-hole, which represents a stress concentration area. The moments at the solder terminal plane are

The reaction moments are

$$M_{ST} \, [1\sigma] = 1.49 \ \ Nm \tag{31}$$

$$M_{ST} \, [2\sigma] = 2.98 \ \ Nm \tag{32}$$

$$M_{ST} \, [3\sigma] = 4.47 \ \ Nm \tag{33}$$

The bending stress S_b is given by

$$S_b = \frac{K \ M \ C}{I} \tag{34}$$

The variable K is the stress concentration factor. Assume that the bracket has small mounting holes for the power supply. The stress concentration factor is 3.0 for small holes.

The variable C is the distance from the neutral axis to the outer fiber of the beam. The cross-section is uniform in the sample problem. Thus C is equal to one-half the thickness, or 0.003 m.

The variable I is the area moment of inertia. Recall that for the sample problem

$$I = 9.0\,(10^{-10})\ m^4 \tag{35}$$

The 1-sigma stress at the solder terminal is thus

$$S_b\,[1\sigma] = \frac{3\,[1.49\ \ Nm\ ]\,[\,0.003\ m]}{9.0\,(\,10^{-10}\,)\ m^4} \tag{36}$$

$$S_b\,[1\sigma]\ =\ 1.49\,\left(10^7\right)\ \frac{N}{m^2} \tag{37}$$

$$S_b\,[2\sigma]\ =\ 2.98\,\left(10^7\right)\ \frac{N}{m^2} \tag{38}$$

$$S_b\,[3\sigma]\ =\ 4.47\,\left(10^7\right)\ \frac{N}{m^2} \tag{39}$$

Relevant statistical properties are given in Tables 2 and 3.

Table 2. Statistical Probabilities for a Normal Distribution
Probability inside $\pm 1\sigma$ Limits = 68.27% Probability outside $\pm 1\sigma$ Limits = 31.73%
Probability inside $\pm 2\sigma$ Limits = 95.45% Probability outside $\pm 2\sigma$ Limits = 4.55%
Probability inside $\pm 3\sigma$ Limits = 99.73% Probability outside $\pm 3\sigma$ Limits = 0.27%

Table 3. Statistical Probabilities for a Normal Distribution
Probability inside $\pm$ [1σ to 2σ] Limits = 27.18%
Probability inside $\pm$ [2σ to 3σ] Limits = 4.28%

A semi-conservative approach is to assume that the stresses act for the times shown in Table 4.

This approach is conservative because it assumes that all of the stress within $\pm 1\sigma$ actually occurs exactly at 1σ.

Similarly, it assumes that any stress occurs at the next highest sigma level.

On the other hand, the approach fails to meet complete conservatism because it assumes that no stresses occur beyond $\pm 3\sigma$.

Table 4. Stress versus Time, Solder Terminal Location	
Stress Level	Percent of Time
$S_b\ [1\sigma] = 14.9$ MPa	68.27%
$S_b\ [2\sigma] = 29.8$ MPa	27.18%
$S_b\ [3\sigma] = 44.7$ MPa	4.28%

The fatigue life of the power supply bracket can be calculated from the number of stress reversals and the magnitude of the bending stress.

A simplifying assumption is made for this example that the single-degree-freedom system only vibrates at its natural frequency.

Let n be the number of stress cycles accumulated during the vibration testing.

Let N be the number of stress cycles to produce a fatigue failure.

The number of stress-reversal cycles required to produce a failure at the 1σ, 2σ, and 3σ stresses are determined from an S-N fatigue curve for the 6061-T4 aluminum bracket.

Miner's cumulative damage index R_n is given by

$$R_n = \frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \cdots \tag{40}$$

In theory, the part should fail when

$$R_{n\ (\text{theory})} = 1.0 \tag{41}$$

For aerospace electronic structures, however, a more conservative limit is used

$$R_{n(\text{aero})} = 0.7 \tag{42}$$

The fatigue curve for aluminum 6061-T4 is shown in Figure 5, as taken from Reference 4. Note that a "less conservative" fatigue curve for this same alloy is given in Reference 1.

The curve in Figure 5 is characterized by the following two equations, where S is the bending stress in MPa.

$$S = -17 \log N + 240 \quad \text{MPa} \tag{43}$$

$$N = 10 \wedge \left[\frac{240 - S}{17} \right] \tag{44}$$

Equations (42) and (43) should be regarded as approximations. Note that S-N curves are empirical. Furthermore, authoritative fatigue curves are difficult to find.

S-N FATIGUE CURVE FOR ALUMINUM 6061-T4
This curve is adapted from: Reference 4. It is intended for "Reference only."

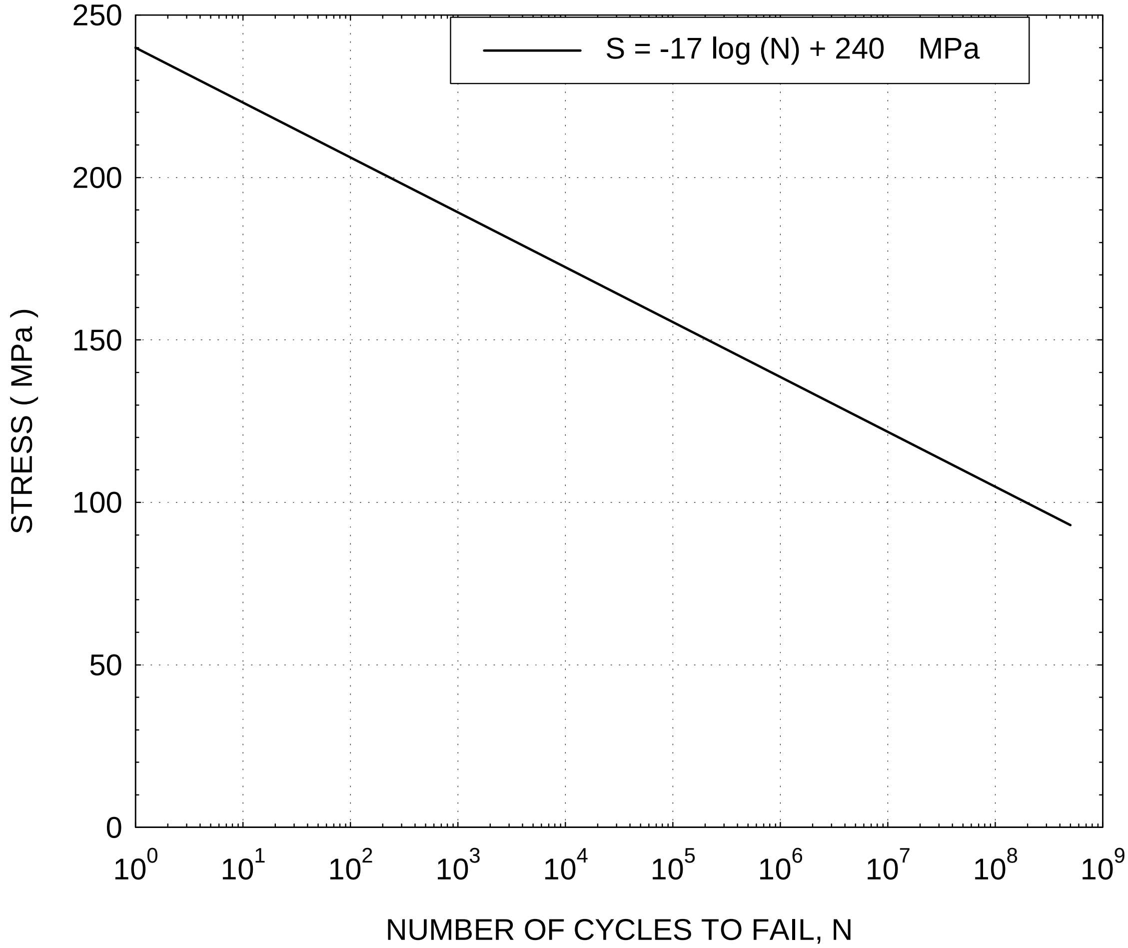

Figure 5.

Nominal Level

The actual time at each stress level for the example problem is calculated in Table 5. Note that

$$\text{Number of cycles} = [\text{time ratio}]\ [\text{natural frequency (Hz)}]\ [\text{duration(sec)}] \qquad (44)$$

Table 5.

Stress versus Time, Solder Terminal Location, Nominal Input

Duration = 3 minutes (180 sec)

Natural frequency = 88.0 Hz

Material = Aluminum 6061-T4

Input = 6.1 GRMS, as shown in Figure 3

Stress Level	Time Ratio	Test Cycles	Limit Cycles from S-N curve
$S_b\ [1\sigma] = 14.9$ MPa	0.6827	$n_1 = 10{,}814$	$N_1 > 50\ (10^7)$
$S_b\ [2\sigma] = 29.8$ MPa	0.2718	$n_2 = 4305$	$N_2 > 50\ (10^7)$
$S_b\ [3\sigma] = 44.7$ MPa	0.0428	$n_3 = 678$	$N_3 > 50\ (10^7)$

Note that aluminum alloys do not have a true endurance limit. Nevertheless, for the purpose of this example 93 MPa shall be considered as the "practical endurance limit," since this value corresponds to $50\ (10^7)$ cycles.

The stress values in Table 5 are well below the assumed endurance limit. Thus, no further calculation is required. The bracket should pass the vibration test with tremendous margin.

Nominal Level +6 dB

Now assume that the input level in Figure 3 is increased uniformly by 6 dB. The new level is 12.2 GRMS[1]. The response stress levels likewise increase by 6 dB. The new levels are shown in Table 6.

Table 6.

Stress versus Time, Solder Terminal Location, 6 dB Increase over Nominal Input

Duration = 3 minutes (180 sec)

Natural frequency = 88.0 Hz

Material = Aluminum 6061-T4

Input = 12.2 GRMS (PSD in Figure 3 + 6 dB)

Stress Level	Time Ratio	Test Cycles	Limit Cycles from S-N curve
$S_b\ [1\sigma] = 29.8$ MPa	0.6827	$n_1 = 10{,}814$	$N_1 > 50\ (10^7)$
$S_b\ [2\sigma] = 59.6$ MPa	0.2718	$n_2 = 4305$	$N_2 > 50\ (10^7)$
$S_b\ [3\sigma] = 89.4$ MPa	0.0428	$n_3 = 678$	$N_3 > 50\ (10^7)$

The stress values in Table 5 are still below the assumed endurance limit, 93 MPa. Thus, no further calculation is required. The bracket should pass the +6 dB vibration test.

[1] $\Delta dB = 20\ \log(A / B)$, where A and B are both in terms of GRMS.

Nominal Level +9 dB

Now assume that the input level in Figure 3 is increased uniformly by 9 dB. The new level is 17.2 GRMS. The response stress levels likewise increase by 9 dB. The new levels are shown in Table 7.

Table 7.

Stress versus Time, Solder Terminal Location, 9 dB Increase over Nominal Input

Duration = 3 minutes (180 sec)

Natural frequency = 88.0 Hz

Material = Aluminum 6061-T4

Input = 17.2 GRMS (PSD in Figure 3 + 9 dB)

Stress Level	Time Ratio	Test Cycles	Limit Cycles from S-N curve
$S_b\ [1\sigma] = 42.1$ MPa	0.6827	$n_1 = 10{,}814$	$N_1 > 50\ (10^7)$
$S_b\ [2\sigma] = 84.3$ MPa	0.2718	$n_2 = 4305$	$N_2 > 50\ (10^7)$
$S_b\ [3\sigma] = 126.4$ MPa	0.0428	$n_3 = 678$	$N_3 = 4.81\text{e+06}$

The 1-sigma and 2-sigma stress levels are below the assumed endurance limit, 93 MPa. The 3-sigma stress level exceeds the limit, however.

The fatigue calculation is thus

$$R_n = \frac{n_3}{N_3} \tag{45}$$

$$R_n = \frac{678}{4.81e+06} \tag{46}$$

$$R_n \approx 0.00 \quad << \quad 0.7 \quad \text{(aerospace limit)} \tag{47}$$

The part should thus pass the test.

Nominal Level +12 dB

Now assume that the input level in Figure 3 is increased uniformly by 12 dB. The new level is 24.4 GRMS. The response stress levels likewise increase by 12 dB. The new levels are shown in Table 8.

Table 8.

Stress versus Time, Solder Terminal Location, 12 dB Increase over Nominal Input

Duration = 3 minutes (180 sec)

Natural frequency = 88.0 Hz

Material = Aluminum 6061-T4

Input = 17.2 GRMS (PSD in Figure 3 + 12 dB)

Stress Level	Time Ratio	Test Cycles	Limit Cycles from S-N curve
$S_b\ [1\sigma]$ = 59.6 MPa	0.6827	$n_1 = 10{,}814$	$N_1 > 50\ (10^7)$
$S_b\ [2\sigma]$ = 129.2 MPa	0.2718	$n_2 = 4305$	N_2 = 3.29e+06
$S_b\ [3\sigma]$ = 178.8 MPa	0.0428	$n_3 = 678$	N_3 = 3981

The 1-sigma stress level is below the assumed endurance limit, 93 MPa. The 2-sigma and 3-sigma stress levels exceeds the limit, however.

The fatigue calculation is thus

$$R_n = \frac{n_2}{N_2} + \frac{n_3}{N_3} \qquad (48)$$

$$R_n = \frac{4305}{3.29e+06} + \frac{678}{3981} \qquad (49)$$

$$R_n = 0.17 \ll 0.7 \qquad \text{(aerospace limit)} \tag{50}$$

The part should thus pass the +12 dB test.

Now consider the expected time for failure at the +12 dB level. Use only the 3-sigma stress level as a rough approximation.

The expected time for failure at the +12 dB level is

$$\left[\frac{0.7}{0.17}\right]\left[3 \text{ min}\right] \approx 12.4 \text{ min} \tag{51}$$

Again, the presence of the solder terminal hole increases the stress level by a factor of 3. The absence of this hole would reduce the 3-sigma stress level to a level below the assumed endurance limit.

A good design compromise would be to move the solder terminal closer to the power supply. The bending stress would thus be reduced at the hole location.

In addition, some general information about fatigue is given in Appendix A, as taken from Reference 5.

As an aside, the instantaneous level of the single-degree-of-freedom response was assumed to follow a normal distribution. This is a good assumption. The individual response peaks, however, may follow a Rayleigh distribution. A more rigorous analysis would account for these peaks.

<u>References</u>

1. Dave Steinberg, Vibration Analysis for Electronic Equipment, Second Edition, Wiley, New York, 1988.

2. T. Irvine, Bending Frequencies of Beams, Rods, and Pipes; Vibrationdata.com Publications, 2000.

3. T. Irvine, An Introduction to the Vibration Response Spectrum, Vibrationdata.com Publications, 2000.

4. Shigley, Mechanical Engineering Design, Third Ed, Table A-18, McGraw-Hill, New York, 1977.

5. V. Adams and A. Askenazi, Building Better Products with Finite Element Analysis, OnWord Press, Santa Fe, N.M., 1999.

APPENDIX A

Fatigue Cracks

A ductile material subjected to fatigue loading experiences basic structural changes. The changes occur in the following order;

1. *Crack Initiation.* A crack begins to form within the material.

2. *Localized crack growth*. Local extrusions and intrusions occur at the surface of the part because plastic deformations are not completely reversible.

3. *Crack growth on planes of high tensile stress.* The crack propagates across the section at those points of greatest tensile stress.

4. *Ultimate ductile failure.* The sample ruptures by ductile failure when the crack reduces the effective cross section to a size that cannot sustain the applied loads.

Design and Environmental Variables affecting Fatigue Life

The following factors decrease fatigue life.

1. *Stress concentrators.* Holes, notches, fillets, steps, grooves, and other irregular features will cause highly localized regions of concentrated stress, and thus reduce fatigue life.

2. *Surface roughness.* Smooth surfaces are more crack resistant because roughness creates stress concentrators.

3. *Surface conditioning.* Hardening processes tend to increase fatigue strength, while plating and corrosion protection tend to diminish fatigue strength.

4. *Environment.* A corrosive environment greatly reduces fatigue strength. A combination of corrosion and cyclical stresses is called *corrosion fatigue.*