VIBRATION OF A BEAM-COLUMN VIA THE FINITE ELEMENT METHOD Rev A

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Introduction

Consider a beam-column as shown in Figure 1.

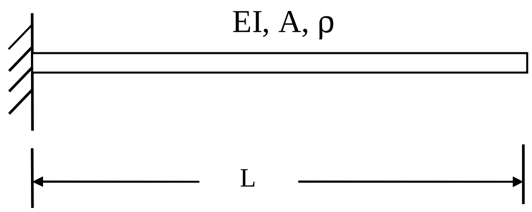

Figure 1.

E	is the modulus of elasticity
I	is the area moment of inertia
A	is the cross-sectional area
L	is the length
ρ	is mass per length

The product EI is the bending stiffness.

Let u(x) be the longitudinal displacement. Let y(x) be the transverse displacement.

Assume a linear problem with small displacements such that there is no coupling or interaction between the axial and bending effects.

Determine the mass and stiffness matrices of the beam-column. Note that only the upper triangular components are shown in the following matrices due to symmetry.

Beam Bending

The beam bending matrices are taken from Reference 1. The displacement matrix for beam bending is

$$\begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix} \tag{1}$$

The stiffness matrix for beam bending is

$$K_j = \left(\frac{EI}{h^3}\right)\begin{bmatrix} 12 & 6h & -12 & 6h \\ & 4h^2 & -6h & 2h^2 \\ & & 12 & -6h \\ & & & 4h^2 \end{bmatrix} \tag{2}$$

The mass matrix for beam bending is

$$M_j = \left(\frac{h\rho}{420}\right)\begin{bmatrix} 156 & 22h & 54 & -13h \\ & 4h^2 & 13h & -3h^2 \\ & & 156 & -22h \\ & & & 4h^2 \end{bmatrix} \tag{3}$$

Column Displacement

The column displacement matrices are taken from Reference 2. The displacement vector is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4}$$

The stiffness matrix for a column is

$$K_j = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \tag{5}$$

The mass matrix for a column is

$$M_j = \frac{hm}{6}\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} \tag{6}$$

The beam bending and column matrices can be combined into respective beam-column matrices using the method in Reference 3.

<u>Beam-Column</u>

The elemental stiffness matrix for the beam-column is

$$K_j = \left(\frac{E}{h^3}\right)\begin{bmatrix} 12I & 6Ih & -12I & 6Ih & 0 & 0 \\ & 4Ih^2 & -6Ih & 2Ih^2 & 0 & 0 \\ & & 12I & -6Ih & 0 & 0 \\ & & & 4Ih^2 & 0 & 0 \\ & & & & Ah^2 & -Ah^2 \\ & & & & & Ah^2 \end{bmatrix} \tag{7}$$

The elemental mass matrix for the beam-column is

$$M_j = \left(\frac{h\rho}{420}\right)\begin{bmatrix} 156 & 22h & 54 & -13h & 0 & 0 \\ & 4h^2 & 13h & -3h^2 & 0 & 0 \\ & & 156 & -22h & 0 & 0 \\ & & & 4h^2 & 0 & 0 \\ & & & & 140 & 70 \\ & & & & & 140 \end{bmatrix} \tag{8}$$

There are three degrees-of-freedom at each node. The elemental displacement vector is

$$\begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \\ u_1 \\ u_2 \end{bmatrix} \tag{9}$$

Example 1

Model the cantilever beam in Figure 1 as a single element using the mass and stiffness matrices in equations (7) and (8). The model consists of one element and two nodes as shown in Figure 2.

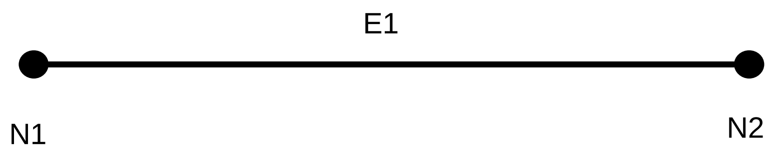

Figure 2.

Note that h=L.

Assume that the beam is aluminum in the form of a solid cylinder, with a diameter of 6 inch and a length of 120 inch.

$$\begin{aligned} E &= 1e{+}07 \text{ lbf/in}^2 \\ I &= 63.62 \text{ in}^4 \\ A &= 28.27 \text{ in}^2 \\ L &= 120 \text{ in} \\ \rho &= 2.827 \text{ lbm/in} = 0.00732 \text{ lbf sec}^2/\text{in}^2 \end{aligned}$$

Let

$$c = \sqrt{\frac{EA}{\rho}} \tag{10}$$

The longitudinal wave speed for the aluminum beam is

$$c = 196{,}500 \text{ in/sec} \tag{11}$$

The generalized eigenvalue problem prior to the application of the boundary conditions is

$$\left\{ \left(\frac{E}{h^3}\right) \begin{bmatrix} 12I & 6Ih & -12I & 6Ih & 0 & 0 \\ & 4Ih^2 & -6Ih & 2Ih^2 & 0 & 0 \\ & & 12I & -6Ih & 0 & 0 \\ & & & 4Ih^2 & 0 & 0 \\ & & & & Ah^2 & -Ah^2 \\ & & & & & Ah^2 \end{bmatrix} \right.$$

$$\left. -\omega^2 \left(\frac{h\rho}{420}\right) \begin{bmatrix} 156 & 22h & 54 & -13h & 0 & 0 \\ & 4h^2 & 13h & -3h^2 & 0 & 0 \\ & & 156 & -22h & 0 & 0 \\ & & & 4h^2 & 0 & 0 \\ & & & & 140 & 70 \\ & & & & & 140 \end{bmatrix} \right\} \begin{Bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \\ u_1 \\ u_2 \end{Bmatrix} = 0 \tag{12}$$

$$
\left\{
\begin{bmatrix}
12I & 6Ih & -12I & 6Ih & 0 & 0 \\
 & 4Ih^2 & -6Ih & 2Ih^2 & 0 & 0 \\
 & & 12I & -6Ih & 0 & 0 \\
 & & & 4Ih^2 & 0 & 0 \\
 & & & & Ah^2 & -Ah^2 \\
 & & & & & Ah^2
\end{bmatrix}
\right.
$$

$$
\left.
-\omega^2\left(\frac{h^4\rho}{420E}\right)
\begin{bmatrix}
156 & 22h & 54 & -13h & 0 & 0 \\
 & 4h^2 & 13h & -3h^2 & 0 & 0 \\
 & & 156 & -22h & 0 & 0 \\
 & & & 4h^2 & 0 & 0 \\
 & & & & 140 & 70 \\
 & & & & & 140
\end{bmatrix}
\right\}
\begin{Bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \\ u_1 \\ u_2 \end{Bmatrix} = 0
\tag{13}
$$

$$
\left\{
\begin{bmatrix}
12I & 6Ih & -12I & 6Ih & 0 & 0 \\
 & 4Ih^2 & -6Ih & 2Ih^2 & 0 & 0 \\
 & & 12I & -6Ih & 0 & 0 \\
 & & & 4Ih^2 & 0 & 0 \\
 & & & & Ah^2 & -Ah^2 \\
 & & & & & Ah^2
\end{bmatrix}
\right.
$$

$$
\left.
-\left(\frac{\omega}{c}\right)^2\left(\frac{A\,h^4}{420}\right)
\begin{bmatrix}
156 & 22h & 54 & -13h & 0 & 0 \\
 & 4h^2 & 13h & -3h^2 & 0 & 0 \\
 & & 156 & -22h & 0 & 0 \\
 & & & 4h^2 & 0 & 0 \\
 & & & & 140 & 70 \\
 & & & & & 140
\end{bmatrix}
\right\}
\begin{Bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \\ u_1 \\ u_2 \end{Bmatrix} = 0
\tag{14}
$$

Apply the boundary conditions.

$$y_1 = 0 \tag{15}$$

$$\theta_1 = 0 \tag{16}$$

$$u_1 = 0 \tag{17}$$

Omit the corresponding rows and columns in the eigenvalue problem.

$$\left\{ \begin{bmatrix} 12I & -6Ih & 0 \\ & 4Ih^2 & 0 \\ & & Ah^2 \end{bmatrix} - \left(\frac{\omega}{c}\right)^2 \left(\frac{Ah^4}{420}\right) \begin{bmatrix} 156 & -22h & 0 \\ & 4h^2 & 0 \\ & & 140 \end{bmatrix} \right\} \begin{Bmatrix} y_2 \\ \theta_2 \\ u_2 \end{Bmatrix} = 0 \tag{18}$$

$$\left\{ \begin{bmatrix} 763.4 & -4.581e+04 & 0 \\ & 3.665e+06 & 0 \\ & & 4.704e+05 \end{bmatrix} - \left(\frac{\omega}{c}\right)^2 \left(\frac{Ah^4}{420}\right) \begin{bmatrix} 156 & -2640 & 0 \\ & 5.760e+04 & 0 \\ & & 140 \end{bmatrix} \right\} \begin{Bmatrix} y_2 \\ \theta_2 \\ u_2 \end{Bmatrix} = 0 \tag{19}$$

$$\begin{bmatrix} \omega_1{}^2 \\ \omega_2{}^2 \\ \omega_3{}^2 \end{bmatrix} = \frac{420\,c^2}{A\,h^4} \begin{bmatrix} 1.8899 \\ 183.54 \\ 3360 \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 72.3 \\ 712.5 \\ 3048.6 \end{bmatrix} \text{ rad/sec} \tag{21}$$

$$
\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 11.5 \\ 113.4 \\ 485.2 \end{bmatrix} \text{ Hz} \tag{22}
$$

The finite element results are compared to the classical results in Table 1.

Table 1. Finite Element Results for Example 1

Index	Mode Shape	Finite Element Model fn (Hz)	Classical Solution fn (Hz)	Error
1	First Bending	11.5	11.5	0.0%
2	Second Bending	113.4	71.8	57.9%
3	First Longitudinal	485.2	409.3	18.5%

Example 2

Model the cantilever beam in Example 1 with two elements. Let each element have equal length.

The model consists of two elements and three nodes as shown in Figure 3.

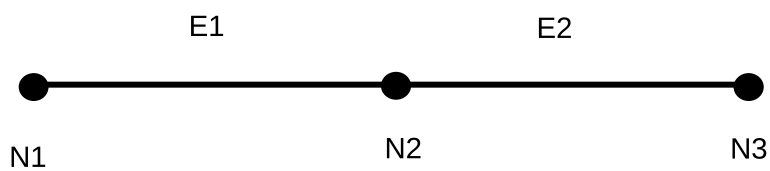

Figure 3.

There are several keys to this problem. One is that h=L/2. The other is that node N2 receives mass and stiffness contributions from both elements E1 and E2. Thus, the resulting global matrices have dimension 6 x 6 prior to the application of the boundary conditions.

Furthermore, rearrange the displacement vector as

$$\begin{bmatrix} y_1 \\ \theta_1 \\ u_1 \\ y_2 \\ \theta_2 \\ u_2 \end{bmatrix} \tag{23}$$

The elemental stiffness matrix for the beam-column with the modified displacement vector is

$$\hat{K}_j = \left(\frac{E}{h^3}\right)\begin{bmatrix} 12I & 6Ih & 0 & -12I & 6Ih & 0 \\ & 4Ih^2 & 0 & -6Ih & 2Ih^2 & 0 \\ & & Ah^2 & 0 & 0 & -Ah^2 \\ & & & 12I & -6Ih & 0 \\ & & & & 4Ih^2 & 0 \\ & & & & & Ah^2 \end{bmatrix} \tag{24}$$

The elemental mass matrix for the beam-columns with the modified displacement vector is

$$\hat{M}_j = \left(\frac{h\rho}{420}\right)\begin{bmatrix} 156 & 22h & 0 & 54 & -13h & 0 \\ & 4h^2 & 0 & 13h & -3h^2 & 0 \\ & & 140 & 0 & 0 & 70 \\ & & & 156 & -22h & 0 \\ & & & & 4h^2 & 0 \\ & & & & & 140 \end{bmatrix} \tag{25}$$

The stiffness matrix for element 1 in example 2 is

$$\hat{K}_1 =$$

$$\begin{bmatrix} 3.534e+04 & 1.060e+06 & 0 & -3.534e+04 & 1.060e+06 & 0 \\ & 4.241e+07 & 0 & -1.060e+06 & 2.121e+07 & 0 \\ & & 4.712e+06 & 0 & 0 & -4.712e+06 \\ & & & 3.534e+04 & -1.060e+06 & 0 \\ & & & & 4.241e+07 & 0 \\ & & & & & 4.712e+06 \end{bmatrix}$$

(26)

The stiffness matrix for element 2 in example 2 is the same as the matrix in equation (26).

The mass matrix for element 1 in example 2 is

$$\hat{M}_1 = \begin{bmatrix} 0.1631 & 1.38 & 0 & 0.05647 & -0.8157 & 0 \\ & 15.06 & 0 & 0.8157 & -11.29 & 0 \\ & & 0.1464 & 0 & 0 & 0.0732 \\ & & & 0.1631 & -1.38 & 0 \\ & & & & 15.06 & 0 \\ & & & & & 0.1464 \end{bmatrix}$$

(27)

The mass matrix for element 2 in example 2 is the same as the matrix in equation (27).

The generalized eigenvalue problem after the boundary conditions are applied is

$$\left\{\begin{bmatrix} 7.069e+04 & 0 & 0 & -3.534e+04 & 1.060e+06 & 0 \\ & 8.483e+07 & 0 & -1.060e+06 & 2.121e+07 & 0 \\ & & 9.423e+06 & 0 & 0 & -4.712e+06 \\ & & & 3.534e+04 & -1.060e+06 & 0 \\ & & & & 4.241e+07 & 0 \\ & & & & & 4.712e+06 \end{bmatrix}\right.$$

$$\left. -\omega^2 \begin{bmatrix} 0.3263 & 0 & 0 & 0.05647 & -0.8157 & 0 \\ & 30.12 & 0 & 0.8157 & -11.29 & 0 \\ & & 0.2928 & 0 & 0 & 0.0732 \\ & & & 0.1631 & -1.38 & 0 \\ & & & & 15.06 & 0 \\ & & & & & 0.1464 \end{bmatrix}\right\} \begin{Bmatrix} y_2 \\ \theta_2 \\ u_2 \\ y_3 \\ \theta_3 \\ u_3 \end{Bmatrix} = 0 \tag{28}$$

The natural frequencies are found via the Jacobi method.

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \end{bmatrix} = \begin{bmatrix} 71.789 \\ 454.52 \\ 1538.4 \\ 2638.5 \\ 4458.5 \\ 9217.3 \end{bmatrix} \text{ rad/sec} \tag{29}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} 11.43 \\ 72.3 \\ 244.8 \\ 419.9 \\ 709.6 \\ 1467.0 \end{bmatrix} \text{ Hz} \tag{30}$$

Table 2. Finite Element Results for Example 2

Index	Mode Shape	Finite Element Model fn (Hz)	Classical Solution fn (Hz)	Error
1	First Bending	11.5	11.5	0.0 %
2	Second Bending	72.3	71.8	0.7 %
3	Third Bending	244.8	201.0	21.8 %
4	First Longitudinal	419.9	409.3	2.6 %
5	Fourth Bending	709.6	394.5	79.9 %
6	Second Longitudinal	1467	1228	19.5 %

The next step would be to solve for the eigenvectors, which represent the mode shapes. A greater number of elements would be required to obtain accurate mode shapes, however.

References

1. T. Irvine, Transverse Vibration of a Beam via the Finite Element Method, Revision D, Vibrationdata, 2004.
2. T. Irvine, Longitudinal Vibration of a Rod via the Finite Element Method, Revision A, Vibrationdata, 2004.
3. C. Desai, Elementary Finite Element Method, Prentice-Hall, New Jersey, 1979.