TRANSVERSE VIBRATION OF A BEAM VIA THE FINITE ELEMENT METHOD
Revision E

By Tom Irvine
Email: tomirvine@aol.com

November 18, 2008

Introduction

Many structures are too complex for analysis via classical method. Closed-form solutions are thus unavailable for these structures.

For example, a structure may be composed of several different materials. Some of the materials may be anisotropic. Furthermore, the structure may be an assembly of plates, beams, and other components.

Consider three examples:

1. A circuit board has numerous chips, crystal oscillators, diodes, connectors, capacitors, jump wires, and other piece parts.
2. A large aircraft consisting of a fuselage, wing sections, tail section, engines, etc.
3. A building has plies, foundation, beams, floor sections, and load-bearing walls.

The finite element method is a numerical method that can be used to analyze complex structures, such as the three examples.

The purpose of this tutorial is to derive for a method for analyzing beam vibration using the finite element method. The method is based on Reference 1.

Theory

Consider a beam, such as the cantilever beam in Figure 1.

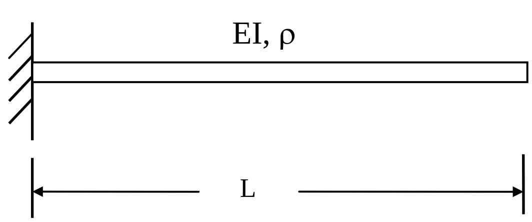

Figure 1.

where

E is the modulus of elasticity.
I is the area moment of inertia.

L is the length.
ρ is mass per length.

The product EI is the bending stiffness.

The vibration modes of the cantilever beam can be found by classical methods. Specifically, the fundamental frequency is

$$\omega_1 = \left[\frac{1.87510}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{1}$$

This problem presents a good opportunity to compare the accuracy of the finite element method to the classical solution.

Let y(x,t) represent the displacement of the beam as a function of space and time.

The free, transverse vibration of the beam is governed by the equation:

$$\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{d^2}{\partial x^2}y(x,t)\right\} = -\rho(x)\frac{\partial^2\ y(x,t)}{\partial t^2} \tag{2}$$

Equation (2) neglects rotary inertia and shear deformation. Note that it is also independent of the boundary conditions, which are applied as constraint equations.

Assume that the solution of equation (1) is separable in time and space.

$$y(x,t) = Y(x)f(t) \tag{3}$$

$$\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{\partial^2}{\partial x^2}Y(x)f(t)\right\} = -\rho(x)\frac{\partial^2\ Y(x)f(t)}{\partial t^2} \tag{4a}$$

$$f(t)\ \frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{\partial^2}{\partial x^2}Y(x)\right\} = -Y(x)\ \rho(x)\ \frac{\partial^2\ f(t)}{\partial t^2} \tag{4b}$$

The partial derivatives change to ordinary derivatives.

$$f(t)\ \frac{d^2}{dx^2}\left\{EI\frac{d^2}{dx^2}Y(x)\right\} = -Y(x)\ \rho(x)\ \frac{d^2\ f(t)}{d\,t^2} \tag{5}$$

$$\frac{1}{Y(x)\rho(x)}\frac{d^2}{dx^2}\left\{EI\frac{d^2}{dx^2}Y(x)\right\} = -\frac{1}{f(t)}\frac{d^2 f(t)}{dt^2} \tag{6}$$

The left-hand side of equation (6) depends on x only. The right hand side depends on t only. Both x and t are independent variables. Thus equation (6) only has a solution if both sides are constant. Let ω^2 be the constant.

$$\frac{1}{Y(x)\rho(x)}\frac{d^2}{dx^2}\left\{EI\frac{d^2}{dx^2}Y(x)\right\} = -\frac{1}{f(t)}\frac{d^2 f(t)}{dt^2} = \omega^2 \tag{7}$$

Equation (7) yields two independent equations.

$$\frac{d^2}{dx^2}\left\{EI(x)\frac{d^2}{dx^2}Y(x)\right\} - \rho(x)\,\omega^2\,Y(x) = 0 \tag{8}$$

$$\frac{d^2}{dt^2}\,f(t) + \omega^2 f(t) = 0 \tag{9}$$

Equation (8) is a homogeneous, forth order, ordinary differential equation.

The weighted residual method is applied to equation (8). This method is suitable for boundary value problems. An alternative method would be the energy method. The energy method is introduced in Appendix A.

There are numerous techniques for applying the weighted residual method. Specifically, the Galerkin approach is used in this tutorial.

The differential equation (8) is multiplied by a test function $\phi(x)$. Note that the test function $\phi(x)$ must satisfy the homogeneous essential boundary conditions. The essential boundary conditions are the prescribed values of Y and its first derivative.

The test function is not required to satisfy the differential equation, however.

The product of the test function and the differential equation is integrated over the domain. The integral is set equation to zero.

$$\int \phi(x)\left\{\frac{d^2}{dx^2}\left[EI(x)\frac{d^2}{dx^2}Y(x)\right] - \rho(x)\,\omega^2\,Y(x)\right\}dx = 0 \tag{10}$$

The test function $\phi(x)$ can be regarded as a virtual displacement. The differential equation in the brackets represents an internal force. This term is also regarded as the residual. Thus, the integral represents virtual work, which should vanish at the equilibrium condition.

Define the domain over the limits from a to b. These limits represent the boundary points of the entire beam.

$$\int_a^b \phi(x) \left\{ \frac{d^2}{dx^2}\left[EI(x) \frac{d^2}{dx^2} Y(x) \right] - \rho(x)\, \omega^2\, Y(x) \right\} dx = 0 \tag{11}$$

$$\int_a^b \phi(x) \left\{ \frac{d^2}{dx^2}\left[EI(x) \frac{d^2}{dx^2} Y(x) \right] \right\} dx - \int_a^b \phi(x) \left\{ \rho(x)\, \omega^2\, Y(x) \right\} dx = 0 \tag{12}$$

Integrate the first integral by parts.

$$\int_a^b \frac{d}{dx} \left\{ \phi(x) \frac{d}{dx}\left[EI(x) \frac{d^2}{dx^2} Y(x) \right] \right\} dx - \int_a^b \left\{ \frac{d}{dx} \phi(x) \right\} \left\{ \frac{d}{dx}\left[EI(x) \frac{d^2}{dx^2} Y(x) \right] \right\} dx$$
$$- \int_a^b \phi(x) \left\{ \rho(x)\, \omega^2\, Y(x) \right\} dx = 0 \tag{13}$$

$$\left. \left\{ \phi(x) \frac{d}{dx}\left[EI(x) \frac{d^2}{dx^2} Y(x) \right] \right\} \right|_a^b - \int_a^b \left\{ \frac{d}{dx} \phi(x) \right\} \left\{ \frac{d}{dx}\left[EI(x) \frac{d^2}{dx^2} Y(x) \right] \right\} dx$$
$$- \int_a^b \phi(x) \left\{ \rho(x)\, \omega^2\, Y(x) \right\} dx = 0 \tag{14}$$

$$\left\{\phi(x)\,\frac{d}{dx}\left[EI(x)\frac{d^2}{dx^2}Y(x)\right]\right\}\Bigg|_a^b - \int_a^b \left\{\frac{d}{dx}\phi(x)\right\}\left\{\frac{d}{dx}\left[EI(x)\frac{d^2}{dx^2}Y(x)\right]\right\}dx$$

$$-\int_a^b \phi(x)\left\{\rho(x)\,\omega^2\,Y(x)\right\}dx = 0 \tag{15}$$

Integrate by parts again.

$$\left\{\phi(x)\,\frac{d}{dx}\left[EI(x)\frac{d^2}{dx^2}Y(x)\right]\right\}\Bigg|_a^b - \int_a^b \frac{d}{dx}\left\{\left[\frac{d}{dx}\phi(x)\right]\left[EI(x)\frac{d^2}{dx^2}Y(x)\right]\right\}dx$$

$$+\int_a^b \left\{\left[\frac{d^2}{dx^2}\phi(x)\right]\left[EI(x)\frac{d^2}{dx^2}Y(x)\right]\right\}dx - \int_a^b \phi(x)\left\{\rho(x)\,\omega^2\,Y(x)\right\}dx = 0$$

$$\tag{16}$$

$$\left\{\phi(x)\,\frac{d}{dx}\left[EI(x)\frac{d^2}{dx^2}Y(x)\right]\right\}\Bigg|_a^b - \left\{\left[\frac{d}{dx}\phi(x)\right]\left[EI(x)\frac{d^2}{dx^2}Y(x)\right]\right\}\Bigg|_a^b$$

$$+\int_a^b \left\{\left[\frac{d^2}{dx^2}\phi(x)\right]\left[EI(x)\frac{d^2}{dx^2}Y(x)\right]\right\}dx - \int_a^b \phi(x)\left\{\rho(x)\,\omega^2\,Y(x)\right\}dx = 0$$

$$\tag{17}$$

The essential boundary conditions for a cantilever beam are

$$Y(a) = 0 \tag{18}$$

$$\left.\frac{dY}{dx}\right|_{x=a} = 0 \tag{19}$$

Thus, the test functions must satisfy

$$\phi(a) = 0 \tag{20}$$

$$\left.\frac{d\phi}{dx}\right|_{x=a} = 0 \tag{21}$$

The natural boundary conditions are

$$\left.\frac{d}{dx}\left[EI(x)\frac{d^2}{dx^2}Y(x)\right]\right|_{x=b} = 0 \tag{22}$$

$$\left.\left[EI(x)\frac{d^2}{dx^2}Y(x)\right]\right|_{x=b} = 0 \tag{23}$$

Equation (23) requires

$$\left.\left[EI(x)\frac{d^2}{dx^2}\phi(x)\right]\right|_{x=b} = 0 \tag{24}$$

Apply equations (20), (21), and (24) to equation (17). The result is

$$\int_a^b \left\{ \left[\frac{d^2}{dx^2}\phi(x)\right]\left[EI(x)\frac{d^2}{dx^2}Y(x)\right] \right\} dx - \int_a^b \phi(x) \left\{\rho(x)\,\omega^2\,Y(x)\right\} dx = 0 \tag{25}$$

Note that equation (25) would also be obtained for other simple boundary condition cases.

Now consider that the beam consists of number of segments, or elements. The elements are arranged geometrically in series form.

Furthermore, the endpoints of each element are called nodes.

The following equation must be satisfied for each element.

$$\int \left\{ \left[\frac{d^2}{dx^2} \phi(x) \right] \left[EI(x) \frac{d^2}{dx^2} Y(x) \right] \right\} dx - \int \phi(x) \left\{ \rho(x)\, \omega^2\, Y(x) \right\} dx = 0 \tag{26}$$

Furthermore, consider that the stiffness and mass properties are constant for a given element.

$$EI \int \left\{ \left[\frac{d^2}{dx^2} \phi(x) \right] \left[\frac{d^2}{dx^2} Y(x) \right] \right\} dx - \rho \omega^2 \int \phi(x)\, Y(x)\, dx = 0 \tag{27}$$

Now express the displacement function Y(x) in terms of nodal displacements y_{j-1} and y_j as well as the rotations θ_{j-1} and θ_j .

$$Y(x) = L_1\, y_{j-1} + L_2\, y_j + L_3\, h\theta_{j-1} + L_4\, h\theta_{j-1}\,, \qquad (j-1)h \le x \le jh \tag{28}$$

Note that h is the element length. In addition, each L coefficients is a function of x.

Now introduce a nondimensional natural coordinate ξ .

$$\xi = j - x/h \tag{29}$$

Note that h is the segment length.

The displacement function becomes.

$$Y(\xi) = L_1\, y_{j-1} + L_2\, h\theta_{j-1} + L_3\, y_j + L_4\, h\theta_{j-1}\,, \qquad 0 \le \xi \le 1 \tag{30}$$

The slope equation is

$$Y\,'(\xi) = L_1'\, y_{j-1} + L_2'\, h\theta_{j-1} + L_3'\, y_j + L_4'\, h\theta_{j-1}\,, \qquad 0 \le \xi \le 1 \tag{31}$$

The displacement function is represented terms of natural coordinates in Figure 2.

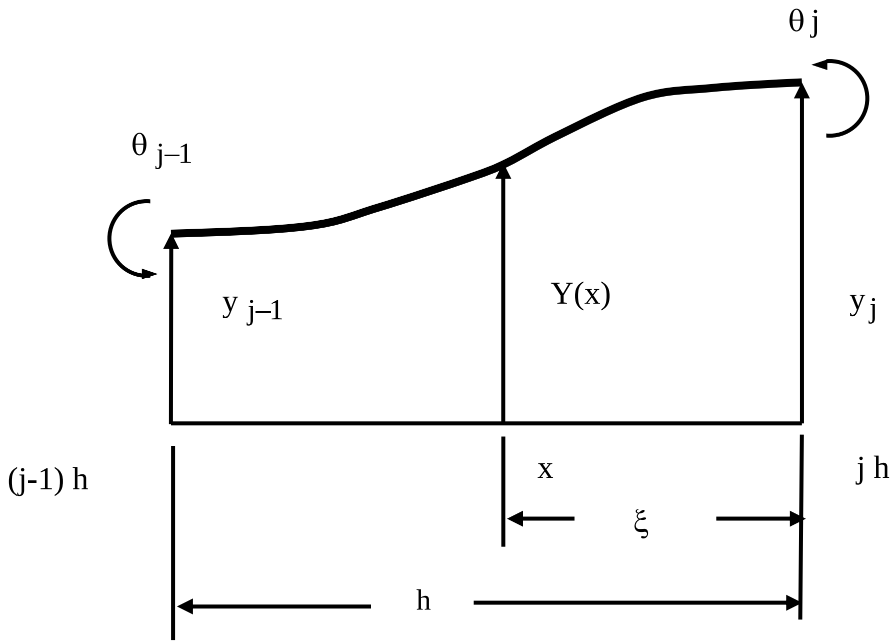

Figure 2.

Represent each L coefficient in terms of a cubic polynomial.

$$L_i = c_{i1} + c_{i2}\,\xi + c_{i3}\,\xi^2 + c_{i4}\,\xi^3, \qquad i = 1, 2, 3, 4 \tag{32}$$

$$\begin{aligned} Y(\xi) = {} & \left\{ c_{11} + c_{12}\,\xi + c_{13}\,\xi^2 + c_{14}\,\xi^3 \right\} y_{j-1} \\ & + \left\{ c_{21} + c_{22}\,\xi + c_{23}\,\xi^2 + c_{24}\,\xi^3 \right\} h\,\theta_{j-1} \\ & + \left\{ c_{31} + c_{32}\,\xi + c_{33}\,\xi^2 + c_{34}\,\xi^3 \right\} y_j \\ & + \left\{ c_{41} + c_{42}\,\xi + c_{43}\,\xi^2 + c_{44}\,\xi^3 \right\} h\,\theta_j\,, \qquad 0 \le \xi \le 1 \end{aligned} \tag{33}$$

$$
\begin{aligned}
Y'(\xi) = &\left\{c_{12} + 2c_{13}\xi + 3c_{14}\xi^2\right\} y_{j-1} \\
&+ \left\{c_{22} + 2c_{23}\xi + 3c_{24}\xi^2\right\} h\theta_{j-1} \\
&+ \left\{c_{32} + 2c_{33}\xi + 3c_{34}\xi^2\right\} y_j \\
&+ \left\{c_{42} + 2c_{43}\xi + 3c_{44}\xi^2\right\} h\theta_j, \qquad 0 \le \xi \le 1
\end{aligned}
\tag{34}
$$

Solve for the coefficients c_{ij}. The constraint equations are

$$Y(0) = y_j \tag{35}$$

$$Y(1) = y_{j-1} \tag{36}$$

$$Y'(0) = -h\theta_j \tag{37}$$

$$Y'(1) = -h\theta_{j-1} \tag{38}$$

Evaluate the displacement at $\xi = 0$.

$$Y(0) = \left\{c_{11}\right\} y_{j-1} + \left\{c_{21}\right\} h\theta_{j-1} + \left\{c_{31}\right\} y_j + \left\{c_{41}\right\} h\theta_j \tag{39}$$

Boundary condition (35) requires

$$\left\{c_{11}\right\} y_{j-1} + \left\{c_{21}\right\} h\theta_{j-1} + \left\{c_{31}\right\} y_j + \left\{c_{41}\right\} h\theta_j = y_j \tag{39}$$

$$c_{11} = 0 \tag{40}$$

$$c_{21} = 0 \tag{41}$$

$$c_{31} = 1 \tag{42}$$

$$c_{41} = 0 \tag{43}$$

The displacement equations becomes

$$
\begin{aligned}
Y(\xi) = &+ \left\{ c_{12}\xi + c_{13}\xi^2 + c_{14}\xi^3 \right\} y_{j-1} \\
&+ \left\{ c_{22}\xi + c_{23}\xi^2 + c_{24}\xi^3 \right\} h\theta_{j-1} \\
&+ \left\{ 1 + c_{32}\xi + c_{33}\xi^2 + c_{34}\xi^3 \right\} y_j \\
&+ \left\{ c_{42}\xi + c_{43}\xi^2 + c_{44}\xi^3 \right\} h\theta_j \quad , \qquad 0 \le \xi \le 1
\end{aligned}
\tag{44}
$$

The slope equations becomes

$$
\begin{aligned}
Y'(\xi) = &\left\{ c_{12} + 2c_{13}\xi + 3c_{14}\xi^2 \right\} y_{j-1} \\
&+ \left\{ c_{22} + 2c_{23}\xi + 3c_{24}\xi^2 \right\} h\theta_{j-1} \\
&+ \left\{ c_{32} + 2c_{33}\xi + 3c_{34}\xi^2 \right\} y_j \\
&+ \left\{ c_{42} + 2c_{43}\xi + 3c_{44}\xi^2 \right\} h\theta_j \quad , \quad 0 \le \xi \le 1
\end{aligned}
\tag{45}
$$

Evaluate the slope at $\xi = 0$.

$$
Y'(0) = \left\{ c_{12} \right\} y_{j-1} + \left\{ c_{22} \right\} h\theta_{j-1} + \left\{ c_{32} \right\} y_j + \left\{ c_{42} \right\} h\theta_j \tag{46}
$$

Boundary condition (37) requires.

$$
\left\{ c_{12} \right\} y_{j-1} + \left\{ c_{22} \right\} h\theta_{j-1} + \left\{ c_{32} \right\} y_j + \left\{ c_{42} \right\} h\theta_j = -h\theta_j \tag{47}
$$

$$
c_{12} = 0 \tag{48}
$$

$$
c_{22} = 0 \tag{49}
$$

$$
c_{32} = 0 \tag{50}
$$

$$
c_{42} = -1 \tag{51}
$$

The displacement equations becomes

$$
\begin{aligned}
Y(\xi) = &+ \left\{c_{13}\xi^2 + c_{14}\xi^3\right\} y_{j-1} \\
&+ \left\{c_{23}\xi^2 + c_{24}\xi^3\right\} h\theta_{j-1} \\
&+ \left\{1 + c_{33}\xi^2 + c_{34}\xi^3\right\} y_j \\
&+ \left\{-\xi + c_{43}\xi^2 + c_{44}\xi^3\right\} h\theta_j \ , \qquad 0 \le \xi \le 1
\end{aligned}
$$

(52)

The slope equations becomes

$$
\begin{aligned}
Y'(\xi) = &\left\{2c_{13}\xi + 3c_{14}\xi^2\right\} y_{j-1} + \left\{2c_{23}\xi + 3c_{24}\xi^2\right\} h\theta_{j-1} \\
&+ \left\{2c_{33}\xi + 3c_{34}\xi^2\right\} y_j + \left\{-1 + 2c_{43}\xi + 3c_{44}\xi^2\right\} h\theta_j \ , \\
&\qquad\qquad 0 \le \xi \le 1
\end{aligned}
$$

(53)

$$
\begin{aligned}
Y(1) = &\left\{c_{13} + c_{14}\right\} y_{j-1} + \left\{c_{23} + c_{24}\right\} h\theta_{j-1} \\
&+ \left\{1 + c_{33} + c_{34}\right\} y_j + \left\{-1 + c_{43} + c_{44}\right\} h\theta_j
\end{aligned}
$$

(54)

Boundary condition (36) requires

$$
\begin{aligned}
&\left\{c_{13} + c_{14}\right\} y_{j-1} + \left\{c_{23} + c_{24}\right\} h\theta_{j-1} \\
&+ \left\{1 + c_{33} + c_{34}\right\} y_j + \left\{-1 + c_{43} + c_{44}\right\} h\theta_j = y_{j-1}
\end{aligned} \tag{55}
$$

$$c_{13} + c_{14} = 1 \tag{56}$$

$$c_{13} = -c_{14} + 1 \tag{57}$$

$$c_{23} + c_{24} = 0 \tag{58}$$

$$c_{23} = -c_{24} \tag{59}$$

$$1 + c_{33} + c_{34} = 0 \tag{60}$$

$$c_{33} = -1 - c_{34} \tag{61}$$

$$-1 + c_{43} + c_{44} = 0 \tag{62}$$

$$c_{43} = 1 - c_{44} \tag{63}$$

The displacement equation becomes

$$\begin{aligned} Y(\xi) = &+ \left\{ \left[1 - c_{14}\right]\xi^2 + c_{14}\xi^3 \right\} y_{j-1} \\ &+ \left\{ \left[- c_{24}\right]\xi^2 + c_{24}\xi^3 \right\} h\,\theta_{j-1} \\ &+ \left\{ 1 + \left[-1 - c_{34}\right]\xi^2 + c_{34}\xi^3 \right\} y_j \\ &+ \left\{ -\xi + \left[1 - c_{44}\right]\xi^2 + c_{44}\xi^3 \right\} h\,\theta_j \;, \qquad 0 \le \xi \le 1 \end{aligned} \tag{64}$$

The slope equation becomes

$$\begin{aligned} Y'(\xi) = &+ \left\{ 2\left[1 - c_{14}\right]\xi + 3c_{14}\xi^2 \right\} y_{j-1} \\ &+ \left\{ -2\,c_{24}\,\xi + 3\,c_{24}\xi^2 \right\} h\,\theta_{j-1} \\ &+ \left\{ 2\left[-1 - c_{34}\right]\xi + 3\,c_{34}\xi^2 \right\} y_j \\ &+ \left\{ -1 + 2\left[1 - c_{44}\right]\xi + 3c_{44}\xi^2 \right\} h\,\theta_j \;, \qquad 0 \le \xi \le 1 \end{aligned} \tag{65}$$

The slope equation becomes

$$\begin{aligned} Y\ '(1) = &+\left\{ 2\left[1-c_{14}\right]+3c_{14}\right\}y_{j-1} \\ &+\left\{-2\,c_{24}+3\,c_{24}\right\}h\,\theta_{j-1} \\ &+\left\{2\left[-1-c_{34}\right]+3\,c_{34}\right\}y_j \\ &+\left\{-1+2\left[1-\ c_{44}\right]+3c_{44}\right\}h\,\theta_j \end{aligned} \tag{66}$$

Boundary condition (38) requires

$$\begin{aligned} &+\left\{ 2\left[1-c_{14}\right]+3c_{14}\right\}y_{j-1} \\ &+\left\{-2\,c_{24}+3\,c_{24}\right\}h\,\theta_{j-1} \\ &+\left\{2\left[-1-c_{34}\right]+3\,c_{34}\right\}y_j \\ &+\left\{-1+2\left[1-\ c_{44}\right]+3c_{44}\right\}h\,\theta_j = -h\,\theta_{j-1} \end{aligned} \tag{67}$$

$$\begin{aligned} &+\left\{ \left[2-2c_{14}\right]+3c_{14}\right\}y_{j-1} \\ &+\left\{-2\,c_{24}+3\,c_{24}\right\}h\,\theta_{j-1} \\ &+\left\{\left[-2-2c_{34}\right]+3\,c_{34}\right\}y_j \\ &+\left\{-1+\left[2-2c_{44}\right]+3c_{44}\right\}h\,\theta_j = -h\,\theta_{j-1} \end{aligned} \tag{68}$$

$$2+c_{14}=0 \tag{69}$$

$$c_{14}=-2 \tag{70}$$

$$c_{24}=-1 \tag{71}$$

$$-2+c_{34}=0 \tag{72}$$

$$c_{34}=2 \tag{73}$$

$$1 + c_{44} = 0 \tag{74}$$

$$c_{44} = -1 \tag{75}$$

The displacement equation becomes

$$\begin{aligned} Y(\xi) = &+ \left\{ [1-(-2)]\,\xi^2 - 2\xi^3 \right\} y_{j-1} \\ &+ \left\{ [-(-1)]\,\xi^2 - 1\xi^3 \right\} h\,\theta_{j-1} \\ &+ \left\{ 1 + [-1-2]\ \xi^2 + 2\xi^3 \right\} y_j \\ &+ \left\{ -\xi + [1-\ (-1)]\xi^2 + (-1)\xi^3 \right\} h\,\theta_j \ , \qquad 0 \ \le \xi \le 1 \end{aligned} \tag{76}$$

$$\begin{aligned} Y(\xi) = &+ \left\{ 3\,\xi^2 - 2\xi^3 \right\} y_{j-1} + \left\{ \ \xi^2 - \xi^3 \right\} h\,\theta_{j-1} \\ &+ \left\{ 1 - 3\ \xi^2 + 2\xi^3 \right\} y_j \ + \left\{ -\xi + 2\xi^2 - \xi^3 \right\} h\,\theta_j \ , \qquad 0 \ \le \xi \le 1 \end{aligned} \tag{77}$$

Recall

$$\xi = j - x/h \tag{78}$$

Thus

$$d\xi = -dx/h \tag{79a}$$

$$-h\,d\xi = dx \tag{79b}$$

$$\frac{d\xi}{dx} = -1/h \tag{80}$$

Note

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} \tag{81}$$

$$Y(x) = +\left\{3\,\xi^2 - 2\xi^3\right\} y_{j-1} + \left\{\ \xi^2 - \xi^3\right\} h\,\theta_{j-1}$$
$$+\left\{1 - 3\ \xi^2 + 2\xi^3\right\} y_j\ + \left\{-\xi + 2\,\xi^2 - \xi^3\right\} h\,\theta_j\ ,$$

$$(j-1)h \le x \le jh, \quad \xi = j - x/h, \quad 0\ \le \xi \le 1 \tag{82}$$

$$\frac{d}{dx} Y(x) = \{-1/h\}\left\{\left[6\,\xi - 6\xi^2\right] y_{j-1} + \left[2\,\xi - 3\xi^2\right] h\,\theta_{j-1}\right.$$
$$\left. + \left[1 - 6\ \xi + 6\xi^2\right] y_j\ + \left[-1 + 4\xi - 3\xi^2\right] h\,\theta_j\ \right\},$$

$$(j-1)h \le x \le jh, \quad \xi = j - x/h, \quad 0\ \le \xi \le 1 \tag{83}$$

$$\frac{d^2}{dx^2} Y(x) = \left\{1/h^2\right\}\left\{\left[\ 6 - 12\xi\right] y_{j-1} + \left[2 - 6\xi\right] h\,\theta_{j-1}\right.$$
$$\left. + \left[\ -6\ + 12\xi\right] y_j\ + \left[\ 4 - 6\ \xi\right] h\,\theta_j\ \right\},$$

$$(j-1)h \le x \le jh, \quad \xi = j - x/h, \quad 0\ \le \xi \le 1 \tag{84}$$

Now Let

$$Y(x) = \underline{L}^{\,T}\ \ \bar{a}\ , \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{85}$$

where

$$L_1 = 3\,\xi^2 - 2\xi^3 \tag{86}$$

$$L_2 = \xi^2 - \xi^3 \tag{87}$$

$$L_3 = 1 - 3\ \xi^2 + 2\xi^3 \tag{88}$$

$$L_4 = -\xi + 2\xi^2 - \xi^3 \tag{89}$$

$$\bar{a} = \begin{bmatrix} y_{j-1} & h\theta_{j-1} & y_j & h\theta_j \end{bmatrix}^T \tag{90}$$

The derivative terms are

$$\frac{d}{dx} Y(x) = \left(\frac{-1}{h}\right) \underline{L}'^{\,T} \bar{a}, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{91}$$

$$\frac{d^2}{dx^2} Y(x) = \left(\frac{1}{h^2}\right) \underline{L}''^{\,T} \bar{a}, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{92}$$

Note that primes indicate derivatives with respect to ξ.

In summary.

$$\underline{L} = \begin{bmatrix} 3\xi^2 - 2\xi^3 \\ \xi^2 - \xi^3 \\ 1 - 3\xi^2 + 2\xi^3 \\ -\xi + 2\xi^2 - \xi^3 \end{bmatrix} \tag{90}$$

$$\underline{L}' = \begin{bmatrix} 6\xi - 6\xi^2 \\ 2\xi - 3\xi^2 \\ -6\xi + 6\xi^2 \\ -1 + 4\xi - 3\xi^2 \end{bmatrix} \tag{91}$$

$$\underline{L}'' = \begin{bmatrix} 6 - 12\xi \\ 2 - 6\xi \\ -6 + 12\xi \\ 4 - 6\xi \end{bmatrix} \tag{92}$$

Recall

$$EI \int \left\{ \left[\frac{d^2}{dx^2} \phi(x) \right] \left[\frac{d^2}{dx^2} Y(x) \right] \right\} dx - \rho \omega^2 \int \phi(x)\ Y(x)\ dx = 0 \tag{93}$$

The essence of the Galerkin method is that the test function is chosen as

$$\phi(x) = Y(x) \tag{94}$$

Thus

$$EI \int \left\{ \left[\frac{d^2}{dx^2} Y(x) \right] \left[\frac{d^2}{dx^2} Y(x) \right] \right\} dx - \rho \omega^2 \int \left[Y(x) \right]^2 dx = 0 \tag{95}$$

Change the integration variable using equation (79b). Also, apply the integration limits.

$$h\ EI \int_0^1 \left\{ \left[\frac{d^2}{dx^2} Y(x) \right] \left[\frac{d^2}{dx^2} Y(x) \right] \right\} d\xi - h \rho \omega^2 \int_0^1 \left[Y(x) \right]^2 d\xi = 0 \tag{96}$$

$$h\ EI \int_0^1 \left\{ \left[\left(\frac{1}{h^2} \right) \underline{L''}^{\,T}\ \bar{a} \right] \left[\left(\frac{1}{h^2} \right) \underline{L''}^{\,T}\ \bar{a} \right] \right\} d\xi$$

$$- h\, \rho \omega^2 \int_0^1 \left[\underline{L}^{\,T}\ \bar{a} \right] \left[\underline{L}^{\,T}\ \bar{a} \right] d\xi = 0 \tag{97}$$

$$\left(\frac{1}{h^3}\right) EI \int_0^1 \left\{ \left[\underline{L''}^{\,T} \quad \bar{a} \right] \left[\underline{L''}^{\,T} \quad \bar{a} \right] \right\} d\xi$$

$$- h\rho\omega^2 \int_0^1 \left[\underline{L}^{\,T} \quad \bar{a} \right] \left[\underline{L}^{\,T} \quad \bar{a} \right] \; d\xi = 0$$

(98)

$$\left(\frac{1}{h^3}\right) EI \int_0^1 \left\{ \left[\bar{a}^{\,T} \underline{L''} \right] \left[\underline{L''}^{\,T} \quad \bar{a} \right] \right\} d\xi$$

$$- h\rho\omega^2 \int_0^1 \left[\bar{a}^T \underline{L} \right] \left[\underline{L}^{\,T} \quad \bar{a} \right] \; d\xi = 0$$

(99)

$$\left(\frac{1}{h^3}\right) EI \int_0^1 \left\{ \bar{a}^{\,T} \underline{L''} \; \underline{L''}^{\,T} \quad \bar{a} \right\} d\xi \;-\; h\rho\omega^2 \int_0^1 \left\{ \bar{a}^{\,T} \underline{L} \, \underline{L}^T \quad \bar{a} \right\} d\xi = 0$$

(100)

$$\bar{a}^{\,T} \left\{ \left(\frac{EI}{h^3}\right) \int_0^1 \left\{ \underline{L''} \; \underline{L''}^{\,T} \right\} d\xi - h\rho\omega^2 \int_0^1 \left\{ \underline{L} \, \underline{L}^T \right\} d\xi \right\} \bar{a} = 0 \qquad (101)$$

$$\left(\frac{EI}{h^3}\right) \int_0^1 \left\{ \underline{L''} \; \underline{L''}^{\,T} \right\} d\xi - h\rho\omega^2 \int_0^1 \left\{ \underline{L} \, \underline{L}^T \right\} d\xi = 0 \qquad (102)$$

For a system of n elements,

$$K_j - \omega^2 \, M_j = 0 \,, \qquad j = 1, 2, \ldots, n \qquad (103)$$

where

$$K_j \;=\; \left(\frac{EI}{h^3}\right) \int_0^1 \left\{ \underline{L''} \; \underline{L''}^{\,T} \right\} d\xi \qquad (104)$$

$$M_j = h\rho \int_0^1 \left\{ \underline{L}\,\underline{L}^T \right\} d\xi \tag{105}$$

$$\underline{L}''\,\underline{L}''^T = \begin{bmatrix} 6-12\xi \\ 2-6\xi \\ -6+12\xi \\ 4-6\xi \end{bmatrix} \begin{bmatrix} 6-12\xi & 2-6\xi & -6+12\xi & 4-6\xi \end{bmatrix} \tag{106}$$

$$\underline{L}''\,\underline{L}''^T =$$

$$\begin{bmatrix} (6-12\xi)(6-12\xi) & (6-12\xi)(2-6\xi) & (6-12\xi)(-6+12\xi) & (6-12\xi)(4-6\xi) \\ & (2-6\xi)(2-6\xi) & (2-6\xi)(-6+12\xi) & (2-6\xi)(4-6\xi) \\ & & (-6+12\xi)(-6+12\xi) & (-6+12\xi)(4-6\xi) \\ & & & (4-6\xi)(4-6\xi) \end{bmatrix} \tag{107}$$

Note that only the upper triangular components are shown due to symmetry.

$$\underline{L}''\,\underline{L}''^T =$$

$$\begin{bmatrix} (6-12\xi)(6-12\xi) & (6-12\xi)(2-6\xi) & (6-12\xi)(-6+12\xi) & (6-12\xi)(4-6\xi) \\ & (2-6\xi)(2-6\xi) & (2-6\xi)(-6+12\xi) & (2-6\xi)(4-6\xi) \\ & & (-6+12\xi)(-6+12\xi) & (-6+12\xi)(4-6\xi) \\ & & & (4-6\xi)(4-6\xi) \end{bmatrix} \tag{108}$$

$\underline{L}''\,\underline{L}''^{T} =$

$$\begin{bmatrix} (6-12\xi)(6-12\xi) & (6-12\xi)(2-6\xi) & (6-12\xi)(-6+12\xi) & (6-12\xi)(4-6\xi) \\ & (2-6\xi)(2-6\xi) & (2-6\xi)(-6+12\xi) & (2-6\xi)(4-6\xi) \\ & & (-6+12\xi)(-6+12\xi) & (-6+12\xi)(4-6\xi) \\ & & & (4-6\xi)(4-6\xi) \end{bmatrix}$$

(109)

$\underline{L}''\,\underline{L}''^{T} =$

$$\begin{bmatrix} 36-144\xi+144\xi^2 & 12-60\xi+72\xi^2 & -36+144\xi-144\xi^2 & 24-84\xi+72\xi^2 \\ & 4-24\xi+36\xi^2 & -12+60\xi-72\xi^2 & 8-36\xi+36\xi^2 \\ & & 36-144\xi+144\xi^2 & -24+84\xi-72\xi^2 \\ & & & 16-48\xi+36\xi^2 \end{bmatrix}$$

(110)

$K_j =$

$$\left(\frac{EI}{h^3}\right)\int_0^1 \begin{bmatrix} 36-144\xi+144\xi^2 & 12-60\xi+72\xi^2 & -36+144\xi-144\xi^2 & 24-84\xi+72\xi^2 \\ & 4-24\xi+36\xi^2 & -12+60\xi-72\xi^2 & 8-36\xi+36\xi^2 \\ & & 36-144\xi+144\xi^2 & -24+84\xi-72\xi^2 \\ & & & 16-48\xi+36\xi^2 \end{bmatrix} d\xi$$

(111)

$$K_j =$$

$$\left(\frac{EI}{h^3}\right)\left.\begin{bmatrix} 36\xi-72\xi^2+48\xi^3 & 12\xi-30\xi^2+24\xi^3 & -36\xi+72\xi^2-48\xi^3 & 24\xi-42\xi^2+24\xi^3 \\ & 4\xi-12\xi^2+12\xi^3 & -12\xi+30\xi^2-24\xi^3 & 8\xi-18\xi^2+12\xi^3 \\ & & 36\xi-72\xi^2+48\xi^3 & -24\xi+42\xi^2-24\xi^3 \\ & & & 16\xi-24\xi^2+12\xi^3 \end{bmatrix}\right|_0^1$$

(112)

$$K_j =$$

$$\left(\frac{EI}{h^3}\right)\begin{bmatrix} 36-72+48 & 12-30+24 & -36+72-48 & 24-42+24 \\ & 4-12+12 & -12+30-24 & 8-18+12 \\ & & 36-72+48 & -24+42-24 \\ & & & 16-24+12 \end{bmatrix}$$

(113)

$$K_j = \left(\frac{EI}{h^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ & & & 4 \end{bmatrix}$$

(114)

$$\underline{L}\ \underline{L}^T = \begin{bmatrix} 3\xi^2 - 2\xi^3 \\ \xi^2 - \xi^3 \\ 1 - 3\ \xi^2 + 2\xi^3 \\ -\xi + 2\xi^2 - \xi^3 \end{bmatrix} \begin{bmatrix} 3\xi^2 - 2\xi^3 & \xi^2 - \xi^3 & 1 - 3\ \xi^2 + 2\xi^3 & -\xi + 2\xi^2 - \xi^3 \end{bmatrix}$$

(115)

$$\underline{L}\ \underline{L}^T =$$

$$\begin{bmatrix} \left(3\xi^2 - 2\xi^3\right)^2 & \left(3\xi^2 - 2\xi^3\right)\left(\xi^2 - \xi^3\right) & \left(3\xi^2 - 2\xi^3\right)\left(1 - 3\ \xi^2 + 2\xi^3\right) & \left(3\xi^2 - 2\xi^3\right)\left(-\xi + 2\xi^2 - \xi^3\right) \\ & \left(\xi^2 - \xi^3\right)^2 & \left(\xi^2 - \xi^3\right)\left(1 - 3\ \xi^2 + 2\xi^3\right) & \left(\xi^2 - \xi^3\right)\left(-\xi + 2\xi^2 - \xi^3\right) \\ & & \left(1 - 3\ \xi^2 + 2\xi^3\right)^2 & \left(1 - 3\ \xi^2 + 2\xi^3\right)\left(-\xi + 2\xi^2 - \xi^3\right) \\ & & & \left(-\xi + 2\xi^2 - \xi^3\right)^2 \end{bmatrix}$$

(116)

$$\underline{L}\ \underline{L}^T = \begin{bmatrix} a11 & a12 & a13 & a14 \\ & a22 & a23 & a24 \\ & & a33 & a34 \\ & & & a44 \end{bmatrix}$$

(117)

$$a11 = 9\xi^4 - 12\xi^5 + 4\xi^6 \qquad (118)$$

$$a12 = 3\xi^4 - 5\xi^5 + 2\xi^6 \qquad (119)$$

$$a13 = 3\xi^2 - 2\xi^3 - 9\xi^4 + 12\xi^5 - 4\xi^6 \qquad (120)$$

$$a14 = -3\xi^3 + 8\xi^4 - 7\xi^5 + 2\xi^6 \tag{121}$$

$$a22 = \xi^4 - 2\xi^5 + \xi^6 \tag{122}$$

$$a23 = \xi^2 - \xi^3 - 3\xi^4 + 5\xi^5 - 2\xi^6 \tag{123}$$

$$a24 = -\xi^3 + 3\xi^4 - 3\xi^5 + \xi^6 \tag{124}$$

$$a33 = 1 - 6\xi^2 + 4\xi^3 + 9\xi^4 - 12\xi^5 + 4\xi^6 \tag{125}$$

$$a34 = -\xi + 2\xi^2 + 2\xi^3 - 8\xi^4 + 7\xi^5 - 2\xi^6 \tag{126}$$

$$a44 = \xi^2 - 4\xi^3 + 6\xi^4 - 4\xi^5 + \xi^6 \tag{127}$$

Recall

$$M_j = h\rho \int_0^1 \left\{ \underline{L}\, \underline{L}^T \right\} d\xi \tag{128}$$

$$M_{j,\,11} = \rho h \int_0^1 \left\{ 9\xi^4 - 12\xi^5 + 4\xi^6 \right\} d\xi \tag{129}$$

$$M_{j,\,11} = \rho h \left[\frac{9}{5}\xi^5 - 2\xi^6 + \frac{4}{7}\xi^7 \right] \Bigg|_0^1 \tag{130}$$

$$M_{j,\,11} = \rho h \left[\frac{9}{5} - 2 + \frac{4}{7} \right] \tag{131}$$

$$M_{j,\,11} = \rho h \left(\frac{13}{35} \right) \tag{132}$$

$$M_{j,\,11} = \rho h \left(\frac{156}{420} \right) \tag{133}$$

$$M_{j,\,12} \;=\; \rho h \int_0^1 \left\{ 3\xi^4 - 5\xi^5 + 2\xi^6 \right\} d\xi \tag{134}$$

$$M_{j,\,12} \;=\; \rho h \left[\frac{3}{5}\xi^5 - \frac{5}{6}\xi^6 + \frac{2}{7}\xi^7 \right]\Bigg|_0^1 \tag{135}$$

$$M_{j,\,12} \;=\; \rho h \left[\frac{3}{5} - \frac{5}{6} + \frac{3}{7} \right] \tag{136}$$

$$M_{j,\,12} \;=\; \rho h \left(\frac{11}{210} \right) \tag{137}$$

$$M_{j,\,12} \;=\; \rho h \left(\frac{22}{420} \right) \tag{138}$$

$$M_{j,\,13} \;=\; \rho h \int_0^1 \left\{ 3\xi^2 - 2\xi^3 - 9\xi^4 + 12\xi^5 - 4\xi^6 \right\} d\xi \tag{139}$$

$$M_{j,\,13} \;=\; \rho h \left[\xi^3 - \left(\frac{1}{2}\right)\xi^4 - \left(\frac{9}{5}\right)\xi^5 + 2\xi^6 - \left(\frac{4}{7}\right)\xi^7 \right]\Bigg|_0^1 \tag{140}$$

$$M_{j,\,13} \;=\; \rho h \left[1 - \left(\frac{1}{2}\right) - \left(\frac{9}{5}\right) + 2 - \left(\frac{4}{7}\right) \right] \tag{141}$$

$$M_{j,\,13} \;=\; \rho h \left(\frac{9}{70} \right) \tag{142}$$

$$M_{j,\,13} \;=\; \rho h \left(\frac{54}{420} \right) \tag{143}$$

$$M_{j,\,14} = \rho h \int_0^1 \left\{ -3\xi^3 + 8\xi^4 - 7\xi^5 + 2\xi^6 \right\} d\xi \tag{139}$$

$$M_{j,\,14} = \rho h \left[-\left(\frac{3}{4}\right)\xi^4 + \left(\frac{8}{5}\right)\xi^5 - \left(\frac{7}{6}\right)\xi^6 + \left(\frac{2}{7}\right)\xi^7 \right]\Bigg|_0^1 \tag{140}$$

$$M_{j,\,14} = \rho h \left[-\left(\frac{3}{4}\right) + \left(\frac{8}{5}\right) - \left(\frac{7}{6}\right) + \left(\frac{2}{7}\right) \right] \tag{141}$$

$$M_{j,\,14} = \rho h \left(\frac{-13}{420}\right) \tag{142}$$

$$M_{j,\,22} = \rho h \int_0^1 \left\{ \xi^4 - 2\xi^5 + \xi^6 \right\} d\xi \tag{143}$$

$$M_{j,\,22} = \rho h \left[\left(\frac{1}{5}\right)\xi^5 - \left(\frac{1}{3}\right)\xi^6 + \left(\frac{1}{7}\right)\xi^7 \right]\Bigg|_0^1 \tag{144}$$

$$M_{j,\,22} = \rho h \left[\left(\frac{1}{5}\right) - \left(\frac{1}{3}\right) + \left(\frac{1}{7}\right) \right] \tag{145}$$

$$M_{j,\,22} = \rho h \left(\frac{1}{105}\right) \tag{146}$$

$$M_{j,\,22} = \rho h \left(\frac{4}{420}\right) \tag{147}$$

$$M_{j,\,23} = \rho h \int_0^1 \left\{ \xi^2 - \xi^3 - 3\xi^4 + 5\xi^5 - 2\xi^6 \right\} d\xi \tag{148}$$

$$M_{j,\,23} \;=\; \rho h \left[\left(\frac{1}{3}\right)\xi^3 - \left(\frac{1}{4}\right)\xi^4 - \left(\frac{3}{5}\right)\xi^5 + \left(\frac{5}{6}\right)\xi^6 - \left(\frac{2}{7}\right)\xi^7\right]\Bigg|_0^1 \tag{149}$$

$$M_{j,\,23} \;=\; \rho h \left[\left(\frac{1}{3}\right) - \left(\frac{1}{4}\right) - \left(\frac{3}{5}\right) + \left(\frac{5}{6}\right) - \left(\frac{2}{7}\right)\right] \tag{150}$$

$$M_{j,\,23} \;=\; \rho h \left(\frac{13}{420}\right) \tag{151}$$

$$M_{j,\,24} \;=\; \rho h \int_0^1 \left\{-\xi^3 + 3\xi^4 - 3\xi^5 + \xi^6\right\} d\xi \tag{152}$$

$$M_{j,\,24} \;=\; \rho h \left[-\left(\frac{1}{4}\right)\xi^4 + \left(\frac{3}{5}\right)\xi^5 - \left(\frac{1}{2}\right)\xi^6 + \left(\frac{1}{7}\right)\xi^7\right]\Bigg|_0^1 \tag{153}$$

$$M_{j,\,24} \;=\; \rho h \left[-\left(\frac{1}{4}\right) + \left(\frac{3}{5}\right) - \left(\frac{1}{2}\right) + \left(\frac{1}{7}\right)\right] \tag{154}$$

$$M_{j,\,24} \;=\; \rho h \left(\frac{-3}{420}\right) \tag{155}$$

$$M_{j,\,33} \;=\; \rho h \int_0^1 \left\{1 - 6\xi^2 + 4\xi^3 + 9\xi^4 - 12\xi^5 + 4\xi^6\right\} d\xi \tag{156}$$

$$M_{j,\,33} \;=\; \rho h \left[\xi - 2\xi^3 + \xi^4 + \left(\frac{9}{5}\right)\xi^5 - 2\xi^6 + \left(\frac{4}{7}\right)\xi^7\right]\Bigg|_0^1 \tag{157}$$

$$M_{j,\,33} \;=\; \rho h \left[1 - 2 + 1 + \left(\frac{9}{5}\right) - 2 + \left(\frac{4}{7}\right)\right] \tag{158}$$

$$M_{j,\ 33} = \rho h \left(\frac{156}{420}\right) \tag{159}$$

$$M_{j,\ 34} = \rho h \int_0^1 \left\{-\xi + 2\xi^2 + 2\xi^3 - 8\xi^4 + 7\xi^5 - 2\xi^6 \right\} d\xi \tag{160}$$

$$M_{j,\ 34} = \rho h \left[-\left(\frac{1}{2}\right)\xi^2 + \left(\frac{2}{3}\right)\xi^3 + \left(\frac{1}{2}\right)\xi^4 - \left(\frac{8}{5}\right)\xi^5 + \left(\frac{7}{6}\right)\xi^6 - \left(\frac{2}{7}\right)\xi^7 \right]\Bigg|_0^1 \tag{161}$$

$$M_{j,\ 34} = \rho h \left[-\left(\frac{1}{2}\right) + \left(\frac{2}{3}\right) + \left(\frac{1}{2}\right) - \left(\frac{8}{5}\right) + \left(\frac{7}{6}\right) - \left(\frac{2}{7}\right) \right] \tag{162}$$

$$M_{j,\ 34} = \rho h \left(\frac{-22}{420}\right) \tag{163}$$

$$M_{j,\ 44} = \rho h \int_0^1 \left\{\xi^2 - 4\xi^3 + 6\xi^4 - 4\xi^5 + \xi^6 \right\} d\xi \tag{164}$$

$$M_{j,\ 44} = \rho h \left[\left(\frac{1}{3}\right)\xi^3 - \xi^4 + \left(\frac{6}{5}\right)\xi^5 - \left(\frac{2}{3}\right)\xi^6 + \left(\frac{1}{7}\right)\xi^7 \right]\Bigg|_0^1 \tag{165}$$

$$M_{j,\ 44} = \rho h \left[\left(\frac{1}{3}\right) - 1 + \left(\frac{6}{5}\right) - \left(\frac{2}{3}\right) + \left(\frac{1}{7}\right) \right] \tag{166}$$

$$M_{j,\ 44} = \rho h \left(\frac{4}{420}\right) \tag{167}$$

Recall

$$K_j = \left(\frac{EI}{h^3}\right) \int_0^1 \left\{ \underline{L}'' \ \underline{L}''^{\,T} \right\} d\xi \tag{168}$$

$$M_j = h\rho \int_0^1 \left\{ \underline{L}\ \underline{L}^T \right\} d\xi \tag{169}$$

$$K_j = \left(\frac{EI}{h^3}\right) \begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ & & & 4 \end{bmatrix} \tag{170}$$

$$M_j = \left(\frac{h\rho}{420}\right) \begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & & 156 & -22 \\ & & & 4 \end{bmatrix} \tag{171}$$

<u>Example 1</u>

Model the cantilever beam in Figure 1 as a single element using the mass and stiffness matrices in equations (170) and (171). The model consists of one element and two nodes as shown in Figure 3.

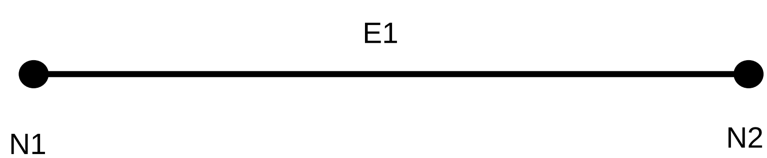

Figure 3.

Note that h=L.

The eigen problem is

$$\left(\frac{EI}{L^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix} = \left(\frac{L\rho}{420}\right)\omega^2 \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix} \tag{172}$$

$$\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix} = \lambda \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix} \tag{173}$$

where

$$\lambda = \left(\frac{L^4\rho}{420\ EI}\right)\omega^2 \tag{174a}$$

$$\omega = \left[\sqrt{\frac{420\ EI}{L^4\rho}}\right]\sqrt{\lambda} \tag{174b}$$

The boundary conditions at node 1 are

$$y_1 = 0 \tag{175}$$

$$\theta_1 = 0 \tag{176}$$

The first two columns and the first two rows of each matrix in equation (173) can thus be struck out.

The resulting eigen equation is thus

$$\begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} y_2 \\ h\,\theta_2 \end{bmatrix} = \lambda \begin{bmatrix} 156 & -22 \\ -22 & 4 \end{bmatrix}\begin{bmatrix} y_2 \\ h\,\theta_2 \end{bmatrix} \tag{177}$$

The eigenvalues are found using the method in Reference 2.

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.029715 \\ 2.8846 \end{bmatrix} \tag{178a}$$

$$\begin{bmatrix} \sqrt{\lambda_1} \\ \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 0.1724 \\ 1.6984 \end{bmatrix} \tag{178b}$$

The finite element results for the natural frequencies are thus

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \sqrt{\frac{420\ EI}{\rho L^4}} \begin{bmatrix} 0.1724 \\ 1.6984 \end{bmatrix} \tag{179}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \sqrt{\frac{EI}{\rho L^4}} \begin{bmatrix} 3.5331 \\ 34.807 \end{bmatrix} \tag{180}$$

The finite element results are compared to the classical results in Table 1.

Table 1. Natural Frequency Comparison, 1 Element

Index	Finite Element Model $\omega \sqrt{\frac{\rho L^4}{EI}}$	Classical Solution $\omega \sqrt{\frac{\rho L^4}{EI}}$
1	3.5331	3.5160
2	34.807	22.034

The classical results are taken from Reference 3. The finite element results thus over-predicted the natural frequencies. Nevertheless, good agreement is obtained for the first frequency.

Example 2

Model the cantilever beam in Figure 1 with two elements using the mass and stiffness matrices in equations (170) and (171). Let each element have equal length.

The model consists of two elements and three nodes as shown in Figure 4.

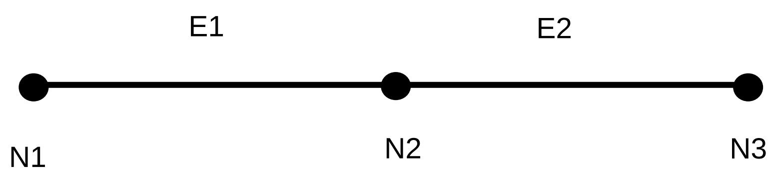

Figure 4.

There are several keys to this problem. One is that h=L/2. The other is that node N2 receives mass and stiffness contributions from both elements E1 and E2. Thus, the resulting global matrices have dimension 6 x 6 prior to the application of the boundary conditions.

The local stiffness matrix for element 1 is

$$\left(\frac{EI}{(L/2)^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\theta_1 \\ y_2 \\ h\theta_2 \end{bmatrix} \tag{181}$$

The displacement vector is also shown in equation (181) for reference.

The local stiffness matrix for element 2 is

$$\left(\frac{EI}{(L/2)^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} y_2 \\ h\theta_2 \\ y_3 \\ h\theta_3 \end{bmatrix} \tag{182}$$

The local mass matrix for element 1 is

$$\left(\frac{(L/2)\rho}{420}\right)\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\theta_1 \\ y_2 \\ h\theta_2 \end{bmatrix} \tag{183}$$

The local mass matrix for element 2 is

$$\left(\frac{(L/2)\rho}{420}\right)\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} y_2 \\ h\theta_2 \\ y_3 \\ h\theta_3 \end{bmatrix} \tag{184}$$

The global eigen problem assembled from the local matrices is

$$\left(\frac{EI}{(L/2)^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\theta_1 \\ y_2 \\ h\theta_2 \\ y_3 \\ h\theta_3 \end{bmatrix}$$

$$= \left(\frac{(L/2)\rho}{420}\right)\omega^2\begin{bmatrix} 156 & 22 & 54 & -13 & 0 & 0 \\ 22 & 4 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\theta_1 \\ y_2 \\ h\theta_2 \\ y_3 \\ h\theta_3 \end{bmatrix} \tag{185}$$

Again, the boundary conditions at node 1 are

$$y_1 = 0 \tag{186}$$

$$\theta_1 = 0 \tag{187}$$

The first two columns and the first two rows of each matrix in equation (185) can thus be struck out.

$$\left(\frac{EI}{(L/2)^3}\right)\begin{bmatrix} 24 & 0 & -12 & 6 \\ 0 & 8 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} y_2 \\ h\,\theta_2 \\ y_3 \\ h\,\theta_3 \end{bmatrix} = \left(\frac{(L/2)\rho}{420}\right)\omega^2 \begin{bmatrix} 312 & 0 & 54 & -13 \\ 0 & 8 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} y_2 \\ h\,\theta_2 \\ y_3 \\ h\,\theta_3 \end{bmatrix} \tag{188}$$

Let

$$\lambda = \frac{\left(\frac{(L/2)\rho}{420}\right)\omega^2}{\left(\frac{EI}{(L/2)^3}\right)} \tag{189}$$

$$\lambda = \left(\frac{(L/2)(L/2)^3\rho}{420EI}\right)\omega^2 \tag{190}$$

$$\lambda = \left(\frac{L^4\,\rho}{6720\,EI}\right)\omega^2 \tag{191a}$$

$$\omega = \left[\sqrt{\frac{6720\,EI}{L^4\rho}}\right]\sqrt{\lambda} \tag{191b}$$

The eigen problem in equation (188) becomes

$$\begin{bmatrix} 24 & 0 & -12 & 6 \\ 0 & 8 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} y_2 \\ h\,\theta_2 \\ y_3 \\ h\,\theta_3 \end{bmatrix} = \lambda\begin{bmatrix} 312 & 0 & 54 & -13 \\ 0 & 8 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} y_2 \\ h\,\theta_2 \\ y_3 \\ h\,\theta_3 \end{bmatrix} \tag{192}$$

The eigenvalues are found using the method in Reference 2. Equation (192) yields four eigenvalues.

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0.0018414 \\ 0.073481 \\ 0.84056 \\ 7.0810 \end{bmatrix} \tag{193a}$$

$$\begin{bmatrix} \sqrt{\lambda_1} \\ \sqrt{\lambda_2} \\ \sqrt{\lambda_3} \\ \sqrt{\lambda_4} \end{bmatrix} = \begin{bmatrix} 0.042912 \\ 0.27107 \\ 0.91682 \\ 2.6610 \end{bmatrix} \tag{193b}$$

The finite element results for the first two natural frequencies are thus

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \sqrt{\frac{6720\ EI}{\rho L^4}}\begin{bmatrix} 0.042912 \\ 0.27107 \end{bmatrix} \tag{194}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \sqrt{\frac{EI}{\rho L^4}}\begin{bmatrix} 3.5177 \\ 22.221 \end{bmatrix} \tag{195}$$

The finite element results are compared to the classical results in Table 2.

Table 2. Natural Frequency Comparison, 2 Elements		
Index	Finite Element Model $\omega\sqrt{\frac{\rho L^4}{EI}}$	Classical Solution $\omega\sqrt{\frac{\rho L^4}{EI}}$
1	3.5177	3.5160
2	22.221	22.034

Excellent agreement is obtained for the first two roots.

The next step would be to solve for the eigenvectors, which represent the mode shapes. A greater number of elements would be required to obtain accurate mode shapes, however.

Example 3

Repeat example 1 with 16 elements. Let each element have equal length. The global stiffness and mass matrix are omitted for brevity.

The eigenvalue scale factor is

$$\omega = \left[\sqrt{\frac{420\left(16^4\right) EI}{L^4 \rho}}\right]\sqrt{\lambda} \tag{196}$$

The model yields 32 eigenvalues. The first four are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 4.4913e-007 \\ 1.7639e-005 \\ 0.0001383 \\ 0.00053121 \end{bmatrix} \tag{197}$$

$$\begin{bmatrix} \sqrt{\lambda_1} \\ \sqrt{\lambda_2} \\ \sqrt{\lambda_3} \\ \sqrt{\lambda_4} \end{bmatrix} = \begin{bmatrix} 0.00067017 \\ 0.0041999 \\ 0.01176 \\ 0.023048 \end{bmatrix} \tag{198}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega3 \\ \omega4 \end{bmatrix} = \sqrt{\frac{420(16^4)EI}{\rho L^4}} \begin{bmatrix} 0.00067017 \\ 0.0041999 \\ 0.01176 \\ 0.023048 \end{bmatrix} \tag{199}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega3 \\ \omega4 \end{bmatrix} = \sqrt{\frac{EI}{\rho L^4}} \begin{bmatrix} 3.5160 \\ 22.035 \\ 61.698 \\ 120.92 \end{bmatrix} \tag{200}$$

The finite element results are compared to the classical results in Table 3.

Table 3. Natural Frequency Comparison, 16 Elements		
Index	Finite Element Model $\omega \sqrt{\frac{\rho L^4}{EI}}$	Classical Solution $\omega \sqrt{\frac{\rho L^4}{EI}}$
1	3.5160	3.5160
2	22.035	22.034
3	61.698	61.697
4	120.92	120.90

Excellent agreement is obtained for the first four roots.

The corresponding mode shapes are shown in Figures 5 through 8. Note that each mode shape is multiplied by an arbitrary amplitude scale factor. The absolute amplitude scale is thus omitted from the plots.

CANTILEVER BEAM, 16 ELEMENT MODEL, MODE 1

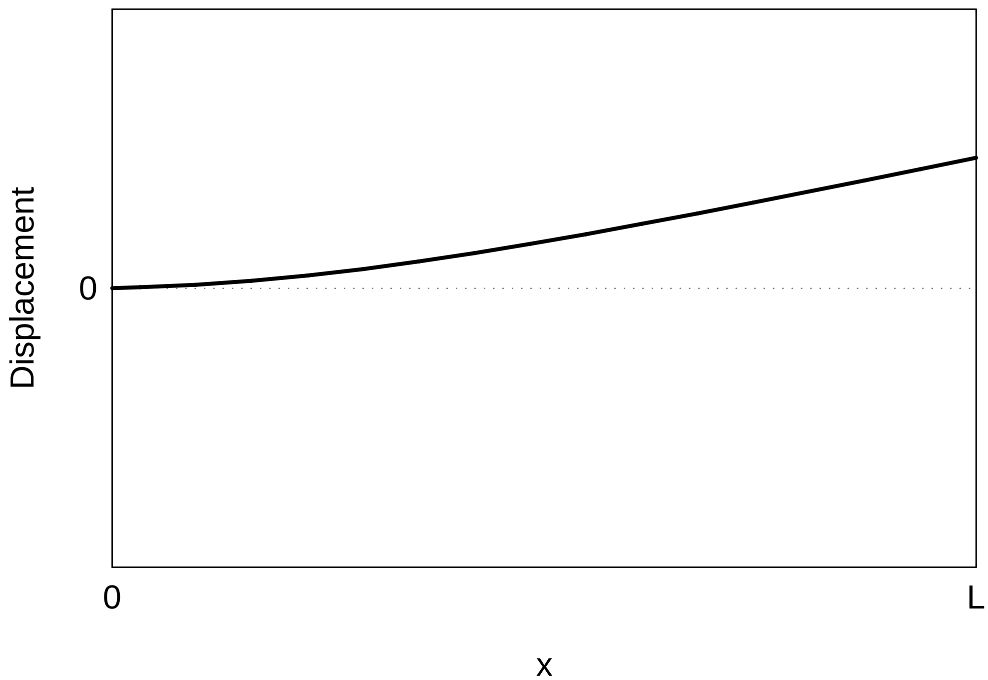

Figure 5.

CANTILEVER BEAM, 16 ELEMENT MODEL, MODE 2

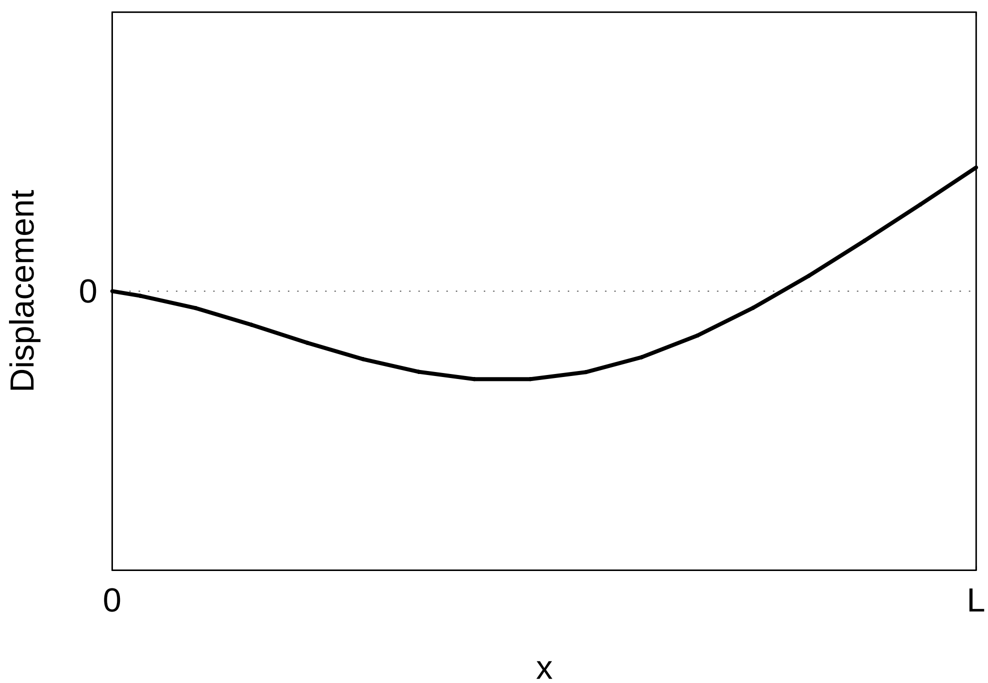

Figure 6.

CANTILEVER BEAM, 16 ELEMENT MODEL, MODE 3

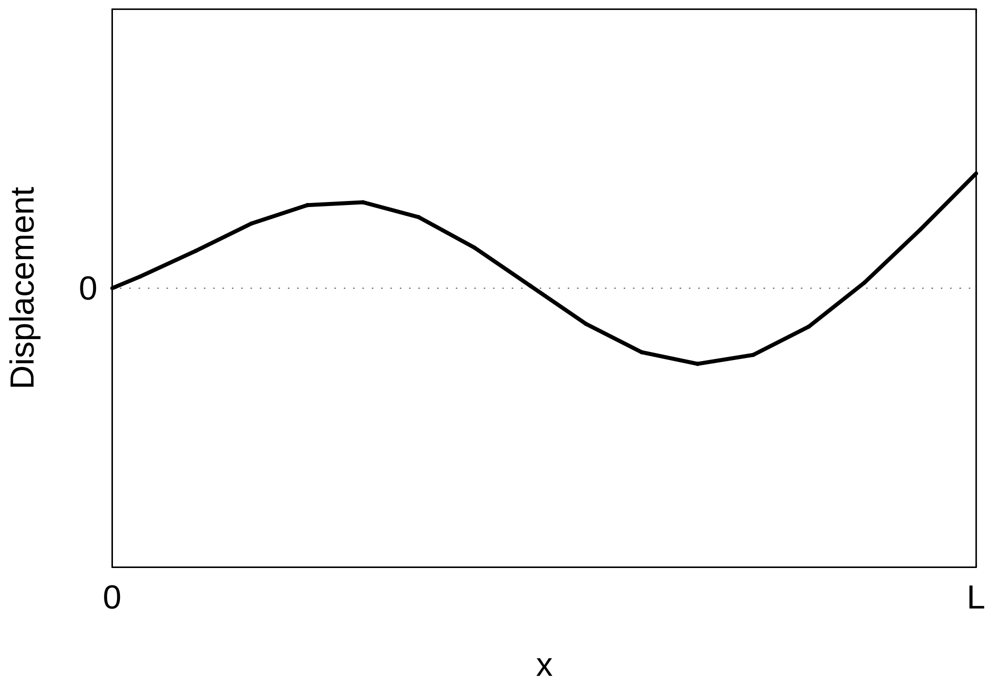

Figure 7.

CANTILEVER BEAM, 16 ELEMENT MODEL, MODE 4

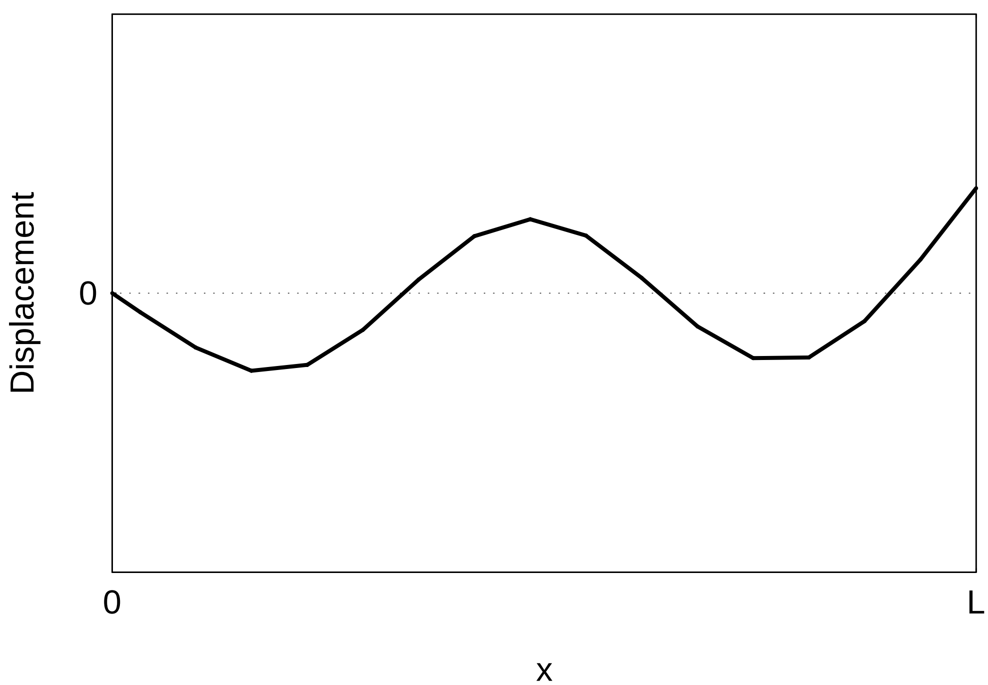

Figure 8.

Additional examples are given in Appendix C.

Reference

1. L. Meirovitch, Computational Methods in Structural Dynamics, Sijthoff & Noordhoff, The Netherlands, 1980.

2. T. Irvine, The Generalized Eigenvalue Problem, 1999.

3. W. Thomson, Theory of Vibration with Applications, Second Edition, Prentice-Hall, New Jersey, 1981.

4. K. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982.

APPENDIX A

Energy Method

The total strain energy P of a beam is

$$P = \frac{1}{2}\int_0^L EI \left(\frac{d^2y}{dx^2}\right)^2 dx \tag{A-1}$$

The total kinetic energy T of a beam is

$$T = \frac{1}{2}\omega_n{}^2 \int_0^L \rho \, [y]^2 dx \tag{A-2}$$

Again let

$$Y(x) = \underline{L}^{\,T} \quad \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{A-3}$$

$$\frac{d}{dx}Y(x) = \left(\frac{-1}{h}\right) \underline{L'}^{\,T}\, \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{A-4}$$

$$\frac{d^2}{dx^2}Y(x) = \left(\frac{1}{h^2}\right) \underline{L''}^{\,T} \quad \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{A-5}$$

$$d\xi = -dx/h \tag{A-6}$$

Assume constant mass density and stiffness.

The strain energy is converted to a localized stiffness matrix as

$$K_j = \left(\frac{EI}{h^3}\right) \int_0^1 \left\{ \underline{L''} \;\; \underline{L''}^{\,T} \right\} d\xi \tag{A-7}$$

The kinetic energy is converted to a localized mass matrix as

$$M_j = h\rho \int_0^1 \left\{ \underline{L}\, \underline{L}^T \right\} d\xi \tag{A-8}$$

The total strain energy is set equal to the total kinetic energy per the Rayleigh method. The result is a generalized eigenvalue problem.

For a system of n elements,

$$K_j - \omega^2 M_j = 0 , \qquad j = 1, 2, \ldots, n \tag{A-9}$$

where

$$K_j = \left(\frac{EI}{h^3}\right) \begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ & & & 4 \end{bmatrix} \tag{A-10}$$

$$M_j = \left(\frac{h\rho}{420}\right) \begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & & 156 & -22 \\ & & & 4 \end{bmatrix} \tag{A-11}$$

APPENDIX B

<u>Beam Bending - Alternate Matrix Format</u>

The displacement matrix for beam bending is

$$\begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix} \tag{B-1}$$

The stiffness matrix for beam bending is

$$K_j = \left(\frac{EI}{h^3}\right)\begin{bmatrix} 12 & 6h & -12 & 6h \\ & 4h^2 & -6h & 2h^2 \\ & & 12 & -6h \\ & & & 4h^2 \end{bmatrix} \tag{B-2}$$

The mass matrix for beam bending is

$$M_j = \left(\frac{h\rho}{420}\right)\begin{bmatrix} 156 & 22h & 54 & -13h \\ & 4h^2 & 13 & -3h^2 \\ & & 156 & -22h \\ & & & 4h^2 \end{bmatrix} \tag{B-3}$$

APPENDIX C

Free-Free Beam

Repeat example 1 from the main text with a single element but with free-free boundary conditions.

The eigen problem is

$$\left(\frac{EI}{L^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix} = \left(\frac{L\rho}{420}\right)\omega^2 \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix}$$

(C-1)

$$\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix} = \lambda \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix}$$

(C-2)

where

$$\lambda = \left(\frac{L^4\rho}{420\ EI}\right)\omega^2 \tag{C-3}$$

$$\omega = \left[\sqrt{\frac{420\ EI}{L^4\rho}}\right]\sqrt{\lambda} \tag{C-4}$$

The eigenvalues are found using the method in Reference 2. Equation (C-1) yields four eigenvalues.

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.7143 \\ 20 \end{bmatrix} \tag{C-5}$$

$$\begin{bmatrix} \sqrt{\lambda_1} \\ \sqrt{\lambda_2} \\ \sqrt{\lambda_3} \\ \sqrt{\lambda_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.3093 \\ 4.4721 \end{bmatrix} \tag{C-6}$$

The finite element results for the first four natural frequencies are thus

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega 3 \\ \omega 4 \end{bmatrix} = \sqrt{\frac{420\ EI}{\rho L^4}} \begin{bmatrix} 0 \\ 0 \\ 1.3093 \\ 4.4721 \end{bmatrix} \tag{C-7}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega 3 \\ \omega 4 \end{bmatrix} = \sqrt{\frac{EI}{\rho L^4}} \begin{bmatrix} 0 \\ 0 \\ 26.833 \\ 91.652 \end{bmatrix} \tag{C-8}$$

The finite element results are compared to the classical results in Table C-1.

Table C-1. Natural Frequency Comparison, 2 Elements		
Index	Finite Element Model $\omega\sqrt{\frac{\rho L^4}{EI}}$	Classical Solution $\omega\sqrt{\frac{\rho L^4}{EI}}$
1	0	0
2	0	0
3	26.833	22.373
4	91.652	61.673