MECHANICAL ENGINEERING DEPARTMENT
UNITED STATES NAVAL ACADEMY

EM423 - INTRODUCTION TO MECHANICAL VIBRATIONS

SINGLE DEGREE OF FREEDOM SYSTEMS
PART 2: HARMONIC EXCITATION

INTRODUCTION

The previous SDOF handout concentrated on the unforced mass/spring system, both with and without damping. In this handout we concentrate on a SDOF subject to various forms of harmonic excitation.

HARMONIC FORCE EXCITATION

When a constant amplitude harmonic force excites a linear system, the system responds at the same frequency as the excitation (we are ignoring any transients due to switching on the excitation). We need to clarify the difference between the natural frequency, and the excitation frequency.

$$\boxed{\begin{aligned} w_n &= \text{ Natural Frequency (rad/s)} \\ w &= \text{ Excitation Frequency (rad/s)} \end{aligned}}$$

The natural frequency, w_n, is a property of the SDOF, and <u>does not vary with excitation.</u> The excitation frequency, w depends <u>only</u> on the excitation, and is independent of the SDOF.

While harmonic force excitation is not a common form met in many real situations, solving for it provides the key for solving many other forms of excitation. Also, many standard experimental vibration test methods measure the frequency characteristics of a structure. Even though the excitation for the test may not be steady state harmonic, the effect is the same as measuring the response due to harmonic excitation. Therefore the response due to steady state excitation needs to be understood fully before other forms of excitation are analyzed. The equation of motion is:

$$m\ddot{x} + c\dot{x} + kx = F_O \sin(wt)$$

The solution to this equation is in two parts: The Complementary Function and the Particular Integral. The Complementary Function represents the response of a damped free vibration, and was discussed in the first SDOF handout. The Particular Integral represents the *steady-state solution,* and can be presented in a number of ways. One solution has the form:

$$x(t) = X.\sin(wt - j)$$

On this course we have already used the complex exponential form of harmonic notation. For vibrations this is often the best method to use, so we now use it to solve the equation of motion. The forcing function can be written as $F_O e^{iwt}$ and the displacement response can be written as:

$$x(t) = |X| e^{i(wt-j)} = X.e^{iwt}$$

where X is a complex displacement, holding both the amplitude of motion, and the phase of the response relative to the excitation. Substituting the complex forms for both force and response into the basic equation of motion results in:

$$\left(-w^2 m + iwc + k\right) X.e^{iwt} = F_O.e^{iwt}$$

which by rearranging and using the relationships $w_n^2 = (k/m)$ and $z = c/(2m w_n)$, results in the equation:

$$H(iw) = \frac{X}{F_O} = \frac{1}{m\left\{\left(w_n^2 - w^2\right) + 2izww_n\right\}}$$

H(iw) is called the *Frequency Response Function* (FRF) of the SDOF. The FRF is a property of the SDOF. Its value (per unit mass) depends solely on the excitation frequency, the natural frequency, and the viscous damping ratio. The amplitude and phase of this complex FRF equation are:

$$\boxed{\begin{aligned} &\text{Magnitude} = |H(iw)| = \left|\frac{X}{F_O}\right| = \frac{1}{m\left\{\left(w_n^2 - w^2\right)^2 + 4z^2 w^2 w_n^2\right\}^{\frac{1}{2}}} \\ &\text{Phase} = \tan^{-1}\left(\frac{-2zww_n}{\left(w_n^2 - w^2\right)}\right) \end{aligned}}$$

The maximum value of the FRF is when the forcing frequency is given by:

$$\text{Excitation Frequency for maximum FRF} = w_r = w_n\sqrt{\left(1-2z^2\right)} \text{ rad/s}$$

This frequency, w_r, is called the *circular resonant frequency.*

The low frequency behavior of the FRF (i.e. well below resonance, with $w \ll w_n$) is controlled by the stiffness.

$$\left|H(iw)\right|_{w \ll w_n} = \frac{1}{mw_n^2} = \frac{1}{k}$$

The high frequency behavior with $w \gg w_n$ is controlled by the mass.

$$\left|H(iw)\right|_{w \gg w_n} = \frac{1}{-mw^2}$$

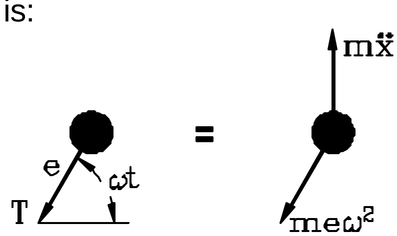
When $w \approx w_n$ (close to resonance) the motion is controlled by the damping. If there were no damping, the FRF would be infinite at the natural frequency. However, even a very small amount of damping (energy dissipation) prevents this happening.
The behavior of these equations, and sketches of the graphs, is given in class.

ROTATING UNBALANCE

One of the most common sources of vibration in engineering is rotating unbalance. This is where a rotating machine is not precisely balanced, and the rotating out of balance mass causes forces on the engine (the idea of centripetal or centrifugal force). We consider the complete engine (including the out of balance mass) to have mass M, and the unbalanced mass itself to be m, rotating with eccentricity e. A real engine unbalance will cause vibrations in at least two planes (vertical and rocking), but for this discussion we limit ourselves to pure vertical motion only. As usual, we measure the (vertical) displacements from the equilibrium position, so:

$$\text{Displacement of non-rotating mass, } (M-m), \ = x$$
$$\text{Displacment of rotating mass, } m, \ = x + e.\sin(wt)$$

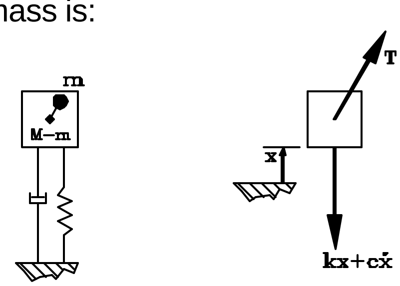
The FBD for the rotating mass is:



Resolve forces vertically upwards

$$-\text{Tension}_{VERTICALCOMPONENT} = m\ddot{x} - mew^2\sin(wt)$$

The FBD for the non rotating mass is:



Resolve forces vertically upwards

$$\text{Tension}_{VERTICALCOMPONENT} - kx - c\dot{x} = (M-m)\ddot{x}$$

hence

$$-m\ddot{x} + mew^2\sin(wt) - kx - c\dot{x} = (M-m)\ddot{x}$$
$$M\ddot{x} + c\dot{x} + kx = mew^2\sin(wt)$$

The last equation is essentially the same as for force excitation, but replacing the forcing function F_o with (me_w^2), which is a function of frequency. We can therefore write down the dynamic response of the SDOF with rotating unbalance by comparing the equation of motion with the force excitation case. The response is relative to the product "*me*".

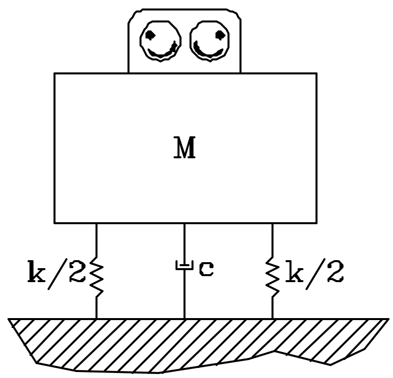
$$\frac{X}{me} = \frac{w^2}{M\left\{\left(w_n^2 - w^2\right) + 2izww_n\right\}}$$

The magnitude and phase of this complex function are:

$$\boxed{\begin{aligned} \left|\frac{X}{me}\right| &= \frac{w^2}{M\left\{\left(w_n^2 - w^2\right)^2 + 4z^2w^2w_n^2\right\}^{1/2}} \\ \text{phase} &= \tan^{-1}\left(\frac{2zww_n}{\left(w_n^2 - w^2\right)}\right) \end{aligned}}$$

Example.

A counter rotating eccentric weight exciter is used to produce forced oscillation of a spring supported mass. By varying the speed of rotation, a resonant amplitude of 0.60 cm was recorded. When the speed of rotation was increased considerably above the resonant frequency, the amplitude approached a fixed value of 0.08 cm. What was the viscous damping ratio of the system?



Solution.

The assumption is that damping is light, and therefore the resonant frequency identified above is the same as (or very close to) the natural frequency of the SDOF. The dynamic response equation for this problem is:

$$\left|\frac{X}{me}\right| = \frac{w^2}{M\left\{\left(w_n^2 - w^2\right)^2 + 4z^2w^2w_n^2\right\}^{1/2}}$$

when $w \approx w_n$

$$\left|\frac{X_1}{me}\right| = \frac{1}{2Mz} \qquad \text{hence} \qquad X_1 = \frac{me}{2Mz} = 0.06 \text{ cm}$$

when $w \gg w_n$

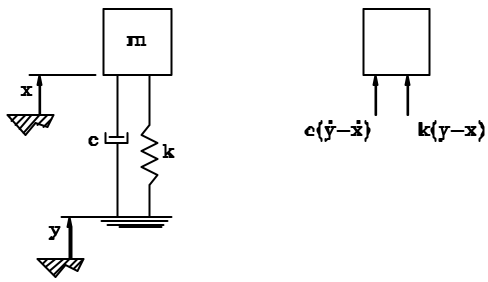
$$X_2 = \frac{me}{M} = 0.08 \text{ cm}$$

Simultaneous solution of these two equations yields the result:

$$z = \frac{0.08}{2 \times 0.60} = 0.0666 = 6.7\%$$

SUPPORT MOTION

A large class of problems concerns structures subject to base (support) motion. Examples include most vehicles (cars, airplanes, trains, ...) as they progress over rough surfaces. Another class of problems includes buildings in earthquake zones. With base motion the excitation is a displacement, which is transmitted to the SDOF mass through a spring element. The force on the mass is therefore dependent on the ***relative*** displacement of the support and the mass. The FBD for the mass is:



Resolving forces vertically gives:

$$k\left(y-x\right)+c\left(\dot{y}-\dot{x}\right)=m\ddot{x}$$

Now *(x-y)* is the relative motion between the mass and the ground. If we make the substitution *z* = *(x-y)* we get:

$$\begin{aligned} k\left(y-x\right)+c\left(\dot{y}-\dot{x}\right)&=m\ddot{x} \\ -kz-c\dot{z}&=m\left(\ddot{z}+\ddot{y}\right) \\ m\ddot{z}+c\dot{z}+kz&=-m\ddot{y} \end{aligned}$$

We will now restrict the problem to harmonic base excitation and we obtain the equation:

$$m\ddot{z}+c\dot{z}+kz=mw^2Y.\sin\left(wt\right)$$

This equation is of the same form as that for rotary unbalance. Therefore we can solve it for *Z* directly using the previous results:

$$Z=\frac{Yw^2}{\left\{\left(w_n^2-w^2\right)+2izww_n\right\}}$$

However, we do not just want to know *Z*, which is the relative motion between the support and mass. We also need to know the <u>absolute</u> motion of the mass, *X*, which is given by *X* = *Z*+*Y*:

$$X=Z+Y=Y\left\{\frac{w^2}{\left\{\left(w_n^2-w^2\right)+2izww_n\right\}}+1\right\}$$

Hence:

$$\frac{X}{Y} = \frac{\left(w_n^2 + 2izww_n\right)}{\left(w_n^2 - w^2 + 2izww_n\right)}$$

The amplitude and phase of this complex equation can be determined:

$$\boxed{\begin{aligned} \left|\frac{X}{Y}\right| &= \frac{\sqrt{w_n^4 + 4z^2 w_n^2 w^2}}{\sqrt{\left(w_n^2 - w^2\right)^2 + 4z^2 w_n^2 w^2}} \\ j &= \tan^{-1}\left(\frac{2zw^3}{w_n\left(w_n^2 - w^2\right) + 4z^2 w_n w^2}\right) \end{aligned}}$$

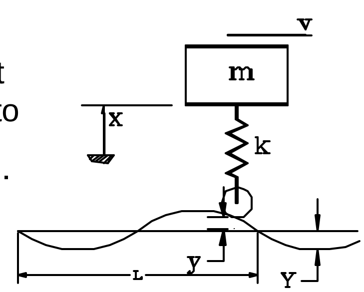
<u>VIBRATION ISOLATION</u>

One important aspect of vibration engineering is the design of vibration isolators. In simplistic form, they are spring/damper systems used to reduce the amount of vibration transmitted through them. A typical example is the suspension system in cars, where the "shock absorber" is a combined spring and damper unit. At sea, most electronic equipment is isolated from the hull by isolators that are usually a natural rubber providing a spring stiffness, supported on steel flanges. These isolators are designed to reduce vibration transmitted <u>to the equipment.</u> Large engines and noisy equipment are also mounted, but this time to reduce the vibration transmitted <u>to the hull.</u>

An incorrect selection of isolator will amplify the vibration, and make things worse. We therefore need to understand the theory of isolators. Before we can make an analysis, we have to determine what we mean by isolation.

<u>ISOLATION FROM GROUND MOTION.</u>

Generally, if the ground is moving or a vehicle is traveling across rough ground, and we wish to protect equipment from that motion, we want to reduce the ratio of transmitted displacement to excitation displacement. That is, we wish to reduce the ratio $\left|\frac{X}{Y}\right|$.



For a vehicle traveling over rough ground approximated by a sine wave, the excitation and response will be:

$$y(t) = Ye^{iwt}$$

$$x(t) = Xe^{iwt}$$

with X being complex to include phase information. The excitation frequency, w, can be determined from the wavelength of the ground roughness, L, and vehicle speed, v:

$$v = fL$$

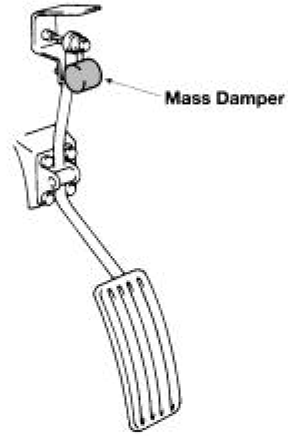
$$\text{hence} \qquad \omega = 2\pi f = \frac{2\pi v}{L}$$

This is an example of support motion, discussed earlier, and we can therefore use the results directly. The equation for the motion of the mass relative to the ground motion is called the *displacement transmissibility*, T_R, of the system. The equation is clearly the same as the result for support motion.

$$\left|T_R\right| = \left|\frac{X}{Y}\right| = \frac{\sqrt{\omega_n^4 + 4\zeta^2\omega_n^2\omega^2}}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

$$\varphi = \tan^{-1}\left(\frac{2\zeta\omega^3}{\omega_n\left(\omega_n^2 - \omega^2\right) + 4\zeta^2\omega_n\omega^2}\right)$$

The response is in two distinct frequency ranges, separated by the frequency at which the displacement transmissibility is unity. The graphical form of the response, and its interpretation, will be given in class.



The vibration (transmitted output) is only less than the input if the transmissibility is less than one (zero dB). This is only the case when the excitation frequency is related to the natural frequency by $\omega > \sqrt{2}\omega_n$. If the excitation frequency is not in this required range, we might be able to modify the structure to achieve this. We would usually want to reduce the natural frequency, so we would consider reducing the stiffness, or increasing the mass, or a combination of the two. Note that changing the amount of damping does NOT change the natural frequency.

As an example, consider an automobile's accelerator pedal. Its vibration can be excessive, and can cause driver discomfort. Toyota, for example, add a mass to the system to reduce its natural frequency well below the typical excitation frequencies observed in a car.

<u>ISOLATION FROM RUNNING EQUIPMENT.</u>

In this class of problems we have a different consideration. Running equipment usually generates dynamic forces, and we want to reduce the **force** transmitted through the isolation mounts to the ground (or ship's hull, or whatever). The isolator is typically a spring/damper system, comparable to the type used for isolation from ground motion. We have already considered the force transmitted through the spring/damper when we considered support motion. This time, however, the displacement is from the mass, but the effect is similar. A full analysis, not included in these notes, shows that the *force transmissibility* of the isolator (ratio of transmitted force to excitation force) is given by:

$$T_R = \frac{F_T}{F} = \frac{\left(\omega_n^2 + 2i\zeta\omega\omega_n\right)}{\left(\omega_n^2 - \omega^2 + 2i\zeta\omega\omega_n\right)}$$

Comparison with the displacement transmissibility shows that the two forms of transmissibility we have considered have the same solution. That is:

$$\left|T_R\right| = \frac{\sqrt{\omega_n^4 + 4\zeta^2\omega_n^2\omega^2}}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

$$\varphi = \tan^{-1}\left(\frac{2\zeta\omega^3}{\omega_n\left(\omega_n^2 - \omega^2\right) + 4\zeta^2\omega_n\omega^2}\right)$$

In other words, the force and displacement transmissibility equations are identical. <u>**HOWEVER, IT MUST BE REMEMBERED THAT EVEN THOUGH THE EQUATIONS ARE THE SAME, THEY ARE USED FOR DIFFERENT SYSTEMS.**</u>

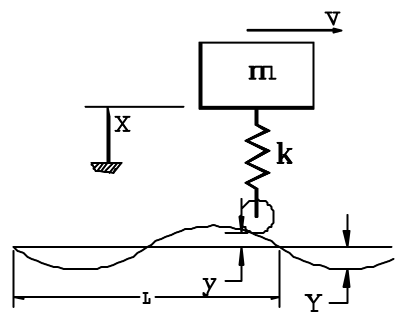
ASSIGNMENTS

1. A machine part of mass 2 kg vibrates in a viscous medium. Determine the damping coefficient, *c*, when a harmonic exciting force of 25 N results in a resonant amplitude of 1.5 cm with a period of 0.20 s.

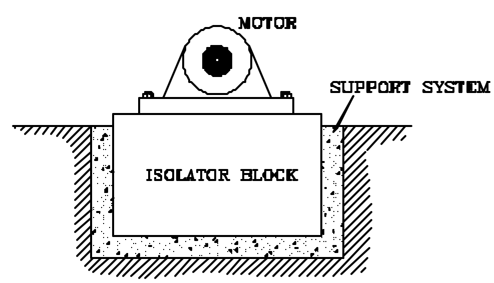
2. A weight attached to a spring of stiffness 500 N/m has a viscous damping device attached. When the weight is displaced and released, the period of vibration is 1.80 s, and the ratio of consecutive amplitudes is 4.2 to 1.0. Determine the amplitude and phase when a force $F_O = 2.\cos(3t)$ acts on the system.

3. A spring-mass system is excited by a force $F_O\sin(\omega t)$. When excited at the natural frequency, the amplitude is measured to be 0.6 cm. At $\omega = (0.8) \times \omega_n$, the amplitude is measured to be 0.5 cm. Determine the viscous damping ratio, ζ, of the system.

4. The figure represents a simplified arrangement for a spring-supported vehicle traveling over a rough road. Determine an equation for the amplitude of motion for m as a function of road speed. What is the worst road speed?



5. A 70 kg motor is rigidly mounted on an isolator block of mass 1200 kg. The natural frequency and viscous damping ratio of the combination on the isolator block's support system are f_n = 160 cpm, and ζ = 0.10. An unbalance in the motor causes a vertical harmonic force of f = 100 sin (30 t). Determine:



a) The amplitude of vibration of the block.

b) The maximum acceleration of the block.

c) The amplitude of the dynamic force transmitted to the ground through the isolator block's support system.

6. A sensitive 100 kg instrument is to be installed at a location where the observed acceleration is 15 cm/s^2 at 20 Hz. It is proposed to mount the instrument on a rubber pad, with k = 280 kN/m and ζ = 0.10. What acceleration will be transmitted to the instrument?

7. Show that for a single-degree-of-freedom system the peak response amplitude occurs when the force excitation frequency is given by:

$$w = w_n\sqrt{1-2z^2}$$

SOLUTIONS

1. A machine part of mass 2 kg vibrates in a viscous medium. Determine the damping coefficient, *c*, when a harmonic exciting force of 25 N results in a resonant amplitude of 1.5 cm with a period of 0.20 s.

$$\left|H(w)\right| = \left|\frac{X}{F_O}\right| = \frac{1}{m\left\{\left(w_n^2 - w^2\right)^2 + 4z^2 w_n^2 w^2\right\}^{1/2}}$$

Assume light damping for now, so at resonance, $w \approx w_n$

$$\left|\frac{X_{RES}}{F_O}\right| = \frac{1}{m2zw_n^2}$$

so $$z = \frac{F_O}{X_{RES}m2w_n^2}$$ but also $$z = \frac{c}{2mw_n}$$

Combine and rearrange:

$$c = \frac{F_O}{X_{RES}w_n}$$

but $$w_n = 2pf_n = \frac{2p}{t_n}$$

hence $$c = \frac{F_O 0.2}{X_{RES}2p} = \frac{25\times 0.2}{0.015\times 2p} = 53.1 \text{ Ns/m}$$

Check the "light damping" assumption:

$$z = \frac{53.1\times 0.2}{2\times 2\times 2p} = 0.423 = 42.3\%$$

which is not "light". So the answer will have some error. It is left to the student to show that in this specific case, using a resonant frequency of $w_R = w_n\sqrt{\left(1-2z^2\right)}$ actually makes no difference to the final answer.

2. A weight attached to a spring of stiffness 500 N/m has a viscous damping device attached. When the weight is displaced and released, the period of vibration is 1.80 s, and the ratio of consecutive amplitudes is 4.2 to 1.0. Determine the amplitude and phase when a force F_O = 2.cos(3t) acts on the system.

First, find the system characteristics from the free decay (log-decrement) information. Note that damping is not "light" for this problem.

$$d_N = \ln\left(\frac{4.2}{1.0}\right) = 1.435 = \frac{2Npz}{\sqrt{1-z^2}}$$

from which $z = 0.2227$

$$w_d = w_n\sqrt{1-z^2} \quad \text{so} \quad \frac{2p}{1.80} = w_n\sqrt{1-0.2227^2}$$

$$w_n = 3.58 \text{ rad/s} \quad \text{and} \quad m = \frac{k}{w_n^2} = 39 \text{ kg}$$

Now solve for the harmonic force excitation part of the problem:

$$F_O = 2 \text{ N}; \quad w = 3 \text{ rad/s}$$

$$\left|H(w)\right| = \left|\frac{X}{F_O}\right| = \frac{1}{m\left\{\left(w_n^2 - w^2\right)^2 + 4z^2 w_n^2 w^2\right\}^{1/2}}$$

$$\text{so} \quad X = \frac{2}{39\left\{\left(3.58^2 - 3^2\right)^2 + 4\times 0.2227^2 \times 3.58^2 \times 3^2\right\}_{1/2}} = 0.0084 \text{ m}$$

$$\text{Phase of } H(w) = \tan^{-1}\left(\frac{2zww_n}{\left(w_n^2 - w^2\right)}\right) = \tan^{-1}\left(\frac{2\times 0.2227\times 3\times 3.58}{\left(3.58^2 - 3^2\right)}\right) = 51.4^O$$

3. A spring-mass system is excited by a force $F_O\sin(\omega t)$. When excited at the natural frequency, the amplitude is measured to be 0.6 cm. At $\omega = (0.8) \times \omega_n$, the amplitude is measured to be 0.5 cm. Determine the viscous damping ratio, ζ, of the system.

$$\text{At resonance} \quad \left|\frac{X_{RES}}{F_O}\right| = \frac{1}{m2zw_n^2} \quad \text{hence} \quad \frac{0.006}{F_O} = \frac{1}{2mzw_n^2}$$

$$\text{At } w = (0.8)w_n \quad \left|\frac{X}{F_O}\right| = \frac{1}{m\left\{\left(w_n^2 - w^2\right)_2 + 4z^2 w^2 w_n^2\right\}^{1/2}}$$

$$\text{hence} \quad \frac{0.005}{F_O} = \frac{1}{mw_n^2\left\{\left(1-0.8^2\right)^2 + 4z^2 0.8^2\right\}^{1/2}}$$

$$\text{Hence} \quad \left(\frac{F_O}{mw_n^2}\right)^2 = 4z^2(0.006)^2 = (0.005)^2\left\{\left(1-0.8^2\right)^2 + 4z^2 0.8^2\right\}$$

from which $z^2 = 0.0405$ or $z = 0.20 = 20\%$

4. The figure represents a simplified arrangement for a spring-supported vehicle traveling over a rough road. Determine an equation for the amplitude of motion for *m* as a function of road speed. What is the worst road speed?

$$\left|\frac{X}{Y}\right| = \frac{\sqrt{w_n^4 + 4z^2 w_n^2 w^2}}{\sqrt{\left(w_n^2 - w^2\right)^2 + 4z^2 w_n^2 w^2}} \quad \text{and} \quad f = \tan^{-1}\left(\frac{2zw^3}{w_n\left(w_n^2 - w^2\right) + 4z^2 w_n w^2}\right)$$

For this problem:

$$w = \frac{2pv}{L} \quad \text{and} \quad w_n = \sqrt{\frac{k}{m}}$$

Most unfavorable speed is when:

$$\frac{2pv_{WORST}}{L} = w = w_R = w_n\sqrt{1-2z^2}$$

$$\text{hence} \quad v_{WORST} = \frac{w_n L\sqrt{1-2z^2}}{2p} = f_n L\sqrt{1-2z^2}$$

5. A 70 kg motor is rigidly mounted on an isolator block of mass 1200 kg. The natural frequency and viscous damping ratio of the combination on the isolator block's support system are f_n = 160 cpm, and ζ = 0.10. An unbalance in the motor causes a vertical harmonic force of f = 100 sin (30 t). Determine:

a) The amplitude of vibration of the block.

Note that even though the excitation is being caused by rotor unbalance, the way the question is worded makes it a **force excitation problem.**

$$\left|H(w)\right| = \left|\frac{X}{F_O}\right| = \frac{1}{m\left\{\left(w_n^2 - w^2\right)^2 + 4z^2 w_n^2 w^2\right\}^{1/2}}$$

$$w_n = \frac{160}{60}\times 2p = \frac{16}{3}p \text{ rad/s} ; \quad w = 30 \text{ rad/s} ; \quad F_O = 100 \text{ N}$$

Hence

$$X = \frac{100}{1270\left\{\left(\left(\frac{16p}{3}\right)^2 - 30^2\right)^2 + 4\times 0.10^2\times 30^2\times\left(\frac{16p}{3}\right)^2\right\}^{1/2}} = 0.126\times 10^{-3} \text{ m}$$

b) The maximum acceleration of the block.

$$x = A.\sin(30t) \quad \text{with} \quad A = 0.126 \text{ mm}$$

$$\ddot{x} = -30^2 A.\sin(30t)$$

$$\text{Hence} \quad \ddot{X} = 30^2\times 0.126\times 10^{-3} = 0.113 \text{ m/s}^2$$

c) The amplitude of the dynamic force transmitted to the ground through the isolator block's support system.

We now consider the Transmissibility of the isolation system:

$$|T_R|\left|\frac{F_T}{F}\right| = \frac{\sqrt{w_n^4 + 4z^2 w_n^2 w^2}}{\sqrt{\left(w_n^2 - w^2\right)^2 + 4z^2 w_n^2 w^2}} = 0.40$$

Hence $F_T = 100 \times 0.40 = 40$ N

6. A sensitive 100 kg instrument is to be installed at a location where the observed acceleration is 15 cm/s2 at 20 Hz. It is proposed to mount the instrument on a rubber pad, with k = 280 kN/m and ζ = 0.10. What acceleration will be transmitted to the instrument?

$$w_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{280 \times 10^3}{100}} = 52.92 \text{ rad/s}$$

$$w = 2p \times 20 = 125.66 \text{ rad/s}$$

$$|T_R| = \left|\frac{X}{Y}\right| = \left|\frac{\ddot{X}}{\ddot{Y}}\right| = \frac{\sqrt{w_n^4 + 4z^2 w_n^2 w^2}}{\sqrt{\left(w_n^2 - w^2\right)^2 + 4z^2 w_n^2 w^2}} = 0.237$$

Hence $\ddot{X} = 15 \times 0.237 = 3.56 \text{ cm/s}^2$

7. Show that for a single-degree-of-freedom system the peak response amplitude occurs when the force excitation frequency is given by:

$$w = w_n \sqrt{1 - 2z^2}$$

This solution is presented in class.