BENDING FREQUENCIES OF BEAMS, RODS, AND PIPES
Revision S

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November 20, 2012

Introduction

The fundamental frequencies for typical beam configurations are given in Table 1. Higher frequencies are given for selected configurations.

Table 1. Fundamental Bending Frequencies

Configuration	Frequency (Hz)
Cantilever	$f_1 = \frac{1}{2\pi}\left[\frac{3.5156}{L^2}\right]\sqrt{\frac{EI}{\rho}}$ $f_2 = 6.268\, f_1$ $f_3 = 17.456\, f_1$
Cantilever with End Mass m	$f_1 = \frac{1}{2\pi}\sqrt{\frac{3EI}{\left(0.2235\,\rho\,L + m\right)L^3}}$
Simply-Supported at both Ends (Pinned-Pinned)	$f_n = \frac{1}{2\pi}\left[\frac{n\,\pi}{L}\right]^2\sqrt{\frac{EI}{\rho}}$, n=1, 2, 3, ….
Free-Free	$f_1 = 0$ (rigid-body mode) $f_2 = \frac{1}{2\pi}\left[\frac{22.373}{L^2}\right]\sqrt{\frac{EI}{\rho}}$ $f_3 = 2.757\, f_1$ $f_4 = 5.404\, f_1$

Table 1. Fundamental Bending Frequencies (continued)	
Configuration	Frequency (Hz)
Fixed-Fixed	Same as free-free beam except there is no rigid-body mode for the fixed-fixed beam.
Fixed - Pinned	$f_1 = \frac{1}{2\pi}\left[\frac{15.418}{L^2}\right]\sqrt{\frac{EI}{\rho}}$

where

E is the modulus of elasticity

I is the area moment of inertia

L is the length

ρ is the mass density (mass/length)

P is the applied force

Note that the free-free and fixed-fixed have the same formula.

The derivations and examples are given in the appendices per Table 2.

Table 2. Table of Contents

Appendix	Title	Mass	Solution
A	Cantilever Beam I	End mass. Beam mass is negligible	Approximate
B	Cantilever Beam II	Beam mass only	Approximate
C	Cantilever Beam III	Both beam mass and the end mass are significant	Approximate
D	Cantilever Beam IV	Beam mass only	Eigenvalue
E	Beam Simply-Supported at Both Ends I	Center mass. Beam mass is negligible.	Approximate
F	Beam Simply-Supported at Both Ends II	Beam mass only	Eigenvalue

Table 2. Table of Contents (continued)			
Appendix	Title	Mass	Solution
G	Free-Free Beam	Beam mass only	Eigenvalue
H	Steel Pipe example, Simply Supported and Fixed-Fixed Cases	Beam mass only	Approximate
I	Rocket Vehicle Example, Free-free Beam	Beam mass only	Approximate
J	Fixed-Fixed Beam	Beam mass only	Eigenvalue
K	Fixed-Pinned Beam	Beam mass only	Eigenvalue

Reference

1. T. Irvine, Application of the Newton-Raphson Method to Vibration Problems, Revision E, Vibrationdata, 2010.

APPENDIX A

Cantilever Beam I

Consider a mass mounted on the end of a cantilever beam. Assume that the end-mass is much greater than the mass of the beam.

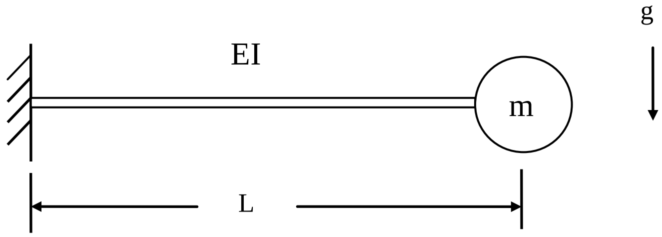


Figure A-1.

E is the modulus of elasticity.
I is the area moment of inertia.
L is the length.
g is gravity.
m is the mass.

The free-body diagram of the system is

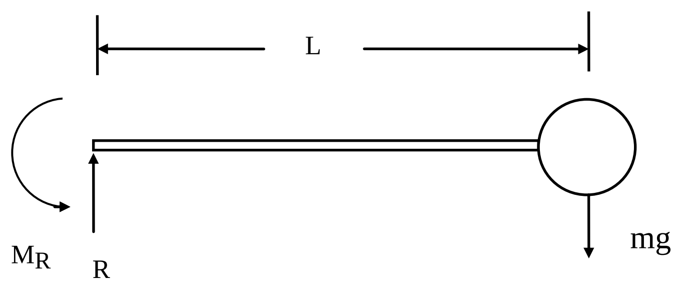


Figure A-2.

R is the reaction force.
M_R is the reaction bending moment.
Apply Newton's law for static equilibrium.

$$+\uparrow \sum \text{forces} = 0 \tag{A-1}$$

$$R - mg = 0 \tag{A-2}$$

$$R = mg \qquad \text{(A-3)}$$

At the left boundary,

$$\circlearrowright + \sum \text{moments} = 0 \qquad \text{(A-4)}$$

$$M_R - mg\,L = 0 \qquad \text{(A-5)}$$

$$M_R = mg\,L \qquad \text{(A-6)}$$

Now consider a segment of the beam, starting from the left boundary.

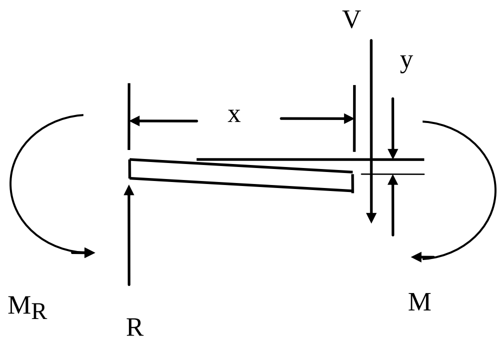


Figure A-3.

V is the shear force.
M is the bending moment.
y is the deflection at position x.

Sum the moments at the right side of the segment.

$$\circlearrowright + \sum \text{moments} = 0 \qquad \text{(A-7)}$$

$$M_R - R\,x - M = 0 \qquad \text{(A-8)}$$

$$M = M_R - R\,x \qquad \text{(A-9)}$$

The moment M and the deflection y are related by the equation

$$M = EI\,y'' \qquad \text{(A-10)}$$

$$EI\,y'' = M_R - R\,x \tag{A-11}$$

$$EI\,y'' = mgL - mg\,x \tag{A-12}$$

$$EI\,y'' = mg(L - x) \tag{A-13}$$

$$y'' = \left[\frac{mg}{EI}\right](L - x) \tag{A-14}$$

Integrating,

$$y' = \left[\frac{mg}{EI}\right]\left[Lx - \left(\frac{x^2}{2}\right)\right] + a \tag{A-15}$$

Note that “a” is an integration constant.

Integrating again,

$$y(x) = \left[\frac{mg}{EI}\right]\left[L\left(\frac{x^2}{2}\right) - \left(\frac{x^3}{6}\right)\right] + a\,x + b \tag{A-16}$$

A boundary condition at the left end is

$$y(0) = 0 \qquad \text{(zero displacement)} \tag{A-17}$$

Thus

$$b = 0 \tag{A-18}$$

Another boundary condition is

$$y'(0) = 0 \qquad \text{(zero slope)} \tag{A-19}$$

Applying the boundary condition to equation (A-16) yields,

$$a = 0 \tag{A-20}$$

The resulting deflection equation is

$$y(x) = \left[\frac{mg}{EI}\right]\left[L\left(\frac{x^2}{2}\right) - \left(\frac{x^3}{6}\right)\right] \tag{A-21}$$

The deflection at the right end is

$$y(L) = \left[\frac{mg}{EI}\right]\left[L\left(\frac{L^2}{2}\right) - \left(\frac{L^3}{6}\right)\right] \tag{A-22}$$

$$y(L) = \left[\frac{mgL^3}{3EI}\right] \tag{A-23}$$

Recall Hooke's law for a linear spring,

$$F = k\, y \tag{A-24}$$

F is the force.
k is the stiffness.

The stiffness is thus

$$k = F / y \tag{A-25}$$

The force at the end of the beam is mg. The stiffness at the end of the beam is

$$k = \left\{\frac{mg}{\left(\left[\frac{mgL^3}{3EI}\right]\right)}\right\} \tag{A-26}$$

$$k = \frac{3EI}{L^3} \tag{A-27}$$

The formula for the natural frequency fn of a single-degree-of-freedom system is

$$fn = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{(A-28)}$$

The mass term m is simply the mass at the end of the beam. The natural frequency of the cantilever beam with the end-mass is found by substituting equation (A-27) into (A-28).

$$fn = \frac{1}{2\pi}\sqrt{\frac{3EI}{mL^3}} \qquad \text{(A-29)}$$

APPENDIX B

Cantilever Beam II

Consider a cantilever beam with mass per length ρ. Assume that the beam has a uniform cross section. Determine the natural frequency. Also find the effective mass, where the distributed mass is represented by a discrete, end-mass.

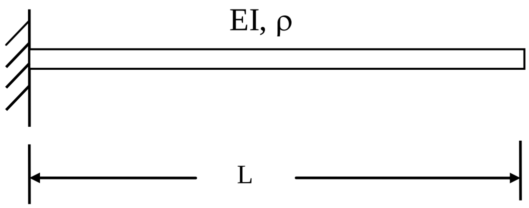


Figure B-1.

The governing differential equation is

$$- EI\frac{\partial^4 y}{\partial x^4} = \rho \frac{\partial^2 y}{\partial t^2} \qquad \text{(B-1)}$$

The boundary conditions at the fixed end x = 0 are

$$y(0) = 0 \qquad \text{(zero displacement)} \qquad \text{(B-2)}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 0 \qquad \text{(zero slope)} \qquad \text{(B-3)}$$

The boundary conditions at the free end x = L are

$$\left.\frac{d^2 y}{dx^2}\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \qquad \text{(B-4)}$$

$$\left.\frac{d^3 y}{dx^3}\right|_{x=L} = 0 \qquad \text{(zero shear force)} \qquad \text{(B-5)}$$

Propose a quarter cosine wave solution.

$$y(x) = y_o\left[1 - \cos\left(\frac{\pi x}{2L}\right)\right] \tag{B-6}$$

$$\frac{dy}{dx} = y_o\left(\frac{\pi}{2L}\right)\sin\left(\frac{\pi x}{2L}\right) \tag{B-7}$$

$$\frac{d^2y}{dx^2} = y_o\left(\frac{\pi}{2L}\right)^2\cos\left(\frac{\pi x}{2L}\right) \tag{B-8}$$

$$\frac{d^3y}{dx^3} = -y_o\left(\frac{\pi x}{2L}\right)^3\sin\left(\frac{\pi x}{2L}\right) \tag{B-9}$$

The proposed solution meets all of the boundary conditions expect for the zero shear force at the right end. The proposed solution is accepted as an approximate solution for the deflection shape, despite one deficiency.

The Rayleigh method is used to find the natural frequency. The total potential energy and the total kinetic energy must be determined.

The total potential energy P in the beam is

$$P = \frac{EI}{2}\int_0^L\left(\frac{d^2y}{dx^2}\right)^2 dx \tag{B-10}$$

By substitution,

$$P = \frac{EI}{2}\int_0^L\left[y_o\left(\frac{\pi}{2L}\right)^2\cos\left(\frac{\pi x}{2L}\right)\right]^2 dx \tag{B-11}$$

$$P = \frac{EI}{2}\left[y_o\left(\frac{\pi}{2L}\right)^2\right]^2\int_0^L\left[\cos\left(\frac{\pi x}{2L}\right)\right]^2 dx \tag{B-12}$$

$$P = \frac{EI}{2}\left[y_o\left(\frac{\pi}{2L}\right)^2\right]^2\int_0^L\left[\frac{1}{2}\right]\left[1+\cos\left(\frac{\pi x}{L}\right)\right]dx \tag{B-13}$$

$$P = \frac{EI}{2}\left[y_o \left(\frac{\pi}{2L} \right)^2 \right]^2 \left[\frac{1}{2} \right]\left[x + \left(\frac{L}{\pi} \right) \sin\left(\frac{\pi x}{L} \right) \right]\Bigg|_0^L \tag{B-14}$$

$$P = \frac{EI}{2}\left[y_o \right]^2 \left[\frac{\pi^4}{32\,L^4} \right] L \tag{B-15}$$

$$P = \frac{1}{64}\pi^4 \left[\frac{EI}{L^3} \right]\left[y_o \right]^2 \tag{B-16}$$

The total kinetic energy T is

$$T = \frac{1}{2}\rho\,\omega_n^2 \int_0^L \left[y \right]^2 dx \tag{B-17}$$

$$T = \frac{1}{2}\rho\,\omega_n^2 \int_0^L \left\{ y_o \left[1 - \cos\left(\frac{\pi x}{2L} \right) \right] \right\}^2 dx \tag{B-18}$$

$$T = \frac{1}{2}\rho\,\omega_n^2 \left[y_o \right]^2 \int_0^L \left[1 - 2\cos\left(\frac{\pi x}{2L} \right) + \cos^2\left(\frac{\pi x}{2L} \right) \right] dx \tag{B-19}$$

$$T = \frac{1}{2}\rho\,\omega_n^2 \left[y_o \right]^2 \int_0^L \left[1 - 2\cos\left(\frac{\pi x}{2L} \right) + \cos^2\left(\frac{\pi x}{2L} \right) \right] dx \tag{B-20}$$

$$T = \frac{1}{2}\rho\,\omega_n^2 \left[y_o \right]^2 \int_0^L \left[1 - 2\cos\left(\frac{\pi x}{2L} \right) + \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi x}{L} \right) \right] dx \tag{B-21}$$

$$T = \frac{1}{2}\rho\,\omega_n^2 \left[y_o \right]^2 \int_0^L \left[\frac{3}{2} - 2\cos\left(\frac{\pi x}{2L} \right) + \cos\left(\frac{\pi x}{L} \right) \right] dx \tag{B-22}$$

$$T = \frac{1}{2}\rho\,\omega_n^2 \left[y_o \right]^2 \left[\frac{3}{2}x - \left(\frac{4L}{\pi} \right) \sin\left(\frac{\pi x}{2L} \right) + \left(\frac{L}{\pi} \right) \sin\left(\frac{\pi x}{L} \right) \right]\Bigg|_0^L \tag{B-23}$$

$$T = \frac{1}{2}\rho\,\omega_n^2 \left[y_o \right]^2 \left[\frac{3}{2}L - \left(\frac{4L}{\pi} \right) \right] \tag{B-24}$$

$$T = \frac{1}{4}\rho\,\omega_n^2\left[y_o\right]^2 L\left[3-\left(\frac{8}{\pi}\right)\right] \tag{B-25}$$

Now equate the potential and the kinetic energy terms.

$$\frac{1}{4}\rho\,\omega_n^2\left[y_o\right]^2 L\left[3-\left(\frac{8}{\pi}\right)\right] = \frac{1}{64}\pi^4\left[\frac{EI}{L^3}\right]\left[y_o\right]^2 \tag{B-26}$$

$$\rho\,\omega_n^2\, L\left[3-\left(\frac{8}{\pi}\right)\right] = \frac{1}{16}\pi^4\left[\frac{EI}{L^3}\right] \tag{B-27}$$

$$\omega_n^2 = \left\{\frac{\pi^4\left[\frac{EI}{L^3}\right]}{16\rho L\left[3-\left(\frac{8}{\pi}\right)\right]}\right\} \tag{B-28}$$

$$\omega_n = \left\{\frac{\pi^4\left[\frac{EI}{L^3}\right]}{16\rho L\left[3-\left(\frac{8}{\pi}\right)\right]}\right\}^{1/2} \tag{B-29}$$

$$f_n = \left\{\frac{1}{2\pi}\right\}\left\{\frac{\pi^4\left[\frac{EI}{L^4}\right]}{16\rho\left[3-\left(\frac{8}{\pi}\right)\right]}\right\}^{1/2} \tag{B-30}$$

$$f_n = \left\{\frac{1}{2\pi}\right\}\left\{\frac{\pi^4\left[\frac{EI}{L^4}\right]}{16\rho\left[3-\left(\frac{8}{\pi}\right)\right]}\right\}^{1/2} \tag{B-31}$$

$$f_n = \left\{\frac{1}{2\pi}\right\}\left\{\frac{\pi^2}{4L^2}\right\}\left\{\frac{EI}{\rho\left[3-\left(\frac{8}{\pi}\right)\right]}\right\}^{1/2} \tag{B-32}$$

$$f_n \approx \left\{\frac{1}{2\pi}\right\}\left\{\frac{3.664}{L^2}\right\}\sqrt{\frac{EI}{\rho}} \tag{B-33}$$

Recall that the stiffness at the free of the cantilever beam is

$$k = \frac{3EI}{L^3} \tag{B-34}$$

The effective mass m_{eff} at the end of the beam is thus

$$m_{eff} = \frac{k}{\left[2\pi\, fn\right]^2} \tag{B-35}$$

$$m_{eff} = \frac{3EI}{L^3\left\{2\pi\left[\frac{1}{2\pi}\right]\left[\frac{3.664}{L^2}\right]\sqrt{\frac{EI}{\rho}}\right\}^2} \tag{B-36}$$

$$m_{eff} = \frac{3EI}{\frac{L^3}{L^4}\{13.425\}\left\{\frac{EI}{\rho}\right\}} \tag{B-37}$$

$$m_{eff} = 0.2235\rho L \tag{B-38}$$

APPENDIX C

Cantilever Beam III

Consider a cantilever beam where both the beam mass and the end-mass are significant.

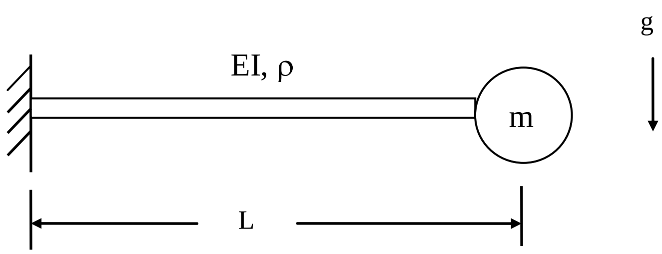


Figure C-1.

The total mass m_t can be calculated using equation (B-38).

$$m_t = 0.2235\rho L + m \qquad \text{(C-1)}$$

Again, the stiffness at the free of the cantilever beam is

$$k = \frac{3EI}{L^3} \qquad \text{(C-2)}$$

The natural frequency is thus

$$fn \approx \frac{1}{2\pi}\sqrt{\frac{3EI}{\left(0.2235\rho L + m\right)L^3}} \qquad \text{(C-3)}$$

APPENDIX D

Cantilever Beam IV

This is a repeat of part II except that an exact solution is found for the differential equation. The differential equation itself is only an approximation of reality, however.

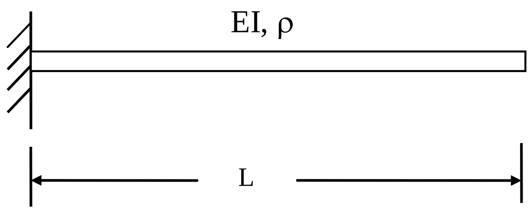


Figure D-1.

The governing differential equation is

$$-EI\frac{\partial^4 y}{\partial x^4} = \rho\frac{\partial^2 y}{\partial t^2} \tag{D-1}$$

Note that this equation neglects shear deformation and rotary inertia.

Separate the dependent variable.

$$y(x,t) = Y(x)T(t) \tag{D-2}$$

$$-EI\frac{\partial^4[Y(x)T(t)]}{\partial x^4} = \rho\frac{\partial^2[Y(x)T(t)]}{\partial t^2} \tag{D-3}$$

$$-EI\ T(t)\left\{\frac{d^4}{dx^4}Y(x)\right\} = \rho\ Y(x)\left\{\frac{d^2}{dt^2}T(t)\right\} \tag{D-4}$$

$$\left\{\frac{-EI}{\rho}\right\}\frac{\left\{\frac{d^4}{dx^4}Y(x)\right\}}{Y(x)} = \frac{\left\{\frac{d^2}{dt^2}T(t)\right\}}{T(t)} \tag{D-5}$$

Let c be a constant

$$\left\{\frac{-EI}{\rho}\right\}\frac{\left\{\frac{d^4}{dx^4}Y(x)\right\}}{Y(x)} = \frac{\left\{\frac{d^2}{dt^2}T(t)\right\}}{T(t)} = -c^2 \tag{D-6}$$

Separate the time variable.

$$\frac{\left\{\frac{d^2}{dt^2}T(t)\right\}}{T(t)} = -c^2 \tag{D-7}$$

$$\frac{d^2}{dt^2}T(t) + c^2\,T(t) = 0 \tag{D-8}$$

Separate the spatial variable.

$$\left\{\frac{-EI}{\rho}\right\}\frac{\left\{\frac{d^4}{dx^4}Y(x)\right\}}{Y(x)} = -c^2 \tag{D-9}$$

$$\frac{d^4}{dx^4}Y(x) - c^2\left\{\frac{\rho}{EI}\right\}Y(x) = 0 \tag{D-10}$$

A solution for equation (D-10) is

$$Y(x) = a_1\sinh(\beta x) + a_2\cosh(\beta x) + a_3\sin(\beta x) + a_4\cos(\beta x) \tag{D-11}$$

$$\frac{dY(x)}{dx} = a_1\beta \cosh(\beta x) + a_2\beta \sinh(\beta x) + a_3\beta \cos(\beta x) - a_4\beta \sin(\beta x) \quad \text{(D-12)}$$

$$\frac{d^2 Y(x)}{dx^2} = a_1\beta^2 \sinh(\beta x) + a_2\beta^2 \cosh(\beta x) - a_3\beta^2 \sin(\beta x) - a_4\beta^2 \cos(\beta x) \quad \text{(D-13)}$$

$$\frac{d^3 Y(x)}{dx^3} = a_1\beta^3 \cosh(\beta x) + a_2\beta^3 \sinh(\beta x) - a_3\beta^3 \cos(\beta x) + a_4\beta^3 \sin(\beta x) \quad \text{(D-14)}$$

$$\frac{d^4 Y(x)}{dx^4} = a_1\beta^4 \sinh(\lambda x) + a_2\beta^4 \cosh(\beta x) + a_3\beta^4 \sin(\beta x) + a_4\beta^4 \cos(\beta x) \quad \text{(D-15)}$$

Substitute (D-15) and (D-11) into (D-10).

$$\left\{a_1\beta^4 \sinh(\beta x) + a_2\beta^4 \cosh(\beta x) + a_3\beta^4 \sin(\beta x) + a_4\beta^4 \cos(\beta x)\right\}$$
$$- c^2\left\{\frac{\rho}{EI}\right\}\left\{a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x)\right\} = 0$$

(D-16)

$$\beta^4\left\{a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x)\right\}$$
$$- c^2\left\{\frac{\rho}{EI}\right\}\left\{a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x)\right\} = 0$$

(D-17)

The equation is satisfied if

$$\beta^4 = c^2\left\{\frac{\rho}{EI}\right\} \quad \text{(D-18)}$$

$$\beta = \left\{c^2 \frac{\rho}{EI}\right\}^{1/4} \quad \text{(D-19)}$$

The boundary conditions at the fixed end x = 0 are

$$Y(0) = 0 \qquad \text{(zero displacement)} \tag{D-20}$$

$$\left.\frac{dY}{dx}\right|_{x=0} = 0 \qquad \text{(zero slope)} \tag{D-21}$$

The boundary conditions at the free end x = L are

$$\left.\frac{d^2Y}{dx^2}\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{D-22}$$

$$\left.\frac{d^3Y}{dx^3}\right|_{x=L} = 0 \qquad \text{(zero shear force)} \tag{D-23}$$

Apply equation (D-20) to (D-11).

$$a_2 + a_4 = 0 \tag{D-24}$$

$$a_4 = -a_2 \tag{D-25}$$

Apply equation (D-21) to (D-12).

$$a_1 + a_3 = 0 \tag{D-26}$$

$$a_3 = -a_1 \tag{D-27}$$

Apply equation (D-22) to (D-13).

$$a_1 \sinh(\beta L) + a_2 \cosh(\beta L) - a_3 \sin(\beta L) - a_4 \cos(\beta L) = 0 \tag{D-28}$$

Apply equation (D-23) to (D-14).

$$a_1 \cosh(\beta L) + a_2 \sinh(\beta L) - a_3 \cos(\beta L) + a_4 \sin(\beta L) = 0 \tag{D-29}$$

Apply (D-25) and (D-27) to (D-28).

$$a_1 \sinh(\beta L) + a_2 \cosh(\beta L) + a_1 \sin(\beta L) + a_2 \cos(\beta L) = 0 \tag{D-30}$$

$$a_1\left\{\sin(\beta L)+\sinh(\beta L)\right\}+a_2\left\{\cos(\beta L)+\cosh(\beta L)\right\}=0 \tag{D-31}$$

Apply (D-25) and (D-27) to (D-29).

$$a_1\cosh(\beta L)+a_2\sinh(\beta L)+a_1\cos(\beta L)-a_2\sin(\beta L)=0 \tag{D-32}$$

$$a_1\left\{\cos(\beta L)+\cosh(\beta L)\right\}+a_2\left\{-\sin(\beta L)+\sinh(\beta L)\right\}=0 \tag{D-33}$$

Form (D-31) and (D-33) into a matrix format.

$$\begin{bmatrix} \sin(\beta L)+\sinh(\beta L) & \cos(\beta L)+\cosh(\beta L) \\ \cos(\beta L)+\cosh(\beta L) & -\sin(\beta L)+\sinh(\beta L) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{D-34}$$

By inspection, equation (D-34) can only be satisfied if $a_1 = 0$ and $a_2 = 0$. Set the determinant to zero in order to obtain a nontrivial solution.

$$\left\{-\sin^2(\beta L)+\sinh^2(\beta L)\right\} - \left\{\cos(\beta L)+\cosh(\beta L)\right\}^2 = 0 \tag{D-35}$$

$$\left\{-\sin^2(\beta L)+\sinh^2(\beta L)\right\}-\left\{\cos^2(\beta L)+2\cos(\beta L)\cosh(\beta L)+\cosh^2(\beta L)\right\}=0 \tag{D-36}$$

$$-\sin^2(\beta L)+\sinh^2(\beta L)-\cos^2(\beta L)-2\cos(\beta L)\cosh(\beta L)-\cosh^2(\beta L)=0 \tag{D-37}$$

$$-2-2\cos(\beta L)\cosh(\beta L)=0 \tag{D-38}$$

$$1+\cos(\beta L)\cosh(\beta L)=0 \tag{D-39}$$

$$\cos(\beta L)\cosh(\beta L)=-1 \tag{D-40}$$

There are multiple roots which satisfy equation (D-40). Thus, a subscript should be added as shown in equation (D-41).

$$\cos(\beta_n L)\cosh(\beta_n L)=-1 \tag{D-41}$$

The subscript is an integer index. The roots can be determined through a combination of graphing and numerical methods. The Newton-Rhapson method is an example of an appropriate numerical method. The roots of equation (D-41) are summarized in Table D-1, as taken from Reference 1.

Table D-1. Roots

Index	$\beta_n L$
n = 1	1.87510
n = 2	4.69409
$n \geq 3$	$(2n-1)\pi/2$

Note: the root value formula for $n \geq 3$ is approximate.

Rearrange equation (D-19) as follows

$$c^2 = \beta_n{}^4 \left[\frac{EI}{\rho} \right] \tag{D-42}$$

Substitute (D-42) into (D-8).

$$\frac{d^2}{dt^2} T(t) + \left[\beta_n{}^4 \left(\frac{EI}{\rho} \right) \right] T(t) = 0 \tag{D-43}$$

Equation (D-43) is satisfied by

$$T(t) = b_1 \sin\left[\left(\beta_n{}^2 \sqrt{\frac{EI}{\rho}} \right) t \right] + b_2 \cos\left[\left(\beta_n{}^2 \sqrt{\frac{EI}{\rho}} \right) t \right] \tag{D-44}$$

The natural frequency term ω_n is thus

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho}} \tag{D-45}$$

Substitute the value for the fundamental frequency from Table D-1.

$$\omega_1 = \left[\frac{1.87510}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{D-46}$$

$$f_1 = \frac{1}{2\pi}\left[\frac{3.5156}{L^2}\right]\sqrt{\frac{EI}{\rho}} \tag{D-47}$$

Substitute the value for the second root from Table D-1.

$$\omega_2 = \left[\frac{4.69409}{L^2}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{D-48}$$

$$f_2 = \frac{1}{2\pi}\left[\frac{22.034}{L^2}\right]\sqrt{\frac{EI}{\rho}} \tag{D-49}$$

$$f_2 = 6.268 f_1 \tag{D-50}$$

Compare equation (D-47) with the approximate equation (B-33).

<u>SDOF Model Approximation</u>

The effective mass m_{eff} at the end of the beam for the fundamental mode is thus

$$m_{eff} = \frac{k}{[2\pi\, fn]^2} \tag{D-51}$$

$$m_{eff} = \frac{3EI}{L^3 \left\{ 2\pi \left[\frac{1}{2\pi}\right] \left[\frac{3.5156}{L^2}\right] \sqrt{\frac{EI}{\rho}} \right\}^2} \tag{D-52}$$

$$m_{eff} = \frac{3EI}{\frac{L^3}{L^4} \{12.3596\} \left\{ \frac{EI}{\rho} \right\}} \tag{D-53}$$

$$m_{eff} = 0.2427 \rho L \qquad \text{(SDOF Approximation)} \tag{D-54}$$

Eigenvalues

n	$\beta_n L$
1	1.875104
2	4.69409
3	7.85476
4	10.99554
5	$(2n-1)\pi/2$

Note that the root value formula for n $\geq$ 5 is approximate.

Normalized Eigenvectors

Mass normalize the eigenvectors as follows

$$\int_0^L \rho Y_n{}^2(x)\,dx = 1 \tag{D-55}$$

The calculation steps are omitted for brevity. The resulting normalized eigenvectors are

$$Y_1(x) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\left\{\left[\cosh(\beta_1 x) - \cos(\beta_1 x)\right] - 0.73410\left[\sinh(\beta_1 x) - \sin(\beta_1 x)\right]\right\} \tag{D-56}$$

$$Y_2(x) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\left\{\left[\cosh(\beta_2 x) - \cos(\beta_2 x)\right] - 1.01847\left[\sinh(\beta_2 x) - \sin(\beta_2 x)\right]\right\} \tag{D-57}$$

$$Y_3(x) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\left\{\left[\cosh(\beta_3 x) - \cos(\beta_3 x)\right] - 0.99922\left[\sinh(\beta_3 x) - \sin(\beta_3 x)\right]\right\} \tag{D-58}$$

$$Y_4(x) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\left\{\left[\cosh(\beta_4 x) - \cos(\beta_4 x)\right] - 1.00003\left[\sinh(\beta_4 x) - \sin(\beta_4 x)\right]\right\} \tag{D-59}$$

The normalized mode shapes can be represented as

$$Y_i(x) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\left\{\left[\cosh(\beta_i x) - \cos(\beta_i x)\right] - D_i\left[\sinh(\beta_i x) - \sin(\beta_i x)\right]\right\} \tag{D-60}$$

where

$$D_i = \frac{\cos(\beta_i L) + \cosh(\beta_i L)}{\sin(\beta_i L) + \sinh(\beta_i L)} \tag{D-61}$$

Participation Factors

The participation factors for constant mass density are

$$\Gamma_n = \rho \int_0^L Y_n(x)\, dx \tag{D-62}$$

The participation factors from a numerical calculation are

$$\Gamma_1 = 0.7830\sqrt{\rho L} \tag{D-63}$$

$$\Gamma_2 = 0.4339\sqrt{\rho L} \tag{D-64}$$

$$\Gamma_3 = 0.2544\sqrt{\rho L} \tag{D-65}$$

$$\Gamma_4 = 0.1818\sqrt{\rho L} \tag{D-66}$$

The participation factors are non-dimensional.

Effective Modal Mass

The effective modal mass is

$$m_{eff\,,\,n} = \frac{\left[\int_0^L \rho\, Y_n(x)dx\right]^2}{\int_0^L \rho\left[Y_n(x)\right]^2 dx} \tag{D-67}$$

The eigenvectors are already normalized such that

$$\int_0^L \rho\left[Y_n(x)\right]^2 dx = 1 \tag{D-68}$$

Thus,

$$m_{eff\,,\,n} = \left[\Gamma_n\right]^2 = \left[\int_0^L \rho\, Y_n(x)dx\right]^2 \tag{D-69}$$

The effective modal mass values are obtained numerically.

$$m_{eff,1} = 0.6131\ \rho L \tag{D-70}$$

$$m_{eff,2} = 0.1883\ \rho L \tag{D-71}$$

$$m_{eff,3} = 0.06474\ \rho L \tag{D-72}$$

$$m_{eff,4} = 0.03306\ \rho L \tag{D-73}$$

APPENDIX E

Beam Simply-Supported at Both Ends I

Consider a simply-supported beam with a discrete mass located at the middle. Assume that the mass of the beam itself is negligible.

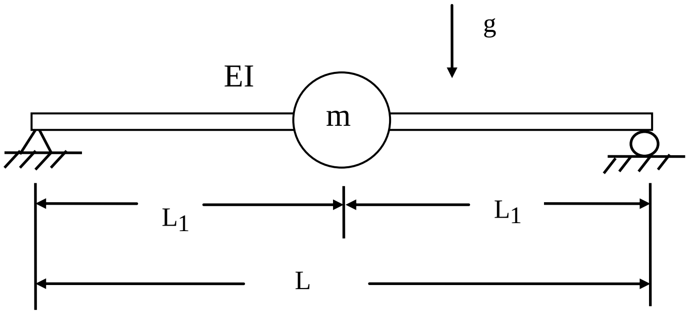


Figure E-1.

The free-body diagram of the system is

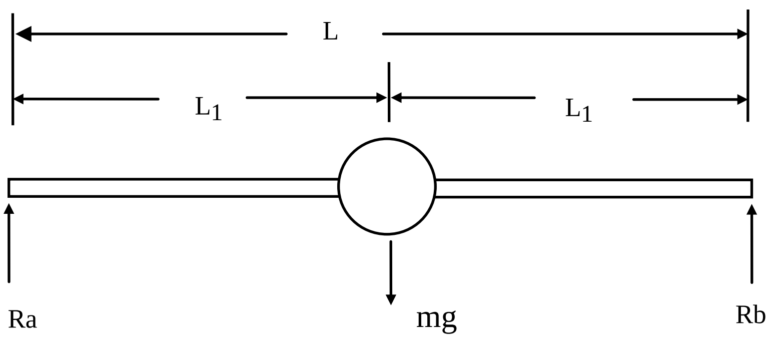


Figure E-2.

Apply Newton's law for static equilibrium.

$$+\uparrow \sum \text{forces} = 0 \tag{E-1}$$

$$Ra + Rb - mg = 0 \tag{E-2}$$

$$Ra = mg - Rb \tag{E-3}$$

At the left boundary,

$$\curvearrowright + \sum \text{moments} = 0 \qquad \text{(E-4)}$$

$$Rb\ L - mg\ L_1 = 0 \qquad \text{(E-5)}$$

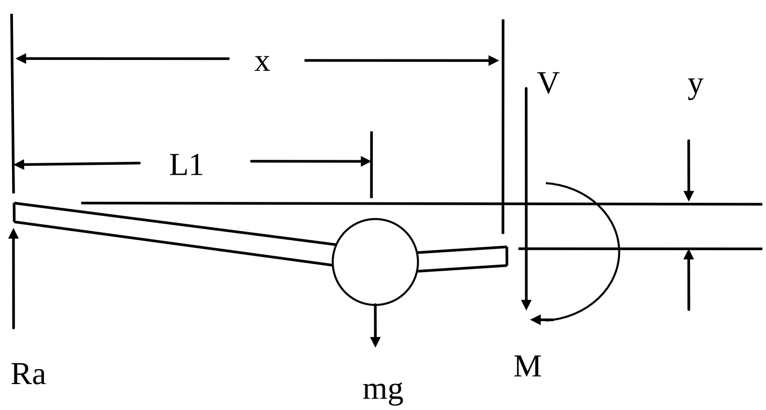
$$Rb = mg\ (\ L_1 / L\ ) \qquad \text{(E-6a)}$$

$$Rb = (1/2)\ mg \qquad \text{(E-6b)}$$

Substitute equation (E-6) into (E-3).

$$Ra = mg - (1/2)mg \qquad \text{(E-7)}$$

$$Ra = (1/2)mg \qquad \text{(E-8)}$$



Sum the moments at the right side of the segment.

$$\curvearrowright + \sum \text{moments} = 0 \qquad \text{(E-9)}$$

$$- R_a\ x + mg\ <x-L_1> - M = 0 \qquad \text{(E-10)}$$

Note that $< x\text{-}L_1 >$ denotes a step function as follows

$$< x - L_1 > = \begin{cases} 0, & \text{for } x < L_1 \\ x - L_1, & \text{for } x \geq L_1 \end{cases} \qquad \text{(E-11)}$$

$$M = -R_a x + mg < x\text{-}L_1 > \qquad \text{(E-12)}$$

$$M = -(1/2)mg\,x + mg < x\text{-}L_1 > \qquad \text{(E-13)}$$

$$M = [-(1/2)\,x + < x\text{-}L_1 >][\,mg\,] \qquad \text{(E-14)}$$

$$EIy'' = [-(1/2)\,x + < x - L_1 >][\,mg\,] \qquad \text{(E-15)}$$

$$y'' = [-(1/2)\,x + < x - L_1 >]\left[\frac{mg}{EI}\right] \qquad \text{(E-16)}$$

$$y' = [-\frac{1}{4}x^2 + \frac{1}{2} < x - L_1 >^2]\left[\frac{mg}{EI}\right] + a \qquad \text{(E-17)}$$

$$y(x) = \left[-\frac{1}{12}x^3 + \frac{1}{6} < x - L_1 >^3\right]\left[\frac{mg}{EI}\right] + ax + b \qquad \text{(E-18)}$$

The boundary condition at the left side is

$$y(0) = 0 \qquad \text{(E-19a)}$$

This requires

$$b = 0 \qquad \text{(E-19b)}$$

Thus

$$y(x) = \left[-\frac{1}{12}x^3 + \frac{1}{6} < x - L_1 >^3\right]\left[\frac{mg}{EI}\right] + ax \qquad \text{(E-20)}$$

The boundary condition on the right side is

$$y(L) = 0 \qquad \text{(E-21)}$$

$$\left[-\frac{1}{12}L^3 + \frac{1}{6} < L - L_1 >^3\right]\left[\frac{mg}{EI}\right] + aL = 0 \quad \text{(E-22)}$$

$$\left[-\frac{1}{12}L^3 + \frac{1}{48}L^3\right]\left[\frac{mg}{EI}\right] + aL = 0 \quad \text{(E-23)}$$

$$\left[-\frac{4}{48}L^3 + \frac{1}{48}L^3\right]\left[\frac{mg}{EI}\right] + aL = 0 \quad \text{(E-24)}$$

$$\left[-\frac{3}{48}L^3\right]\left[\frac{mg}{EI}\right] + aL = 0 \quad \text{(E-25)}$$

$$\left[-\frac{1}{16}L^3\right]\left[\frac{mg}{EI}\right] + aL = 0 \quad \text{(E-26)}$$

$$aL = \left[\frac{1}{16}L^3\right]\left[\frac{mg}{EI}\right] \quad \text{(E-27)}$$

$$a = \left[\frac{1}{16}L^2\right]\left[\frac{mg}{EI}\right] \quad \text{(E-28)}$$

Now substitute the constant into the displacement function

$$y(x) = \left[-\frac{1}{12}x^3 + \frac{1}{6} < x - L_1 >^3\right]\left[\frac{mg}{EI}\right] + \left[\frac{1}{16}L^2\right]\left[\frac{mg}{EI}\right][x] \quad \text{(E-29)}$$

$$y(x) = \left[-\frac{1}{12}x^3 + \frac{1}{16}xL^2 + \frac{1}{6} < x - L_1 >^3\right]\left[\frac{mg}{EI}\right] \quad \text{(E-30)}$$

The displacement at the center is

$$y\left(\frac{L}{2}\right) = \left[-\frac{1}{12}\left(\frac{L}{2}\right)^3 + \frac{1}{16}\left(\frac{L}{2}\right)L^2 + \frac{1}{6} < \left(\frac{L}{2}\right) - L_1 >^3\right]\left[\frac{mg}{EI}\right] \quad \text{(E-31)}$$

$$y\left(\frac{L}{2}\right)=\left[-\frac{1}{96}+\frac{1}{32}\right]\left[\frac{mgL^3}{EI}\right] \tag{E-32}$$

$$y\left(\frac{L}{2}\right)=\left[-\frac{1}{96}+\frac{3}{96}\right]\left[\frac{mgL^3}{EI}\right] \tag{E-33}$$

$$y\left(\frac{L}{2}\right)=\left[\frac{2}{96}\right]\left[\frac{mgL^3}{EI}\right] \tag{E-34}$$

$$y\left(\frac{L}{2}\right)=\left[\frac{1}{48}\right]\left[\frac{mgL^3}{EI}\right] \tag{E-35}$$

Recall Hooke's law for a linear spring,

$$F = k\ y \tag{E-36}$$

F is the force.
k is the stiffness.

The stiffness is thus

$$k = F / y \tag{E-37}$$

The force at the center of the beam is mg. The stiffness at the center of the beam is

$$k=\left\{\frac{mg}{\left[\frac{mgL^3}{48EI}\right]}\right\} \tag{E-38}$$

$$k=\frac{48\,EI}{L^3} \tag{E-39}$$

The formula for the natural frequency fn of a single-degree-of-freedom system is

$$fn = \left(\frac{1}{2\pi}\right)\sqrt{\frac{k}{m}} \qquad \text{(E-40)}$$

The mass term m is simply the mass at the center of the beam.

$$fn = \left(\frac{1}{2\pi}\right)\sqrt{\frac{48\,EI}{mL^3}} \qquad \text{(E-41)}$$

$$fn = \left(\frac{1}{2\pi}\right)(6.928)\sqrt{\frac{EI}{mL^3}} \qquad \text{(E-42)}$$

APPENDIX F

Beam Simply-Supported at Both Ends II

Consider a simply-supported beam as shown in Figure F-1.

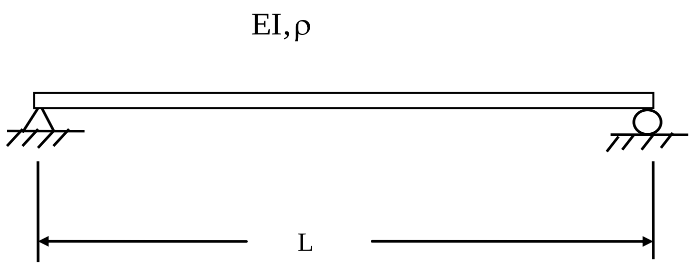


Figure F-1.

Recall that the governing differential equation is

$$-\,EI\frac{\partial^4 y}{\partial x^4} = \rho\frac{\partial^2 y}{\partial t^2} \tag{F-1}$$

The spatial solution from section D is

$$Y(x) = a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x) \tag{F-2}$$

$$\frac{d^2\,Y(x)}{dx^2} = a_1\beta^2 \sinh(\beta x) + a_2\beta^2 \cosh(\beta x) - a_3\beta^2 \sin(\beta x) - a_4\beta^2 \cos(\beta x) \tag{F-3}$$

The boundary conditions at the left end x = 0 are

$$Y(0) = 0 \qquad \text{(zero displacement)} \tag{F-4}$$

$$\left.\frac{d^2Y}{dx^2}\right|_{x=0} = 0 \qquad \text{(zero bending moment)} \tag{F-5}$$

The boundary conditions at the right end x = L are

$$Y(L) = 0 \qquad \text{(zero displacement)} \tag{F-6}$$

$$\left.\frac{d^2Y}{dx^2}\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{F-7}$$

Apply boundary condition (F-4) to (F-2).

$$a_2 + a_4 = 0 \tag{F-8}$$

$$a_4 = -a_2 \tag{F-9}$$

Apply boundary condition (F-5) to (F-3).

$$a_2 - a_4 = 0 \tag{F-10}$$

$$a_2 = a_4 \tag{F-11}$$

Equations (F-8) and (F-10) can only be satisfied if

$$a_2 = 0 \tag{F-12}$$

and

$$a_4 = 0 \tag{F-13}$$

The spatial equations thus simplify to

$$Y(x) = a_1 \sinh(\beta x) + a_3 \sin(\beta x) \tag{F-14}$$

$$\frac{d^2\,Y(x)}{dx^2} = a_1\beta^2 \sinh(\beta x) - a_3\beta^2 \sin(\beta x) \tag{F-15}$$

Apply boundary condition (F-6) to (F-14).

$$a_1 \sinh(\beta L) + a_3 \sin(\beta L) = 0 \tag{F-16}$$

Apply boundary condition (F-7) to (F-15).

$$a_1\beta^2 \sinh(\beta L) - a_3\beta^2 \sin(\beta L) = 0 \tag{F-17}$$

$$a_1 \sinh(\beta L) - a_3 \sin(\beta L) = 0 \tag{F-18}$$

$$\begin{bmatrix} \sinh(\beta L) & \sin(\beta L) \\ \sinh(\beta L) & -\sin(\beta L) \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{F-19}$$

By inspection, equation (F-19) can only be satisfied if $a_1 = 0$ and $a_3 = 0$. Set the determinant to zero in order to obtain a nontrivial solution.

$$-\sin(\beta L)\sinh(\beta L) - \sin(\beta L)\sinh(\beta L) = 0 \tag{F-20}$$

$$-2\sin(\beta L)\sinh(\beta L) = 0 \tag{F-21}$$

$$\sin(\beta L)\sinh(\beta L) = 0 \tag{F-22}$$

Equation (F-22) is satisfied if

$$\beta_n L = n\pi, \quad n = 1, 2, 3, ... \tag{F-23}$$

$$\beta_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, ... \tag{F-24}$$

The natural frequency term ω_n is

$$\omega_n = \beta_n{}^2 \sqrt{\frac{EI}{\rho}} \tag{F-25}$$

$$\omega_n = \left[\frac{n\pi}{L}\right]^2 \sqrt{\frac{EI}{\rho}}, \quad n = 1, 2, 3, ... \tag{F-26}$$

$$f_n = \left[\frac{1}{2\pi}\right]\left[\frac{n\pi}{L}\right]^2 \sqrt{\frac{EI}{\rho}}, \quad n = 1, 2, 3, ... \tag{F-27}$$

$$f_n = \left[\frac{1}{2\pi}\right]\left[\frac{n\pi}{L}\right]^2 \sqrt{\frac{EI}{\rho}}, \quad n = 1, 2, 3, \ldots \tag{F-28}$$

SDOF Approximation

Now calculate effective mass at the center of the beam for the fundamental frequency.

$$\omega_1 = \left[\frac{\pi}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{F-29}$$

Recall the natural frequency equation for a single-degree-of-freedom system.

$$\omega_1 = \sqrt{\frac{k}{m}} \tag{F-30}$$

Recall the beam stiffness at the center from equation (E-39).

$$k = \frac{48EI}{L^3} \tag{F-31}$$

Substitute equation (F-31) into (F-30).

$$\omega_1 = \sqrt{\frac{48EI}{mL^3}} \tag{F-32}$$

Substitute (F-32) into (F-29).

$$\sqrt{\frac{48EI}{mL^3}} = \left[\frac{\pi}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{F-33}$$

$$\frac{48EI}{mL^3} = \left[\frac{\pi}{L}\right]^4 \frac{EI}{\rho} \tag{F-34}$$

$$\frac{48}{mL^3} = \left[\frac{\pi}{L}\right]^4 \frac{1}{\rho} \tag{F-35}$$

$$\frac{1}{m} = \left[\frac{\pi^4}{48\rho L}\right] \tag{F-36}$$

The effective mass at the center of the beam for the first mode is

$$m = \frac{48\rho L}{\pi^4} \qquad \text{(SDOF Approximation)} \tag{F-37}$$

Normalized Eigenvectors

The eigenvector and its second derivative at this point are

$$Y(x) = a_1 \sinh(\beta x) + a_3 \sin(\beta x) \tag{F-38}$$

$$\frac{d^2 Y(x)}{dx^2} = a_1 \beta^2 \sinh(\beta x) - a_3 \beta^2 \sin(\beta x) \tag{F-39}$$

The eigenvector derivation requires some creativity. Recall

$$Y(L) = 0 \qquad \text{(zero displacement)} \tag{F-40}$$

$$\left. \frac{d^2 Y}{dx^2} \right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{F-41}$$

Thus,

$$\frac{d^2 Y}{dx^2} + Y = 0 \quad \text{for x=L and} \quad \beta_n L = n\pi, \quad n=1,2,3, \ldots \tag{F-42}$$

$$\left(1 - \left(\frac{n\pi}{L}\right)^2\right) a_1 \sinh(n\pi) + \left(1 - \left(\frac{n\pi}{L}\right)^2\right) a_3 \sin(n\pi) = 0 \quad , n=1,2,3, \ldots \tag{F-43}$$

The sin(nπ) term is always zero. Thus $a_1 = 0$.

The eigenvector for all n modes is

$$Y_n(x) = a_n \sin(n\pi x / L) \tag{F-44}$$

Mass normalize the eigenvectors as follows

$$\int_0^L \rho Y_n^2(x)\,dx = 1 \tag{F-45}$$

$$\rho a_n^2 \int_0^L \sin^2(n\pi x/L)\,dx = 1 \tag{F-46}$$

$$\frac{\rho a_n^2}{2} \int_0^L [1 - \cos(2n\pi x/L)] = 1 \tag{F-47}$$

$$\frac{\rho a_n^2}{2} \left[x - \frac{1}{2\beta_n} \sin(2n\pi x/L) \right] \Bigg|_0^L = 1 \tag{F-48}$$

$$\frac{\rho a_n^2 L}{2} = 1 \tag{F-49}$$

$$a_n^2 = \frac{2}{\rho L} \tag{F-50}$$

$$a_n = \sqrt{\frac{2}{\rho L}} \tag{F-51}$$

$$Y_n(x) = \sqrt{\frac{2}{\rho L}} \sin(n\pi x/L) \tag{F-52}$$

Participation Factors

The participation factors for constant mass density are

$$\Gamma_n = \rho \int_0^L Y_n(x)\, dx \tag{F-53}$$

$$\Gamma_n = \rho \int_0^L \sqrt{\frac{2}{\rho L}} \sin(n\pi x / L) dx \tag{F-53}$$

$$\Gamma_n = \sqrt{\frac{2\rho}{L}} \int_0^L \sin(n\pi x / L) dx \tag{F-54}$$

$$\Gamma_n = -\sqrt{\frac{2\rho}{L}} \left[\frac{L}{n\pi}\right] \cos(n\pi x / L) \Big|_0^L \tag{F-55}$$

$$\Gamma_n = -\sqrt{2\rho L} \left[\frac{1}{n\pi}\right] [\cos(n\pi) - 1] \quad , n=1, 2, 3, \ldots. \tag{F-56}$$

Effective Modal Mass

The effective modal mass is

$$m_{eff,\,n} = \frac{\left[\int_0^L \rho\, Y_n(x) dx\right]^2}{\int_0^L \rho [Y_n(x)]^2\, dx} \tag{F-57}$$

The eigenvectors are already normalized such that

$$\int_0^L \rho [Y_n(x)]^2\, dx = 1 \tag{F-58}$$

Thus,

$$m_{eff,n} = [\Gamma_n]^2 = \left[\int_0^L \rho\, Y_n(x) dx \right]^2 \qquad \text{(F-59)}$$

$$m_{eff,n} = \left[-\sqrt{2\rho L} \left[\frac{1}{n\pi} \right] [\cos(n\pi) - 1] \right]^2 \qquad \text{(F-60)}$$

$$m_{eff,n} = 2\rho L \frac{1}{(n\pi)^2} [\cos(n\pi) - 1]^2 \quad , n=1, 2, 3, \ldots. \qquad \text{(F-61)}$$

APPENDIX G

Free-Free Beam

Consider a uniform beam with free-free boundary conditions.

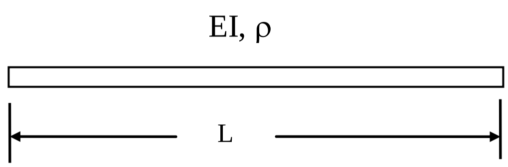


Figure G-1.

The governing differential equation is

$$-EI\frac{\partial^4 y}{\partial x^4} = \rho\frac{\partial^2 y}{\partial t^2} \tag{G-1}$$

Note that this equation neglects shear deformation and rotary inertia.

The following equation is obtain using the method in Appendix D

$$\frac{d^4}{dx^4}Y(x) - c^2\left\{\frac{\rho}{EI}\right\}Y(x) = 0 \tag{G-2}$$

The proposed solution is

$$Y(x) = a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x) \tag{G-3}$$

$$\frac{dY(x)}{dx} = a_1\beta \cosh(\beta x) + a_2\beta \sinh(\beta x) + a_3\beta \cos(\beta x) - a_4\beta \sin(\beta x) \tag{G-4}$$

$$\frac{d^2\,Y(x)}{dx^2} = a_1\beta^2 \sinh(\beta x) + a_2\beta^2 \cosh(\beta x) - a_3\beta^2 \sin(\beta x) - a_4\beta^2 \cos(\beta x) \tag{G-5}$$

$$\frac{d^3Y(x)}{dx^3} = a_1\beta^3 \cosh(\beta x) + a_2\beta^3 \sinh(\beta x) - a_3\beta^3 \cos(\beta x) + a_4\beta^3 \sin(\beta x) \tag{G-6}$$

Apply the boundary conditions.

$$\left.\frac{d^2Y}{dx^2}\right|_{x=0} = 0 \qquad \text{(zero bending moment)} \tag{G-7}$$

$$a_2 - a_4 = 0 \tag{G-8}$$

$$a_4 = a_2 \tag{G-9}$$

$$\left.\frac{d^3Y}{dx^3}\right|_{x=0} = 0 \qquad \text{(zero shear force)} \tag{G-10}$$

$$a_1 - a_3 = 0 \tag{G-11}$$

$$a_3 = a_1 \tag{G-12}$$

$$\frac{d^2Y(x)}{dx^2} = a_1\beta^2[\sinh(\beta x) - \sin(\beta x)] + a_2\beta^2[\cosh(\beta x) - \cos(\beta x)] \tag{G-13}$$

$$\frac{d^3Y(x)}{dx^3} = a_1\beta^3[\cosh(\beta x) - \cos(\beta x)] + a_2\beta^3[\sinh(\beta x) + \sin(\beta x)] \tag{G-14}$$

$$\left.\frac{d^2Y}{dx^2}\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{G-15}$$

$$a_1[\sinh(\beta L) - \sin(\beta L)] + a_2[\cosh(\beta L) - \cos(\beta L)] = 0 \tag{G-16}$$

$$\left.\frac{d^3 Y}{dx^3}\right|_{x=L} = 0 \quad \text{(zero shear force)} \tag{G-17}$$

$$a_1[\cosh(\beta L) - \cos(\beta L)] + a_2[\sinh(\beta L) + \sin(\beta L)] = 0 \tag{G-18}$$

Equation (G-16) and (G-18) can be arranged in matrix form.

$$\begin{bmatrix} \sinh(\beta L) - \sin(\beta L) & \cosh(\beta L) - \cos(\beta L) \\ \cosh(\beta L) - \cos(\beta L) & \sinh(\beta L) + \sin(\beta L) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{G-19}$$

Set the determinant equal to zero.

$$[\sinh(\beta L) - \sin(\beta L)][\sinh(\beta L) + \sin(\beta L)] - [\cosh(\beta L) - \cos(\beta L)]^2 = 0 \tag{G-20}$$

$$\sinh^2(\beta L) - \sin^2(\beta L) - \cosh^2(\beta L) + 2\cosh(\beta L)\cos(\beta L) - \cos^2(\beta L) = 0 \tag{G-21}$$

$$+2\cosh(\beta L)\cos(\beta L) - 2 = 0 \tag{G-22}$$

$$\cosh(\beta L)\cos(\beta L) - 1 = 0 \tag{G-23}$$

The roots can be found via the Newton-Raphson method, Reference 1.

The free-free beam has a rigid-body mode with a frequency of zero, corresponding to $\beta L = 0$.

The second root is

$$\beta L = 4.73004 \tag{G-24}$$

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho}} \tag{G-25}$$

$$\omega_2 = \left[\frac{4.73004}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{G-26}$$

$$\omega_2 = \left[\frac{22.373}{L^2}\right] \sqrt{\frac{EI}{\rho}} \tag{G-27}$$

The third root is

$$\beta L = 7.85320 \tag{G-28}$$

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho}} \tag{G-29}$$

$$\omega_3 = \left[\frac{7.85320}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{G-30}$$

$$\omega_3 = \left[\frac{61.673}{L^2}\right] \sqrt{\frac{EI}{\rho}} \tag{G-31}$$

$$\omega_3 = 2.757\,\omega_2 \tag{G-32}$$

The fourth root is

$$\beta L = 10.9956 \tag{G-33}$$

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho}} \tag{G-34}$$

$$\omega_4 = \left[\frac{10.9956}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{G-35}$$

$$\omega_4 = \left[\frac{120.903}{L^2}\right] \sqrt{\frac{EI}{\rho}} \tag{G-36}$$

$$\omega_4 = 5.404\ \omega_2 \tag{G-37}$$

Eigenvalues

n	$\beta_n L$
1	0
2	4.73004
3	7.85320
4	10.9956

$$\beta_n L \approx \pi\left[n - \frac{1}{2}\right] \quad \text{for } n \geq 5 \tag{G-38}$$

The following mode shape and coefficient derivation applies only to the elastic modes where $\beta L > 0$.

Equation (G-18) can be expressed as

$$a_2 = a_1\left[\frac{-\cosh(\beta L) + \cos(\beta L)}{\sinh(\beta L) + \sin(\beta L)}\right] \tag{G-39}$$

Recall

$$a_4 = a_2 \tag{G-40}$$

$$a_3 = a_1 \tag{G-41}$$

The displacement mode shape is thus

$$Y(x) = a_1[\sinh(\beta x) + \sin(\beta x)] + a_2[\cosh(\beta x) + \cos(\beta x)] \tag{G-42}$$

$$Y(x) = a_1\left\{[\sinh(\beta x) + \sin(\beta x)] + \left[\frac{-\cosh(\beta L) + \cos(\beta L)}{\sinh(\beta L) + \sin(\beta L)}\right][\cosh(\beta x) + \cos(\beta x)]\right\} \tag{G-43}$$

Modify the mode shapes as follows.

$$Y(x) = \frac{\hat{a}_1}{\sqrt{\rho L}}\left\{[\sinh(\beta x) + \sin(\beta x)] + \left[\frac{-\cosh(\beta L) + \cos(\beta L)}{\sinh(\beta L) + \sin(\beta L)}\right][\cosh(\beta x) + \cos(\beta x)]\right\}$$

(G-44)

Normalize the eigenvectors with respect to mass.

$$\int_0^L \rho\left[Y_n(x)\right]^2 dx = 1 \tag{G-45}$$

The eigenvectors are mass-normalized for $\hat{a}_1 = 1$.

Thus

$$Y(x) = \frac{1}{\sqrt{\rho L}}\left\{[\sinh(\beta x) + \sin(\beta x)] + \left[\frac{-\cosh(\beta L) + \cos(\beta L)}{\sinh(\beta L) + \sin(\beta L)}\right][\cosh(\beta x) + \cos(\beta x)]\right\}$$

(G-46)

The first derivative is

$$\frac{dy}{dx} = \frac{\beta}{\sqrt{\rho L}} \left\{ [\cosh(\beta x) + \cos(\beta x)] + \left[\frac{-\cosh(\beta L) + \cos(\beta L)}{\sinh(\beta L) + \sin(\beta L)} \right] [\sinh(\beta x) - \sin(\beta x)] \right\}$$

(G-47)

The second derivative is

$$\frac{d^2 y}{dx^2} = \frac{\beta^2}{\sqrt{\rho L}} \left\{ [\sinh(\beta x) - \sin(\beta x)] + \left[\frac{-\cosh(\beta L) + \cos(\beta L)}{\sinh(\beta L) + \sin(\beta L)} \right] [\cosh(\beta x) - \cos(\beta x)] \right\}$$

(G-48)

The participation factors for constant mass density are

$$\Gamma_n = \rho \int_0^L Y_n(x)\, dx \tag{G-49}$$

The participation factors are calculated numerically.

As a result of the rigid-body mode,

$$\Gamma_n = 0 \qquad \text{for } n > 1 \tag{G-50}$$

APPENDIX H

Pipe Example

Consider a steel pipe with an outer diameter of 2.2 inches and a wall thickness of 0.60 inches. The length is 20 feet. Find the natural frequency for two boundary condition cases: simply-supported and fixed-fixed.

The area moment of inertia is

$$I = \frac{\pi}{64}\left[D_o{}^4 - D_i{}^4\right] \quad \text{(H-1)}$$

$$D_o = 2.2 \text{ in} \quad \text{(H-2)}$$

$$D_i = 2.2 - 2(0.6) \quad \text{in} \quad \text{(H-3)}$$

$$D_i = 2.2 - 1.2 \quad \text{in} \quad \text{(H-4)}$$

$$D_i = 1.0 \quad \text{in} \quad \text{(H-5)}$$

$$I = \frac{\pi}{64}\left[2.2^4 - 1.0^4\right] \quad \text{in}^4 \quad \text{(H-6)}$$

$$I = 1.101 \quad \text{in}^4 \quad \text{(H-7)}$$

The elastic modulus is

$$E = 30\left(10^6\right) \quad \frac{\text{lbf}}{\text{in}^2} \quad \text{(H-8)}$$

The mass density is

$$\rho = \text{mass per unit length.} \quad \text{(H-9)}$$

$$\rho = \left[0.282 \quad \frac{\text{lbm}}{\text{in}^3}\right]\left[\frac{\pi}{4}\left[2.2^2 - 1.0^2\right]\text{in}^2\right] \quad \text{(H-10)}$$

$$\rho = 0.850 \quad \frac{\text{lbm}}{\text{in}} \tag{H-11}$$

$$\sqrt{\frac{EI}{\rho}} = \sqrt{\frac{30\left(10^6\right) \quad \frac{\text{lbf}}{\text{in}^2} \quad 1.101 \quad \text{in}^4 \quad \left(\frac{1\,\text{slug ft/sec}^2}{1\,\text{lbf}}\right)\left(\frac{12\text{ in}}{1\text{ ft}}\right)}{0.850 \quad \frac{\text{lbm}}{\text{in}}\left(\frac{1\,\text{slug}}{32.2\text{ lbm}}\right)}}$$

(H-12)

$$\sqrt{\frac{EI}{\rho}} = 1.225\left(10^5\right) \quad \frac{\text{in}^2}{\text{sec}} \tag{H-13}$$

The natural frequency for the simply-supported case is

$$f_n = \left[\frac{1}{2\pi}\right]\left[\frac{n\pi}{L}\right]^2 \sqrt{\frac{EI}{\rho}}, \quad n = 1, 2, 3, \ldots \tag{H-14}$$

$$f_1 = \left[\frac{1}{2\pi}\right]\left[\frac{\pi}{\left(20\,\text{ft}\right)\left(\frac{12\text{ in}}{1\text{ft}}\right)}\right]^2 1.225\left(10^5\right) \quad \frac{\text{in}^2}{\text{sec}} \tag{H-15}$$

$$f_1 = 3.34 \quad \text{Hz} \qquad \text{(simply-supported)} \tag{H-16}$$

The natural frequency for the fixed-fixed case is

$$f_1 = \left[\frac{1}{2\pi}\right]\left[\frac{22.37}{L^2}\right]\sqrt{\frac{EI}{\rho}} \tag{H-17}$$

$$f_1 = \left[\frac{1}{2\pi}\right]\left[\frac{22.37}{\left[(20\,\text{ft})\left(\frac{12\text{ in}}{1\text{ft}}\right)\right]^2}\right]1.225\left(10^5\right)\ \frac{\text{in}^2}{\text{sec}} \qquad \text{(H-18)}$$

$$f_1 = \ 7.58\ \text{Hz} \qquad \text{(fixed-fixed)} \qquad \text{(H-19)}$$

APPENDIX I

Suborbital Rocket Vehicle

Consider a rocket vehicle with the following properties.

mass = 14078.9 lbm (at time = 0 sec)

L = 372.0 inches.

$$\rho = \frac{14078.9 \text{ lbm}}{372.0 \text{ inches}}$$

$$\rho = 37.847 \ \frac{\text{lbm}}{\text{in}}$$

The average stiffness is

EI = 63034 (10^6) lbf in^2

The vehicle behaves as a free-free beam in flight. Thus

$$f_1 = \frac{1}{2\pi}\left[\frac{22.37}{L^2}\right]\sqrt{\frac{EI}{\rho}} \qquad \text{(I-1)}$$

$$f_1 = \frac{1}{2\pi}\left[\frac{22.37}{(372 \text{ in})^2}\right]\sqrt{\frac{\left[63034e+06 \text{ lbf in}^2\right]\left[\frac{\text{slug ft}/\text{sec}^2}{\text{lbf}}\right]\left[\frac{12 \text{ in}}{\text{ft}}\right]\left[\frac{32.2 \text{ lbm}}{\text{slugs}}\right]}{37.847 \ \frac{\text{lbm}}{\text{in}}}} \qquad \text{(I-2)}$$

$$f_1 = 20.64 \text{ Hz} \qquad \text{(at time = 0 sec)} \qquad \text{(I-3)}$$

Note that the fundamental frequency decreases in flight as the vehicle expels propellant mass.

APPENDIX J

Fixed-Fixed Beam

Consider a fixed-fixed beam with a uniform mass density and a uniform cross-section. The governing differential equation is

$$-EI\frac{\partial^4 y}{\partial x^4} = \rho\frac{\partial^2 y}{\partial t^2} \tag{J-1}$$

The spatial equation is

$$\frac{\partial^4}{\partial x^4}Y(x) - c^2\left\{\frac{\rho}{EI}\right\}Y(x) = 0 \tag{J-2}$$

The boundary conditions for the fixed-fixed beam are:

$$Y(0) = 0 \tag{J-3}$$

$$\left.\frac{dY(x)}{dx}\right|_{x=0} = 0 \tag{J-4}$$

$$Y(L)=0 \tag{J-5}$$

$$\left.\frac{dY(x)}{dx}\right|_{x=L} = 0 \tag{J-6}$$

The eigenvector has the form

$$Y(x) = a_1\sinh(\beta x) + a_2\cosh(\beta x) + a_3\sin(\beta x) + a_4\cos(\beta x) \tag{J-7}$$

$$\frac{dY(x)}{dx} = a_1\beta\cosh(\beta x) + a_2\beta\sinh(\beta x) + a_3\beta\cos(\beta x) - a_4\beta\sin(\beta x) \tag{J-8}$$

$$\frac{d^2\,Y(x)}{dx^2} = a_1\beta^2 \sinh(\beta x) + a_2\beta^2 \cosh(\beta x) - a_3\beta^2 \sin(\beta x) - a_4\beta^2 \cos(\beta x) \qquad \text{(J-9)}$$

$$Y(0) = 0 \qquad \text{(J-10)}$$

$$a_2 + a_4 = 0 \qquad \text{(J-11)}$$

$$- a_2 = a_4 \qquad \text{(J-12)}$$

$$\left.\frac{dY(x)}{dx}\right|_{x=0} = 0 \qquad \text{(J-13)}$$

$$a_1\,\beta + a_3\,\beta = 0 \qquad \text{(J-14)}$$

$$a_1 + a_3 = 0 \qquad \text{(J-15)}$$

$$- a_1 = a_3 \qquad \text{(J-16)}$$

$$Y(x) = a_1[\sinh(\beta x) - \sin(\beta x)] + a_2[\cosh(\beta x) - \cos(\beta x)] \qquad \text{(J-17)}$$

$$\frac{dY(x)}{dx} = a_1\beta[\cosh(\beta x) - \cos(\beta x)] + a_2\beta[\sinh(\beta x) + \sin(\beta x)] \qquad \text{(J-18)}$$

$$Y(L) = 0 \qquad \text{(J-19)}$$

$$a_1[\sinh(\beta L) - \sin(\beta L)] + a_2[\cosh(\beta L) - \cos(\beta L)] = 0 \qquad \text{(J-20)}$$

$$\left.\frac{dY(x)}{dx}\right|_{x=L} = 0 \qquad \text{(J-21)}$$

$$a_1\beta[\cosh(\beta L) - \cos(\beta L)] + a_2\beta[\sinh(\beta L) + \sin(\beta L)] = 0 \qquad \text{(J-22)}$$

$$a_1[\cosh(\beta L)-\cos(\beta L)]+a_2[\sinh(\beta L)+\sin(\beta L)]=0 \quad \text{(J-23)}$$

$$\begin{bmatrix} \sinh(\beta L)-\sin(\beta L) & \cosh(\beta L)-\cos(\beta L) \\ \cosh(\beta L)-\cos(\beta L) & \sinh(\beta L)+\sin(\beta L) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(J-24)}$$

$$\det\begin{bmatrix} \sinh(\beta L)-\sin(\beta L) & \cosh(\beta L)-\cos(\beta L) \\ \cosh(\beta L)-\cos(\beta L) & \sinh(\beta L)+\sin(\beta L) \end{bmatrix} = 0 \quad \text{(J-25)}$$

$$[\sinh(\beta L)-\sin(\beta L)][\sinh(\beta L)+\sin(\beta L)]-[\cosh(\beta L)-\cos(\beta L)]^2 = 0 \quad \text{(J-26)}$$

$$\sinh^2(\beta L)-\sin^2(\beta L)-\cosh^2(\beta L)+2\cos(\beta L)\cosh(\beta L)-\cos^2(\beta L)=0 \quad \text{(J-27)}$$

$$2\cos(\beta L)\cosh(\beta L)-2=0 \quad \text{(J-28)}$$

$$\cos(\beta L)\cosh(\beta L)-1=0 \quad \text{(J-29)}$$

The roots can be found via the Newton-Raphson method, Reference 1. The first root is

$$\beta L = 4.73004 \quad \text{(J-30)}$$

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho}} \quad \text{(J-31)}$$

$$\omega_1 = \left[\frac{4.73004}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \quad \text{(J-32)}$$

$$\omega_1 = \left[\frac{22.373}{L^2}\right] \sqrt{\frac{EI}{\rho}} \quad \text{(J-33)}$$

$$f_1 = \frac{1}{2\pi}\left[\frac{22.373}{L^2}\right]\sqrt{\frac{EI}{\rho}} \tag{J-34}$$

$$a_1[\cosh(\beta L) - \cos(\beta L)] = -a_2[\sinh(\beta L) + \sin(\beta L)] \tag{J-35}$$

$$\text{Let} \quad a_2 = 1 \tag{J-36}$$

$$a_1[\cosh(\beta L) - \cos(\beta L)] = -[\sinh(\beta L) + \sin(\beta L)] \tag{J-37}$$

$$a_1 = \frac{-\sinh(\beta L) - \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)} \tag{J-38}$$

$$Y(x) = [\cosh(\beta x) - \cos(\beta x)] + \left[\frac{-\sinh(\beta L) - \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)}\right][\sinh(\beta x) + \sin(\beta x)] \tag{J-39}$$

$$Y(x) = [\cosh(\beta x) - \cos(\beta x)] - \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)}\right][\sinh(\beta x) + \sin(\beta x)] \tag{J-40}$$

The un-normalized mode shape for a fixed-fixed beam is

$$\hat{Y}_n(x) = [\cosh(\beta_n x) - \cos(\beta_n x)] - \sigma_n\,[\sinh(\beta_n x) - \sin(\beta_n x)] \tag{J-41}$$

where

$$\sigma_n = \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)}\right] \tag{J-42}$$

The eigenvalues are

n	$\beta_n L$
1	4.73004
2	7.85321
3	10.9956
4	14.13717
5	17.27876

For n> 5

$$\beta_n L \approx \pi\left[\frac{1}{2}+n\right] \tag{J-43}$$

Normalized Eigenvectors

Mass normalize the eigenvectors as follows

$$\int_0^L \rho Y_n^{\,2}(x)\,dx = 1 \tag{J-44}$$

The mass normalization is satisfied by

$$Y_n(x) = \frac{1}{\sqrt{\rho L}}\left\{\left[\cosh(\beta_n x) - \cos(\beta_n x)\right] - \sigma_n\left[\sinh(\beta_n x) - \sin(\beta_n x)\right]\right\} \tag{J-45}$$

where

$$\sigma_n = \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)}\right] \tag{J-46}$$

The first derivative is

$$\frac{d}{dx} Y_n(x) = \frac{1}{\sqrt{\rho L}} \{ \beta_n [\sinh(\beta_n x) + \sin(\beta_n x)] - \sigma_n \beta_n \; [\cosh(\beta_n x) - \cos(\beta_n x)] \}$$

(J-47)

The second derivative is

$$\frac{d^2}{dx^2} Y_n(x) = \frac{1}{\sqrt{\rho L}} \left\{ \beta_n{}^2 [\cosh(\beta_n x) + \cos(\beta_n x)] - \sigma_n \; \beta_n{}^2 [\sinh(\beta_n x) + \sin(\beta_n x)] \right\}$$

(J-48)

<u>Participation Factors</u>

The participation factors for constant mass density are

$$\Gamma_n = \rho \int_0^L Y_n(x)\, dx \tag{J-49}$$

$$\Gamma_n = \frac{\rho}{\sqrt{\rho L}} \int_0^L \{ [\cosh(\beta_n x) - \cos(\beta_n x)] - \sigma_n \; [\sinh(\beta_n x) - \sin(\beta_n x)] \} dx \tag{J-50}$$

$$\Gamma_n = \frac{1}{\beta_n} \sqrt{\frac{\rho}{L}} \left\{ [\sinh(\beta_n x) - \sin(\beta_n x)] - \sigma_n [\cosh(\beta_n x) + \cos(\beta_n x)] \right\} \Big|_0^L \tag{J-51}$$

$$\Gamma_n = \frac{1}{\beta_n} \sqrt{\frac{\rho}{L}} \{ [\sinh(\beta_n L) - \sin(\beta_n L)] - \sigma_n [\cosh(\beta_n L) + \cos(\beta_n L)] + 2\sigma_n \} \tag{J-52}$$

$$\Gamma_n = \frac{1}{\beta_n} \sqrt{\frac{\rho}{L}} \{ [\sinh(\beta_n L) - \sin(\beta_n L)] + \sigma_n [2 - \cosh(\beta_n L) - \cos(\beta_n L)] \} \tag{J-53}$$

The participation factors from a numerical calculation are

$$\Gamma_1 = 0.8309\sqrt{\rho L} \quad \text{(J-54)}$$

$$\Gamma_2 = 0 \quad \text{(J-55)}$$

$$\Gamma_3 = 0.3638\sqrt{\rho L} \quad \text{(J-56)}$$

$$\Gamma_4 = 0 \quad \text{(J-57)}$$

$$\Gamma_5 = 0.2315\sqrt{\rho L} \quad \text{(J-58)}$$

The participation factors are non-dimensional.

APPENDIX K

Beam Fixed – Pinned

Consider a fixed – pinned beam as shown in Figure K-1.

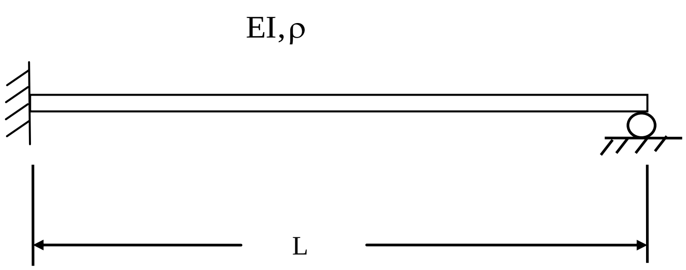


Figure K-1.

Recall that the governing differential equation is

$$-EI\frac{\partial^4 y}{\partial x^4} = \rho\frac{\partial^2 y}{\partial t^2} \tag{K-1}$$

The spatial solution is

$$Y(x) = a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x) \tag{K-2}$$

$$\frac{dY(x)}{dx} = a_1\beta\cosh(\beta x) + a_2\beta\sinh(\beta x) + a_3\beta\cos(\beta x) - a_4\beta\sin(\beta x) \tag{K-3}$$

$$\frac{d^2 Y(x)}{dx^2} = a_1\beta^2 \sinh(\beta x) + a_2\beta^2 \cosh(\beta x) - a_3\beta^2 \sin(\beta x) - a_4\beta^2 \cos(\beta x) \tag{K-4}$$

The boundary conditions at the left end x = 0 are

$$Y(0) = 0 \qquad \text{(zero displacement)} \tag{K-5}$$

$$\left.\frac{dY(x)}{dx}\right|_{x=0} = 0 \qquad \text{(zero slope)} \qquad \text{(K-6)}$$

The boundary conditions at the right end x = L are

$$Y(L) = 0 \qquad \text{(zero displacement)} \qquad \text{(K-7)}$$

$$\left.\frac{d^2Y}{dx^2}\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \qquad \text{(K-8)}$$

Apply boundary condition (K-5).

$$a_2 + a_4 = 0 \qquad \text{(K-9)}$$

$$a_4 = -a_2 \qquad \text{(K-10)}$$

Apply boundary condition (K-6).

$$a_1 + a_3 = 0 \qquad \text{(K-11)}$$

$$a_3 = -a_1 \qquad \text{(K-12)}$$

Apply the left boundary results to the displacement function.

$$Y(x) = a_1[\sinh(\beta x) - \sin(\beta x)] + a_2[\cosh(\beta x) - \cos(\beta x)] \qquad \text{(K-13)}$$

Apply boundary condition (K-7).

$$a_1[\sinh(\beta L) - \sin(\beta L)] + a_2[\cosh(\beta L) - \cos(\beta L)] = 0 \qquad \text{(K-14)}$$

Apply the left boundary results to the second derivative of the displacement function.

$$\frac{d^2Y(x)}{dx^2} = a_1\beta^2[\sinh(\beta x) + \sin(\beta x)] + a_2\beta^2[\cosh(\beta x) + \cos(\beta x)] \tag{K-15}$$

Apply boundary condition (K-7).

$$a_1\beta^2[\sinh(\beta L) + \sin(\beta L)] + a_2\beta^2[\cosh(\beta L) + \cos(\beta L)] = 0 \tag{K-16}$$

$$a_1[\sinh(\beta L) + \sin(\beta L)] + a_2[\cosh(\beta L) + \cos(\beta L)] = 0 \tag{K-17}$$

$$\begin{bmatrix} \sinh(\beta L) - \sin(\beta L) & \cosh(\beta L) - \cos(\beta L) \\ \sinh(\beta L) + \sin(\beta L) & \cosh(\beta L) + \cos(\beta L) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{K-18}$$

$$\det\begin{bmatrix} \sinh(\beta L) - \sin(\beta L) & \cosh(\beta L) - \cos(\beta L) \\ \sinh(\beta L) + \sin(\beta L) & \cosh(\beta L) + \cos(\beta L) \end{bmatrix} = 0 \tag{K-19}$$

$$[\sinh(\beta L) - \sin(\beta L)][\cosh(\beta L) + \cos(\beta L)] - [\sinh(\beta L) + \sin(\beta L)][\cosh(\beta L) - \cos(\beta L)] = 0 \tag{K-20}$$

$$\begin{aligned} &\sinh(\beta L)\cosh(\beta L) + \sinh(\beta L)\cos(\beta L) - \sin(\beta L)\cosh(\beta L) - \sin(\beta L)\cos(\beta L) \\ &- \sinh(\beta L)\cosh(\beta L) + \sinh(\beta L)\cos(\beta L) - \sin(\beta L)\cosh(\beta L) + \sin(\beta L)\cos(\beta L) = 0 \end{aligned} \tag{K-21}$$

$$2\sinh(\beta L)\cos(\beta L) - 2\sin(\beta L)\cosh(\beta L) = 0 \tag{K-22}$$

$$\sinh(\beta L)\cos(\beta L) - \sin(\beta L)\cosh(\beta L) = 0 \qquad \text{(K-23)}$$

$$\tanh(\beta L) - \tan(\beta L) = 0 \qquad \text{(K-24)}$$

The eigenvalues are

n	$\beta_n L$
1	3.9266
2	7.0686
3	10.2102
4	13.3518
5	16.4934

For n>5, $$\beta_n L \approx \pi\left(n + \frac{1}{4}\right) \qquad \text{(K-25)}$$

$$a_2 = -a_1 \frac{[\sinh(\beta L) + \sin(\beta L)]}{[\cosh(\beta L) + \cos(\beta L)]} \qquad \text{(K-26)}$$

The unscaled eigenvector is

$$Y(x) = a_1 \left\{ [\sinh(\beta x) - \sin(\beta x)] - \frac{[\sinh(\beta L) + \sin(\beta L)]}{[\cosh(\beta L) + \cos(\beta L)]} [\cosh(\beta x) - \cos(\beta x)] \right\} \qquad \text{(K-27)}$$

The mass-normalized eigenvector is

$$Y(x) = \frac{1}{\sqrt{\rho L}} \left\{ [\sinh(\beta x) - \sin(\beta x)] - \frac{[\sinh(\beta L) + \sin(\beta L)]}{[\cosh(\beta L) + \cos(\beta L)]} [\cosh(\beta x) - \cos(\beta x)] \right\} \qquad \text{(K-28)}$$

Participation Factors

The participation factors for constant mass density are

$$\Gamma_n = \rho \int_0^L Y_n(x)\,dx \tag{K-29}$$

$$\Gamma_n = \frac{\rho}{\sqrt{\rho L}} \int_0^L \{[\sinh(\beta_n x) - \sin(\beta_n x)] - \sigma_n\, [\cosh(\beta_n x) - \cos(\beta_n x)]\} dx \tag{K-30}$$

$$\sigma_n = \frac{[\sinh(\beta L) + \sin(\beta L)]}{[\cosh(\beta L) + \cos(\beta L)]} \tag{K-31}$$

$$\Gamma_n = \frac{1}{\beta_n} \sqrt{\frac{\rho}{L}} \left\{ [\cosh(\beta_n x) + \cos(\beta_n x)] - \sigma_n [\sinh(\beta_n x) - \sin(\beta_n x)] \right\} \Big|_0^L \tag{K-32}$$

$$\Gamma_n = \frac{1}{\beta_n} \sqrt{\frac{\rho}{L}} \left\{ [\cosh(\beta_n L) + \cos(\beta_n L)] - \sigma_n [\sinh(\beta_n L) - \sin(\beta_n L)] - 2 \right\} \tag{K-33}$$

$$\Gamma_n = \frac{1}{\beta_n} \sqrt{\frac{\rho}{L}} \left\{ [-2 + \cosh(\beta_n L) + \cos(\beta_n L)] - \sigma_n [\sinh(\beta_n L) - \sin(\beta_n L)] \right\} \tag{K-34}$$

The participation factors from a numerical calculation are

$$\Gamma_1 = -0.8593\sqrt{\rho L} \tag{K-35}$$

$$\Gamma_2 = -0.0826\sqrt{\rho L} \tag{K-36}$$

$$\Gamma_3 = -0.3344\sqrt{\rho L} \tag{K-37}$$

$$\Gamma_4 = -0.2070\sqrt{\rho L} \tag{K-38}$$

$$\Gamma_5 = -0.0298\sqrt{\rho L} \tag{K-39}$$

The participation factors are non-dimensional.