THE STEADY-STATE RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A HARMONIC BASE EXCITATION

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EQUATION OF MOTION

Consider a single degree-of-freedom system.

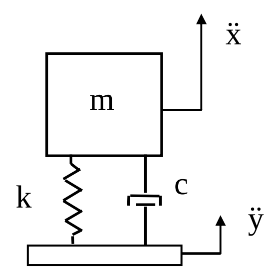

Figure 1.

The variables are

m	=	mass
c	=	viscous damping coefficient
k	=	stiffness
x	=	absolute displacement of the mass
y	=	base displacement

The double-dot denotes acceleration.

The free-body diagram is

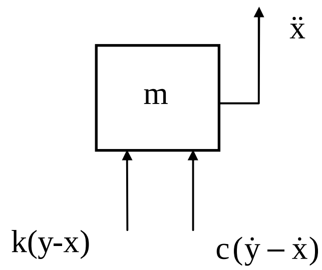

Figure 2.

Summation of forces in the vertical direction,

$$\sum F = m\ddot{x} \tag{1a}$$

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \tag{1b}$$

Define a relative displacement

$$z = x - y \tag{2}$$

Substituting the relative displacement terms into equation (1b) yields

$$m(\ddot{z} + \ddot{y}) = -c\dot{z} - kz \tag{3}$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4}$$

Dividing through by mass yields,

$$\ddot{z} + (c / m)\dot{z} + (k / m)z = -\ddot{y} \tag{5}$$

By convention,

$$(c / m) = 2\xi\omega_n \tag{6}$$

$$(k / m) = \omega_n{}^2 \tag{7}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substituting the convention terms into equation (5) yields

$$\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n{}^2 z = -\ddot{y} \tag{8}$$

$$\int_{-\infty}^{\infty} \left\{ \ddot{z} + 2\xi\omega_n \dot{z} + \omega_n{}^2 z \right\} e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left\{ -\ddot{y} \right\} e^{-j\omega t} dt \tag{9}$$

Note that the approach used here is rigorous. Simpler approaches are often used in other references.

Let

$$Z(\omega) = \int_{-\infty}^{\infty} \{z(t)\} e^{-j\omega t} dt \tag{10}$$

$$Y(\omega) = \int_{-\infty}^{\infty} \{y(t)\} e^{-j\omega t} dt \tag{11}$$

Now take the Fourier transform of the velocity term

$$\int_{-\infty}^{\infty} \{\dot{z}(t)\} e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left\{ \frac{dz(t)}{dt} \right\} e^{-j\omega t} dt \tag{12}$$

Integrate by parts

$$\int_{-\infty}^{\infty} \{\dot{z}(t)\} e^{-j\omega t} dt = \int_{-\infty}^{\infty} d\left\{ z(t) e^{-j\omega t} \right\} - \int_{-\infty}^{\infty} [z(t)](-j\omega) e^{-j\omega t} dt \tag{13}$$

$$\int_{-\infty}^{\infty} \{\dot{z}(t)\} e^{-j\omega t} dt = z(t) e^{-j\omega t} \Big|_{-\infty}^{\infty} + (j\omega) \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt \tag{14}$$

$$z(t) e^{-j\omega t} \Big|_{-\infty}^{\infty} = 0 \text{ as t approaches the } \pm\infty \text{ limits.} \tag{15}$$

$$\int_{-\infty}^{\infty} \{\dot{z}(t)\} e^{-j\omega t} dt = (j\omega) \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt \tag{16}$$

$$\int_{-\infty}^{\infty} \{\dot{z}(t)\} e^{-j\omega t} \, dt = (j\omega) X(\omega) \tag{17}$$

Furthermore

$$\int_{-\infty}^{\infty} \{\ddot{z}(t)\} e^{-j\omega t} \, dt = \int_{-\infty}^{\infty} \left\{ \frac{d^2 z(t)}{dt^2} \right\} e^{-j\omega t} \, dt \tag{18}$$

$$\int_{-\infty}^{\infty} \{\ddot{z}(t)\} e^{-j\omega t} \, dt = \int_{-\infty}^{\infty} d\left\{ \frac{dz(t)}{dt} e^{-j\omega t} \right\} - \int_{-\infty}^{\infty} [\frac{dz(t)}{dt}(-j\omega) e^{-j\omega t} dt \tag{19}$$

$$\int_{-\infty}^{\infty} \{\ddot{z}(t)\} e^{-j\omega t} \, dt = \frac{dz(t)}{dt} e^{-j\omega t} \Big|_{-\infty}^{\infty} + (j\omega) \int_{-\infty}^{\infty} \frac{dz(t)}{dt} e^{-j\omega t} dt \tag{20}$$

$$\frac{dz(t)}{dt} e^{-j\omega t} \Bigg|_{-\infty}^{\infty} = 0 \text{ as t approaches the } \pm\infty \text{ limits.} \tag{21}$$

$$\int_{-\infty}^{\infty} \{\ddot{z}(t)\} e^{-j\omega t} \, dt = (j\omega) \int_{-\infty}^{\infty} \frac{dz(t)}{dt} e^{-j\omega t} dt \tag{22}$$

$$\int_{-\infty}^{\infty} \{\ddot{z}(t)\} e^{-j\omega t} dt = (j\omega)(j\omega) Z(\omega) \tag{23}$$

$$\int_{-\infty}^{\infty} \{\ddot{z}(t)\} e^{-j\omega t} \, dt = -\omega^2 \, Z(\omega) \tag{24}$$

Recall

$$\int_{-\infty}^{\infty} \left\{ \ddot{z} + 2\xi\omega_n \dot{z} + \omega_n{}^2 z \right\} e^{-j\omega t} \, dt = \int_{-\infty}^{\infty} \{-\ddot{y}\} e^{-j\omega t} \, dt \tag{25}$$

Let the subscript A denote acceleration. By substitution,

$$-\omega^2 Z(\omega) + j\omega(2\xi\omega_n)Z(\omega) + \omega_n{}^2 Z(\omega) = -Y_A(\omega) \tag{26}$$

$$\left[(\omega_n{}^2 - \omega^2) + j2\xi\,\omega\omega_n\right]Z(\omega) = -Y_A(\omega) \tag{27}$$

$$Z(\omega) = \frac{-Y_A(\omega)}{[(\omega_n{}^2 - \omega^2) + j2\xi\omega\omega_n]} \tag{28}$$

$$Z_A(\omega) = -\omega^2 Z(\omega) \tag{29}$$

$$Z_A(\omega) = \left[\frac{\omega^2 Y_A(\omega)}{(\omega_n{}^2 - \omega^2) + j2\xi\omega\omega_n}\right] \tag{30}$$

The relative acceleration equation can be expressed in terms of Fourier transforms as

$$Z_A(\omega) = X_A(\omega) - Y_A(\omega) \tag{31}$$

The absolute acceleration is

$$X_A(\omega) = Z_A(\omega) + Y_A(\omega) \tag{32}$$

$$X_A(\omega) = \left[\frac{\omega^2 Y_A(\omega)}{(\omega_n{}^2 - \omega^2) + j2\xi\omega\omega_n}\right] + Y_A(\omega) \tag{33}$$

$$X_A(\omega) = \frac{[\omega^2 + (\omega_n{}^2 - \omega^2) + j2\xi\omega\omega_n]Y_A(\omega)}{[(\omega_n{}^2 - \omega^2) + j2\xi\omega\omega_n]} \tag{34}$$

$$X_A(\omega) = \frac{[\omega_n{}^2 + j2\xi\omega\omega_n]\,Y_A(\omega)}{[(\omega_n{}^2 - \omega^2) + j2\xi\,\omega\,\omega_n]} \tag{35}$$

Define a transfer function $H(\omega)$.

$$X_A(\omega) = H(\omega)\,Y_A(\omega) \tag{36}$$

$$H(\omega) = \frac{\omega_n{}^2 + j\,2\xi\omega\omega_n}{(\omega_n{}^2 - \omega^2) + j\,2\xi\omega\omega_n} \tag{37}$$

Divide the numerator and denominator by $\omega_n{}^2$.

$$H(\omega) = \frac{1 + j\,2\xi(\omega/\omega_n)}{[1 - (\omega/\omega_n)^2\,] + j\,2\xi(\omega/\omega_n)} \tag{38}$$

Let $\rho = \omega/\omega_n$.

$$H(\rho) = \frac{1 + j\,2\xi\rho}{[1 - \rho^2\,] + j\,2\xi\rho} \tag{39}$$

Multiply the numerator and denominator of equation (39) by the complex conjugate of the denominator.

$$H(\rho) = \frac{[1 + j\,2\xi\rho]}{[(1 - \rho^2) + j\,2\xi\rho]}\;\frac{[(1 - \rho^2) - j\,2\xi\rho]}{[(1 - \rho^2) - j\,2\xi\rho]} \tag{40}$$

$$H(\rho) = \frac{[1 + (2\xi\rho)^2 - \rho^2\,]}{[(1 - \rho^2)^2 + (2\xi\rho)^2\,]} + j\,\frac{2\xi\rho(1 - \rho^2) - 2\xi\rho}{[(1 - \rho^2)^2 + (2\xi\rho)^2\,]} \tag{41}$$

$$H(\rho) = \frac{[1+\rho^2(4\xi^2-1)]}{[(1-\rho^2)^2+(2\xi\rho)^2]} - j\frac{2\xi\rho^3}{[(1-\rho^2)^2+(2\xi\rho)^2]} \tag{42}$$

The magnitude is

$$|H(\rho)| = \sqrt{\frac{1+[2\xi\rho]^2}{[1-\rho^2]^2+[2\xi\rho]^2}} \tag{43}$$

The phase angle θ by which the response lags the input is

$$\theta = \arctan\left[\frac{2\xi\rho^3}{1+\rho^2(4\xi^2-1)}\right] \tag{44}$$

The transfer function magnitude is plotted for three damping cases in Figure 3. The phase angle is shown in Figure 4.

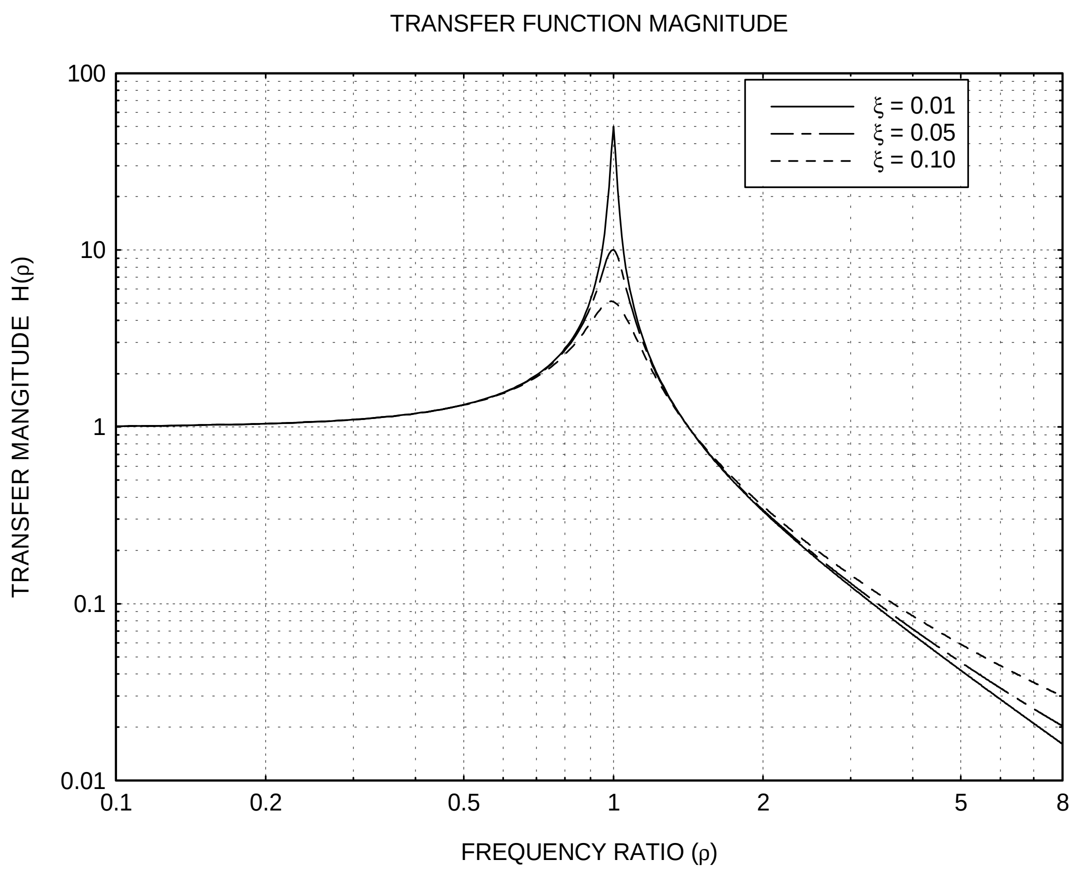
TRANSFER FUNCTION MAGNITUDE
ξ = 0.01
ξ = 0.05
ξ = 0.10
100
10
1
0.1
0.01
0.1
0.2
0.5
1
2
5
8
TRANSFER MANGITUDE H(ρ)
FREQUENCY RATIO (ρ)

Figure 3.

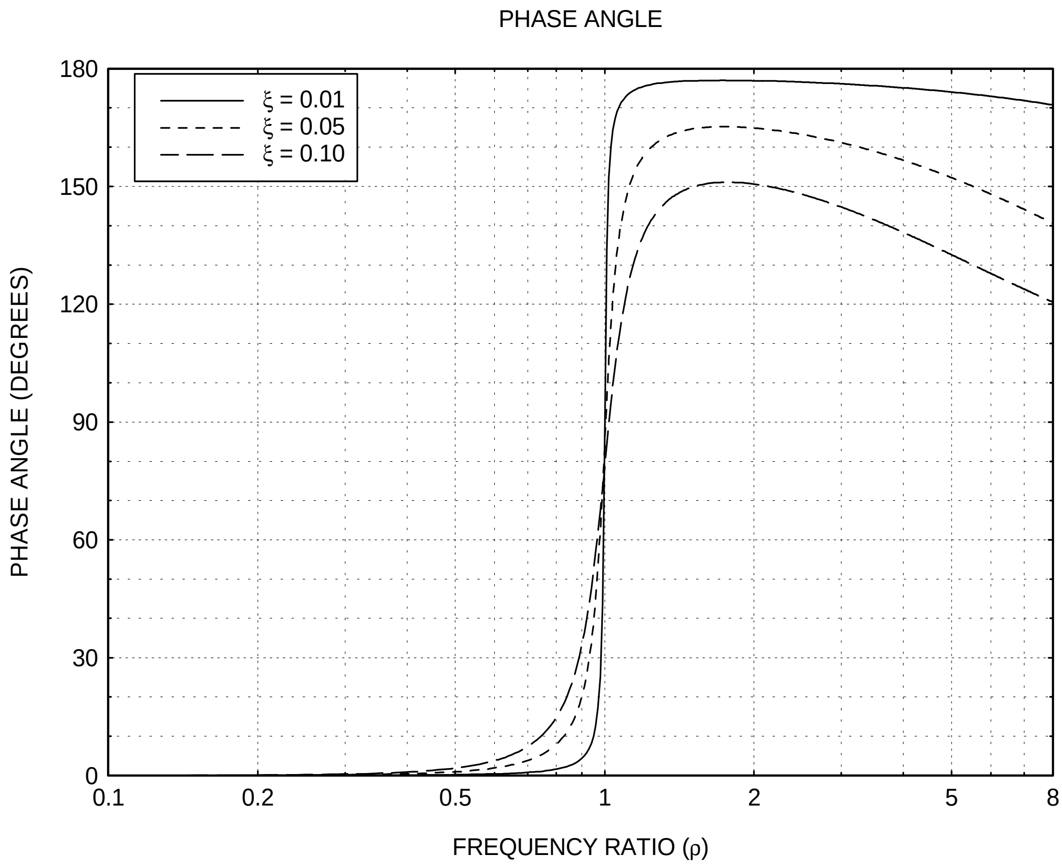
PHASE ANGLE
ξ = 0.01
ξ = 0.05
ξ = 0.10
180
150
120
90
60
30
0
PHASE ANGLE (DEGREES)
0.1
0.2
0.5
1
2
5
8
FREQUENCY RATIO (ρ)

Figure 4.