TRANSVERSE VIBRATION OF A FIXED-FIXED BEAM SUBJECTED TO A CONSTANT AXIAL LOAD VIA THE FINITE ELEMENT METHOD

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Theory

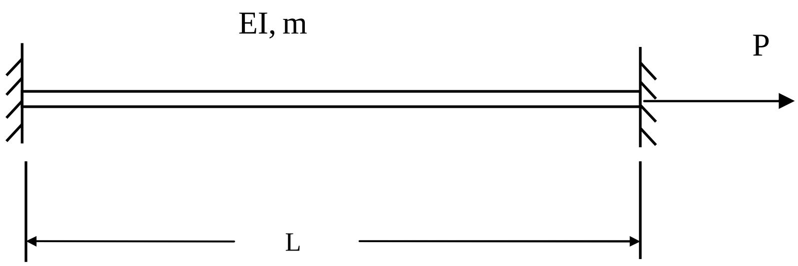

Figure 1.

The governing differential equation for the transverse displacement y(x, t) of a fixed-fixed beam subject to an axial load applied at its free end is

$$\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{d^2}{\partial x^2}y(x,t)\right\}+\frac{\partial}{\partial x}\left[P\frac{\partial}{\partial x}y(x,t)\right]+m\frac{\partial^2 y(x,t)}{\partial t^2}=0 \tag{1}$$

where

E	is the modulus of elasticity
I	is the area moment of inertia
m	is the mass per length
L	is the length
P	is the axial tension load

Equation (1) is taken from Reference 1.

Assume that the load P is constant.

$$\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{d^2}{\partial x^2}y(x,t)\right\} + P\,\frac{d^2}{\partial x^2}\,y(x,t) + m\,\frac{\partial^2\,y(x,t)}{\partial t^2} = 0 \tag{2}$$

The product EI is the bending stiffness.

(8)

Equation (8) yields two independent equations.

$$\frac{d^2}{dx^2}\left\{EI\frac{d^2}{dx^2}Y(x)\right\} + P\left[\frac{d^2}{dx^2}Y(x)\right] - m\,\omega^2\,Y(x) = 0 \tag{9}$$

$$\frac{d^2}{dt^2}\,f(t) + \omega^2 f(t) = 0 \tag{10}$$

Equation (9) is a homogeneous, forth order, ordinary differential equation.

The weighted residual method is applied to equation (9). This method is suitable for boundary value problems. An alternative method would be the energy method.

There are numerous techniques for applying the weighted residual method. Specifically, the Galerkin approach is used in this tutorial.

The differential equation (9) is multiplied by a test function $\phi(x)$. Note that the test function $\phi(x)$ must satisfy the homogeneous essential boundary conditions. The essential boundary conditions are the prescribed values of Y and its first derivative.

The test function is not required to satisfy the differential equation, however.

The product of the test function and the differential equation is integrated over the domain. The integral equation is set to zero.

Final Assembly of Mass and Stiffness Matrices

The elemental mass and stiffness matrices are taken from References 3 and 4.

$$M_j = \left(\frac{h\,m}{420}\right)\begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & & 156 & -22 \\ & & & 4 \end{bmatrix} \tag{36}$$

$$K_j = \left(\frac{EI}{h^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ & & & 4 \end{bmatrix} + \frac{P}{h}\left(\frac{1}{30}\right)\begin{bmatrix} 36 & 3 & -36 & 3 \\ & 4 & -3 & -1 \\ & & 36 & -3 \\ & & & 4 \end{bmatrix} \tag{37}$$

An example is given in Appendix B.

References

1. L. Meirovitch, Analytical Methods in Vibration, Macmillan, New York, 1967.
2. L. Meirovitch, Computational Methods in Structural Dynamics, Sijthoff & Noordhoff, The Netherlands, 1980.
3. T. Irvine, Transverse Vibration of a Beam via the Finite Element Method, Revision A, Vibrationdata, 2000.

4. T. Irvine, Transverse Vibration of a Cantilever Beam Subjected to a Constant Axial Load via the Finite Element Method, Vibrationdata, 2003.

5. T. Irvine, Natural Frequencies of Beams Subjected to a Uniform Axial Load, Revision A, Vibrationdata, 2003.

6. T. Irvine, Bending Frequencies of Beam, Rods, and Pipes, Revision J, Vibrationdata, 2003.

APPENDIX A

Example 1

Model the fixed-fixed beam in Figure A-1 as two elements using the mass and stiffness matrices in equations 36 and 37. The model consists of two elements and three nodes as shown in Figure B-1.

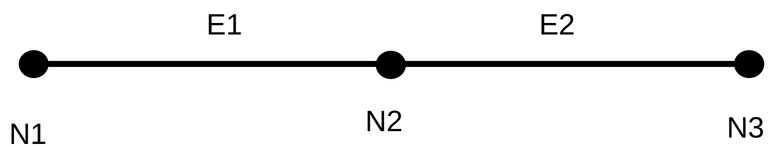

Figure A-1.

Note that h = L/2.

The mass matrix is

$$\underline{M} = \left(\frac{L\,m}{840}\right)\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix} \tag{A-1}$$

The stiffness matrix is

$$K_j = \left(\frac{EI}{h^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ & & & 4 \end{bmatrix}$$

$$+\frac{P}{h}\left(\frac{1}{30}\right)\begin{bmatrix} 36 & 3 & -36 & 3 \\ & 4 & -3 & -1 \\ & & 36 & -3 \\ & & & 4 \end{bmatrix} \tag{A-2}$$

The buckling load for a fixed-fixed beam is

$$P_{cr} = \frac{4\pi^2 EI}{L^2} \tag{A-3}$$

Let

$$P = 0.4\, P_{cr} \tag{A-4}$$

$$P = \frac{1.6\,\pi^2 EI}{L^2} \tag{A-5}$$

$$P = \frac{8\pi^2 EI}{5L^2} \tag{A-6}$$

$$K_j = \left(\frac{EI}{(L/2)^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$+ \frac{8\pi^2 EI}{5L^2}\left(\frac{1}{(L/2)}\right)\left(\frac{1}{30}\right)\begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix} \tag{A-7}$$

$$K_j = \left(\frac{8EI}{L^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$+\frac{8\pi^2}{75}\left(\frac{EI}{L^3}\right)\begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix} \tag{A-8}$$

$$K_j = \left(\frac{EI}{L^3}\right)\begin{bmatrix} 133.9 & 51.16 & -133.9 & 51.16 \\ 51.16 & 36.21 & -51.16 & 14.95 \\ -133.9 & -51.16 & 133.9 & -51.16 \\ 51.16 & 14.95 & -51.16 & 36.21 \end{bmatrix} \tag{A-9}$$

$$\left(\frac{EI}{L^3}\right)\begin{bmatrix} 133.9 & 51.16 & -133.9 & 51.16 & 0 & 0 \\ 51.16 & 36.21 & -51.16 & 14.95 & 0 & 0 \\ -133.9 & -51.16 & 267.8 & 0 & -133.9 & 51.16 \\ 51.16 & 14.95 & 0 & 72.42 & -51.16 & 14.95 \\ 0 & 0 & -133.9 & -51.16 & 133.9 & -51.16 \\ 0 & 0 & 51.16 & 14.95 & -51.16 & 36.21 \end{bmatrix}\begin{bmatrix} y_1 \\ h\theta_1 \\ y_2 \\ h\theta_2 \\ y_3 \\ h\theta_3 \end{bmatrix}$$

$$= \omega^2\left(\frac{Lm}{840}\right)\begin{bmatrix} y_1 \\ h\theta_1 \\ y_2 \\ h\theta_2 \\ y_3 \\ h\theta_3 \end{bmatrix}\begin{bmatrix} 156 & 22 & 54 & -13 & 0 & 0 \\ 22 & 4 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix} \tag{A-10}$$

$$\begin{bmatrix} 133.9 & 51.16 & -133.9 & 51.16 & 0 & 0 \\ 51.16 & 36.21 & -51.16 & 14.95 & 0 & 0 \\ -133.9 & -51.16 & 267.8 & 0 & -133.9 & 51.16 \\ 51.16 & 14.95 & 0 & 72.42 & -51.16 & 14.95 \\ 0 & 0 & -133.9 & -51.16 & 133.9 & -51.16 \\ 0 & 0 & 51.16 & 14.95 & -51.16 & 36.21 \end{bmatrix} \begin{bmatrix} y_1 \\ h\theta_1 \\ y_2 \\ h\theta_2 \\ y_3 \\ h\theta_3 \end{bmatrix}$$

$$= \quad \omega^2 \left(\frac{L^4 m}{840 EI} \right) \begin{bmatrix} y_1 \\ h\theta_1 \\ y_2 \\ h\theta_2 \\ y_3 \\ h\theta_3 \end{bmatrix} \begin{bmatrix} 156 & 22 & 54 & -13 & 0 & 0 \\ 22 & 4 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix}$$

(A-11)

where

$$\lambda = \left(\frac{L^4 m}{840\ EI} \right) \omega^2 \qquad \text{(A-12)}$$

$$\omega = \left[\sqrt{\frac{840\ EI}{L^4 m}} \right] \sqrt{\lambda} \qquad \text{(A-13)}$$

The boundary conditions are

$$y_1 = 0 \qquad \text{(A-14a)}$$

$$h\,\theta_1 = 0 \qquad \text{(A-14b)}$$

$$y_3 = 0 \qquad \text{(A-15a)}$$

$$h\,\theta_3 = 0 \qquad \text{(A-15b)}$$

The first and second rows and columns are struck out to meet the first boundary condition. The fifth and sixth rows and columns are struck out to meet the second boundary condition.

The resulting eigen equation is thus

$$\begin{bmatrix} 267.8 & 0 \\ 0 & 72.42 \end{bmatrix} \begin{bmatrix} y_2 \\ h\theta_2 \end{bmatrix} = \lambda \begin{bmatrix} 312 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} y_2 \\ h\theta_2 \end{bmatrix} \tag{A-17}$$

The eigenvalues are found using the method in Reference 2.

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.8583 \\ 9.053 \end{bmatrix} \tag{A-18}$$

$$\begin{bmatrix} \sqrt{\lambda_1} \\ \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 0.9264 \\ 3.009 \end{bmatrix} \tag{A-19}$$

The finite element results for the natural frequencies are thus

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \sqrt{\frac{840\ EI}{m L^4}} \begin{bmatrix} 0.9264 \\ 3.009 \end{bmatrix} \tag{A-20}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \sqrt{\frac{EI}{m L^4}} \begin{bmatrix} 26.851 \\ 87.204 \end{bmatrix} \tag{A-21}$$

The finite element results are compared to the classical results in Table B-1.

Table B-1. P = 0.4 Pcr Case, Natural Frequency Comparison, 2 Elements		
Mode	Finite Element Model $\omega\sqrt{\frac{mL^4}{EI}}$	Classical Solution $\omega\sqrt{\frac{mL^4}{EI}}$
1	26.851	26.466

The finite element value is 1.45 % higher than the classical solution. The classical result is taken from Reference 4.

Note that $\omega\sqrt{\frac{\rho L^4}{EI}} = 22.37$ for the case where P = 0, per the classical solution in Reference 6.

The analysis is repeated for other model sizes, representing the same beam, in Table B-2.

Table B-2. P = 0.4 Pcr Case, Fundamental Frequency for Various Model Sizes			
Elements in Model	Finite Element Model $\omega\sqrt{\frac{mL^4}{EI}}$	Classical Solution $\omega\sqrt{\frac{mL^4}{EI}}$	Error
2	26.851	26.466	1.45 %
4	26.360	26.466	-0.40 %
8	26.312	26.466	-0.58 %
16	26.500	26.466	0.13 %