CIRCULAR ORBIT ABOUT A CENTRAL BODY Revision A

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Consider two bodies, such as the Earth and a spacecraft.

Assume

1. The spacecraft is in a circular orbit about the Earth.
2. The spacecraft mass is negligible compared to the Earth's mass.
3. The only force acting on the spacecraft is gravity.
4. The Earth is a perfect sphere.
5. Neglect the Earth's rotation.

Calculate the velocity required for the spacecraft to maintain a constant altitude above the Earth.

Let v be the tangential velocity of the spacecraft. Let r be the radius from the center of the Earth to the spacecraft. The spacecraft has centripetal acceleration $v^2/r$, directed radially inward toward the Earth. The spacecraft falls to the Earth with this acceleration due to the unbalanced gravitational force. This observation is made with respect to the Earth's inertial reference frame.

Consider an astronaut inside the spacecraft. The astronaut is in a separate reference frame which is characterized as follows:

1. The reference frame is a rotating frame.
2. The reference frame is an accelerating frame.
3. The reference frame is a noninertial frame.

Furthermore, the astronaut considers the spacecraft to be at rest.

Thus the astronaut must introduce a pseudo force of magnitude $mv^2/r$ acting radially outward in order to balance the gravitational force. This fictional force is called the "centrifugal force."

Newton's law of gravitation states that the magnitude of the force F acting between two masses is

$$F = \frac{G\, m_1\, m_2}{r^2} \tag{1}$$

where

the two masses are given by $m_1$ and $m_2$,
r is the distance from one center of mass to the other center of mass.

Note that G is a universal constant having the same value for all pairs of particles.

$$G = 6.6720\left(10^{-11}\right) \frac{Nm^2}{kg^2} \tag{2}$$

Now let $m_1$ be the spacecraft mass and $m_2$ be the Earth's mass.

Newton's second law is

$$\sum F = m_1 a \tag{3}$$

where a is the acceleration.

Substitute the centrifugal acceleration into the right hand side of equation (3).

$$\sum F = \frac{m_1 v^2}{r} \tag{4}$$

$$\frac{G m_1 m_2}{r^2} = \frac{m_1 v^2}{r} \tag{5}$$

Simplifying,

$$\frac{G m_2}{r} = v^2 \tag{6}$$

$$v^2 = \frac{G m_2}{r} \tag{7}$$

$$v = \sqrt{\frac{G m_2}{r}} \tag{8}$$

Now let,

$$r = r_e + x \tag{9}$$

where

$r_e$ is the Earth’s radius
x is the altitude above the Earth’s surface.

By substitution,

$$v = \sqrt{\frac{G\,m_2}{r_e + x}} \tag{10}$$

Let

$$\mu = G\,m_2 \tag{11}$$

The spacecraft velocity equation becomes

$$v = \sqrt{\frac{\mu}{r_e + x}} \tag{12}$$

The value of $\mu$ for the Earth is

$$\mu = \left\{ 6.672(10^{-11})\ \frac{Nm^2}{kg^2} \right\} \left\{ 5.976(10^{24})\ kg \right\} \tag{13}$$

$$\mu = \left\{ 3.987(10^{14})\ \frac{m^3}{sec^2} \right\} \tag{14}$$

The Earth’s mean radius is

$$r_e = 6.37(10^6) \quad m \tag{15}$$

Note that the Earth's shape is an oblate spheroid. The equatorial radius is slightly greater than the polar radius.

<u>Example 1</u>

A spacecraft is orbiting the Earth at an altitude of 400 km. What is the required velocity?

$$v = \sqrt{\frac{\mu}{r_e + x}} \qquad (16)$$

$$v = \sqrt{\frac{\left\{3.987(10^{14})\ \frac{m^3}{sec^2}\right\}}{6.37(10^6)\ m + 400(10^3)\ m}} \qquad (17)$$

$$v = 7680\ m/sec \qquad (18)$$

<u>Example 2</u>

A communication satellite must be placed in a geostationary orbit about the Earth's equator. The satellite will thus remain above the same point on the equator. What is the required altitude for the circular orbit?

First, calculate the Earth's angular velocity about its own axis.

$$\omega = \frac{2\pi}{T} \qquad (19)$$

where T is the period.

The Earth's rotational period about its own axis is

$$T = 23\ hr\ 56\ min\ 4.09\ sec \qquad (20)$$

$$T = 86164.09\ sec \qquad (21)$$

The Earth's angular velocity is thus

$$\omega = \frac{2\pi}{86164.09\ \ sec} \qquad (22)$$

$$\omega = 7.2921\left(10^{-5}\right) rad/sec \qquad (23)$$

The satellite must maintain the same angular velocity.

Recall the velocity formula.

$$v = \sqrt{\frac{\mu}{r_e + x}} \tag{24}$$

Note that the translational velocity v is related to the angular velocity by

$$v = \omega \left(r_e + x\right) \tag{25}$$

Substitute equation (25) into (24).

$$\omega \left(r_e + x\right) = \sqrt{\frac{\mu}{r_e + x}} \tag{26}$$

Solve for x.

$$\left[\omega \left(r_e + x\right)\right]^2 = \left[\frac{\mu}{r_e + x}\right] \tag{27}$$

$$[\omega]^2 [\ r_e + x]^3 = \mu \tag{28}$$

$$[\ r_e + x]^3 = \frac{\mu}{\omega^2} \tag{29}$$

$$[\ r_e + x] = \left[\frac{\mu}{\omega^2}\right]^{1/3} \tag{30}$$

$$x = \left[\frac{\mu}{\omega^2}\right]^{1/3} - r_e \tag{31}$$

$$x = \left[\frac{\mu}{\omega^2}\right]^{1/3} - r_e \tag{32}$$

The Earth's equatorial radius is

$$r_e = 6{,}378{,}188 \ \ \text{m} \tag{33}$$

$$x = \left[ \frac{3.987(10^{14}) \ \dfrac{\text{m}^3}{\text{sec}^2}}{\left[7.2921\left(10^{-5}\right)\text{rad/sec}\right]^2} \right]^{1/3} - 6{,}378{,}188 \ \ \text{m} \tag{34}$$

Finally, the geostationary altitude is approximately

$$x = 35{,}790{,}000 \ \ \text{m} \tag{35}$$

$$x = 35{,}790 \ \ \text{km} \tag{36}$$