COMPONENT MODE SYNTHESIS, FIXED-INTERFACE MODEL
Revision A

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February 13, 2010

Introduction

Component mode synthesis is a method for analyzing the dynamic behavior of a system consisting of an assembly of substructures. It is particularly useful for analyzing a system consisting of a launch vehicle and a spacecraft. The fixed-interface model is an approach for implementing this component mode synthesis.

Reference 1 explains:

> Fixed-interface modeling is the most commonly employed technique for spacecraft. It is commonly viewed as capable of producing lower modes similar to those occurring in the coupled system because the spacecraft interface points are attached to a relatively stiff structure. Furthermore, the well established base-fixed modal test approach is applicable for model verification purposes.

The resulting spacecraft model is referred to as a Craig-Bampton model.

Derivation

A spacecraft is used as the subsystem.

The dynamic response of the spacecraft is modeled as

$$M\,\ddot{x} + K\,x = F \tag{1}$$

where

M	is the mass matrix
K	is the stiffness matrix
F	is the force vector
x	is the displacement vector

The mass and stiffness coefficients are determined by a finite element model.

The physical coordinates are separated into a set of interface coordinates x_I and a set of non-interface (interior) coordinates x_N.

The matrices and vectors in equation (1) are then partitioned as

$$\begin{bmatrix} M_{NN} & M_{NI} \\ M_{NI}^T & M_{II} \end{bmatrix} \begin{bmatrix} \ddot{x}_N \\ \ddot{x}_I \end{bmatrix} + \begin{bmatrix} K_{NN} & K_{NI} \\ K_{NI}^T & K_{II} \end{bmatrix} \begin{bmatrix} x_N \\ x_I \end{bmatrix} = \begin{bmatrix} F_N \\ F_I \end{bmatrix} \tag{2}$$

$$\bar{x} = \begin{bmatrix} x_N \\ x_I \end{bmatrix} \tag{3}$$

Typically, the non-interface forces are zero, $F_N = 0$. An exception would be the case where direct acoustic excitation of the spacecraft surface was considered.

The generalized eigenvalue problem is

$$\left[K - \Omega_n^2 \, M \right] \Phi_n = 0 \tag{4}$$

where

Ω_n is the diagonal matrix of circular natural frequencies

Φ_n is the matrix of mass-normalized mode shapes or eigenvectors

where

Φ_{NN} equal the matrix of fixed-interface modes

Φ_{CN} equal the matrix of constraint modes

The constraint modes may be rigid-body vectors.

Transform the physical coordinate x as

$$\bar{x} = \begin{bmatrix} x_N \\ x_I \end{bmatrix} = \begin{bmatrix} \phi_{NN} & \phi_{CN} \\ 0 & I \end{bmatrix} \begin{bmatrix} q_N \\ x_I \end{bmatrix} \tag{5}$$

The Craig-Bampton transformation matrix is

$$\begin{bmatrix} \phi_{NN} & \phi_{CN} \\ 0 & I \end{bmatrix} \tag{6}$$

As an important note, the problem size can be greatly reduced by including only the first few modes in the submatrix Φ_{NN}.

Transform the stiffness matrix.

$$\tilde{K} = \begin{bmatrix} \phi_{NN}{}^T & 0 \\ \phi_{CN}{}^T & I \end{bmatrix} \begin{bmatrix} K_{NN} & K_{NI} \\ K_{NI}{}^T & K_{II} \end{bmatrix} \begin{bmatrix} \phi_{NN} & \phi_{CN} \\ 0 & I \end{bmatrix} \tag{7}$$

$$\tilde{K} = \begin{bmatrix} \phi_{NN}{}^T & 0 \\ \phi_{CN}{}^T & I \end{bmatrix} \begin{bmatrix} K_{NN}\phi_{NN} & K_{NN}\phi_{CN} + K_{NI} \\ K_{NI}{}^T\phi_{NN} & K_{NI}{}^T\phi_{CN} + K_{II} \end{bmatrix} \tag{8}$$

$$\tilde{K} = \begin{bmatrix} \phi_{NN}{}^T K_{NN}\,\phi_{NN} & \phi_{NN}{}^T [K_{NN}\phi_{CN} + K_{NI}] \\ \phi_{CN}{}^T K_{NN}\phi_{NN} + K_{NI}{}^T\phi_{NN} & \phi_{CN}{}^T [K_{NN}\phi_{CN} + K_{NI}] + K_{NI}{}^T\phi_{CN} + K_{II} \end{bmatrix} \tag{9}$$

$$\tilde{K} = \begin{bmatrix} \tilde{K}_{NN} & \tilde{K}_{NI} \\ \tilde{K}_{NI}{}^{T} & \tilde{K}_{II} \end{bmatrix} \tag{10}$$

$$\tilde{K}_{NN} = \phi_{NN}{}^{T} K_{NN}\, \phi_{NN} \tag{11}$$

$$\tilde{K}_{NI} = \phi_{NN}{}^{T} \left[K_{NN}\phi_{CN} + K_{NI} \right] \tag{12}$$

$$\tilde{K}_{NI}{}^{T} = \phi_{CN}{}^{T} K_{NN}\phi_{NN} + K_{NI}{}^{T}\phi_{NN} \tag{13}$$

$$\tilde{K}_{II} = \phi_{CN}{}^{T} \left[K_{NN}\phi_{CN} + K_{NI} \right] + K_{NI}{}^{T}\phi_{CN} + K_{II} \tag{14}$$

Note that

$$\phi_{NN}{}^{T} K_{NN}\, \phi_{NN} = \Omega_{n}^{2} \tag{15}$$

where Ω_{n}^{2} is the eigenvalue matrix for the fixed-interface modal analysis.

Let

$$\phi_{NN}{}^{T} \left[K_{NN}\phi_{CN} + K_{NI} \right] = 0 \tag{16}$$

$$\phi_{NN}{}^{T} \left[K_{NN}\phi_{CN} + K_{NI} \right] = 0 \tag{17}$$

$$\left[K_{NN}\phi_{CN} + K_{NI} \right] = 0 \tag{18}$$

$$\phi_{CN} + K_{NN}{}^{-1} K_{NI} = 0 \tag{19}$$

Thus

$$\tilde{K}_{NI} = \phi_{NN}{}^{T}\left[K_{NN}\phi_{CN} + K_{NI}\right] = 0 \tag{20}$$

Again,

$$\tilde{K}_{NI}{}^{T} = \phi_{CN}{}^{T}K_{NN}\phi_{NN} + K_{NI}{}^{T}\phi_{NN} \tag{21}$$

Substitute equation (19) into (21).

$$\tilde{K}_{NI}{}^{T} = -\left[K_{NN}{}^{-1}K_{NI}\right]^{T}K_{NN}\phi_{NN} + K_{NI}{}^{T}\phi_{NN} \tag{22}$$

$$\tilde{K}_{NI}{}^{T} = -K_{NN}{}^{T}K_{NI}{}^{-1}K_{NN}\phi_{NN} + K_{NI}{}^{T}\phi_{NN} \tag{23}$$

$$\tilde{K}_{NI}{}^{T} = -K_{NI}{}^{-T}\phi_{NN} + K_{NI}{}^{T}\phi_{NN} \tag{24}$$

$$\tilde{K}_{NI}{}^{T} = 0 \tag{25}$$

$$\tilde{K}_{II} = \phi_{CN}{}^{T}\left[K_{NN}\phi_{CN} + K_{NI}\right] + K_{NI}{}^{T}\phi_{CN} + K_{II} \tag{26}$$

Substitute equation (19) into (26).

$$\tilde{K}_{II} = K_{NI}{}^{T}\phi_{CN} + K_{II} \tag{27}$$

Thus

$$\tilde{K} = \begin{bmatrix} \Omega_n{}^2 & 0 \\ 0 & K_{NI}{}^{T}\phi_{CN} + K_{II} \end{bmatrix} \tag{28}$$

where

$$\phi_{CN} = -K_{NN}^{-1} K_{NI} \tag{29}$$

Transform the mass matrix. Let

$$\tilde{M} = \begin{bmatrix} \tilde{M}_{NN} & \tilde{M}_{NI} \\ \tilde{M}_{NI}^{T} & \tilde{M}_{II} \end{bmatrix} \tag{30}$$

$$\tilde{M} = \begin{bmatrix} \phi_{NN}^{T} & 0 \\ \phi_{CN}^{T} & I \end{bmatrix} \begin{bmatrix} \tilde{M}_{NN} & \tilde{M}_{NI} \\ \tilde{M}_{NI}^{T} & \tilde{M}_{II} \end{bmatrix} \begin{bmatrix} \phi_{NN} & \phi_{CN} \\ 0 & I \end{bmatrix} \tag{31}$$

$$\tilde{M} = \begin{bmatrix} \phi_{NN}^{T} M_{NN} \phi_{NN} & \phi_{NN}^{T} [M_{NN}\phi_{CN} + M_{NI}] \\ \phi_{CN}^{T} M_{NN}\phi_{NN} + M_{NI}^{T} \phi_{NN} & \phi_{CN}^{T} [M_{NN}\phi_{CN} + M_{NI}] + M_{NI}^{T}\phi_{CN} + M_{II} \end{bmatrix} \tag{32}$$

The eigenvectors are mass-normalized. Thus

$$\tilde{M}_{NN} = \phi_{NN}^{T} M_{NN} \phi_{NN} = I \tag{33}$$

The other terms are

$$\tilde{M}_{NI} = \phi_{NN}^{T} [M_{NN}\phi_{CN} + M_{NI}] \tag{34}$$

$$\tilde{M}_{NI} = \phi_{CN}^{T} M_{NN}\phi_{NN} + M_{NI}^{T} \phi_{NN} \tag{35}$$

$$\tilde{M}_{II} = \phi_{CN}^{T} [M_{NN}\phi_{CN} + M_{NI}] + M_{NI}^{T}\phi_{CN} + M_{II} \tag{36}$$

The transformed load matrix is

$$\tilde{F} = \begin{bmatrix} \phi_{NN}{}^{T} & 0 \\ \phi_{CN}{}^{T} & I \end{bmatrix} \begin{bmatrix} F_N \\ F_I \end{bmatrix} \tag{37}$$

$$\tilde{F} = \begin{bmatrix} \phi_{NN}{}^{T} F_N \\ \phi_{CN}{}^{T} F_N + F_I \end{bmatrix} \tag{38}$$

In summary,

$$\tilde{M} = \begin{bmatrix} I & \tilde{M}_{NI} \\ \tilde{M}_{NI}{}^{T} & \tilde{M}_{II} \end{bmatrix} \tag{39}$$

$$\tilde{K} = \begin{bmatrix} \tilde{K}_{NN} & \tilde{K}_{NI} \\ \tilde{K}_{NI}{}^{T} & \tilde{K}_{II} \end{bmatrix} \tag{40}$$

The Craig-Bampton transformation matrix is

$$\begin{bmatrix} \phi_{NN} & \phi_{CN} \\ 0 & I \end{bmatrix} \tag{41}$$

where

$\tilde{M}_{II}$	is the boundary mass matrix
$\tilde{M}_{NI}$	is the dynamic coupling matrix
$\tilde{K}_{II}$	is the interface stiffness matrix
$\tilde{K}_{II} = 0$	if the boundary point is a single grid.

The transformed equation of motion including damping is

$$\begin{bmatrix} I & \tilde{M}_{NI} \\ \tilde{M}_{NI}^{T} & \tilde{M}_{II} \end{bmatrix} \begin{bmatrix} \ddot{q}_N \\ \ddot{x}_I \end{bmatrix} + \begin{bmatrix} 2\xi_n\Omega_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_N \\ \dot{x}_I \end{bmatrix} + \begin{bmatrix} \Omega_n{}^2 & 0 \\ 0 & \tilde{K}_{II} \end{bmatrix} \begin{bmatrix} q_N \\ x_I \end{bmatrix} = \begin{bmatrix} \phi_{NN}{}^{T} F_N \\ \phi_{CN}{}^{T} F_N + F_I \end{bmatrix} \tag{42}$$

If the non-interface forces are zero, then

$$\begin{bmatrix} I & \tilde{M}_{NI} \\ \tilde{M}_{NI}^{T} & \tilde{M}_{II} \end{bmatrix} \begin{bmatrix} \ddot{q}_N \\ \ddot{x}_I \end{bmatrix} + \begin{bmatrix} 2\xi_n\Omega_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_N \\ \dot{x}_I \end{bmatrix} + \begin{bmatrix} \Omega_n{}^2 & 0 \\ 0 & \tilde{K}_{II} \end{bmatrix} \begin{bmatrix} q_N \\ x_I \end{bmatrix} = \begin{bmatrix} 0 \\ F_I \end{bmatrix} \tag{43}$$

Thus, the fixed-interface component mode approach results in component equations of motion, which have only inertial coupling between the fixed-interface normal modes and the constraint modes.

The next step is to assemble equations of motion for all substructures. This is done by adding the matrices in equation (43). Note that the substructure matrices must be expanded with zeroes to become conformable for addition. The assemble equation may be solved by either the normal mode method or direct numerical integration. The physical displacements for each substructure are then found using equation (5).

References

1. NASA-HDBK-7005, Dynamic Environmental Criteria, 2001.
2. Scott Gordon, The Craig Bampton Method, FEMCI Presentation, 1999.

APPENDIX A

Consider an undamped, homogeneous problem.

The rows and columns are interchanged for compatibility with the author's software programs.

$$\begin{bmatrix} \tilde{M}_{II} & \tilde{M}_{NI} \\ \tilde{M}_{NI}^T & I \end{bmatrix} \begin{bmatrix} \ddot{q}_N \\ \ddot{x}_I \end{bmatrix} + \begin{bmatrix} \tilde{K}_{II} & 0 \\ 0 & \Omega_n{}^2 \end{bmatrix} \begin{bmatrix} q_N \\ x_I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A-1}$$

The Craig-Bampton transformation matrix is

$$\begin{bmatrix} I & 0 \\ \phi_{CN} & \phi_{NN} \end{bmatrix} \tag{A-2}$$

where

$$\phi_{CN} = -K_{NN}{}^{-1} K_{NI} \tag{A-3}$$

Recall

Φ_{NN} equal the matrix of fixed-interface modes

Φ_{CN} equal the matrix of constraint modes

And Ω_n^2 is the eigenvalue matrix for the fixed-interface modal analysis.

The generalized eigenvalue problem can be represented as

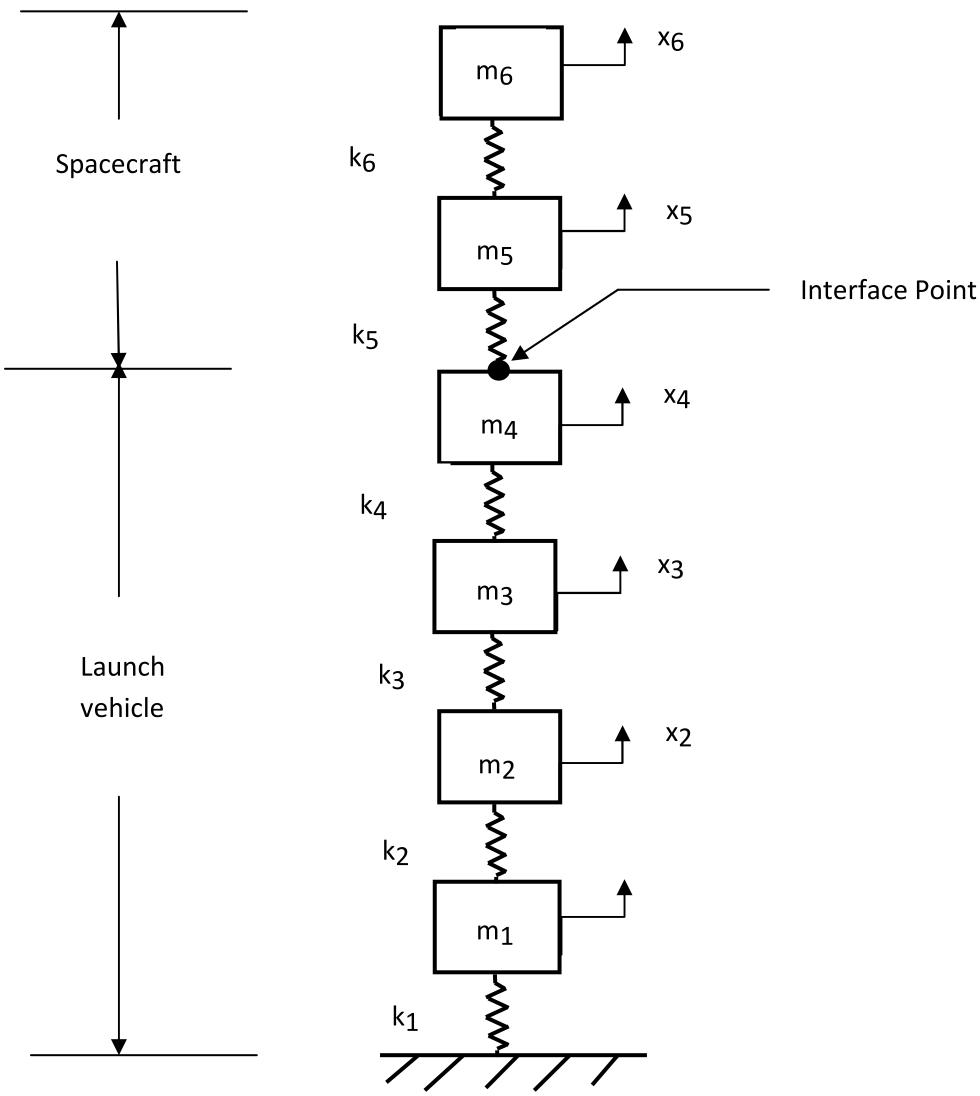
$$\left[\tilde{K} - \Omega_n^2 \, \tilde{M}\right] \tilde{\Phi}_n = 0 \tag{A-4}$$

$$\tilde{K} = \begin{bmatrix} I & 0 \\ \phi_{CN} & \phi_{NN} \end{bmatrix}^T \begin{bmatrix} K_{II} & K_{NI} \\ K_{NI}{}^T & K_{NN} \end{bmatrix} \begin{bmatrix} I & 0 \\ \phi_{CN} & \phi_{NN} \end{bmatrix} \tag{A-5}$$

$$\tilde{M} = \begin{bmatrix} I & 0 \\ \phi_{CN} & \phi_{NN} \end{bmatrix}^T \begin{bmatrix} M_{II} & M_{NI} \\ M_{NI}{}^T & M_{NN} \end{bmatrix} \begin{bmatrix} I & 0 \\ \phi_{CN} & \phi_{NN} \end{bmatrix} \tag{A-6}$$

A sample system is shown in Figure (A-1). Assume that the bottom four masses represent a launch vehicle, and that the two masses represent a spacecraft.

Retain the launch vehicle's four degrees-of-freedom in the reduced model. Restore the spacecraft degree-of-freedom number 5 in the fixed-interface model.



m6
x6
Spacecraft
k6
m5
x5
Interface Point
k5
m4
x4
k4
m3
x3
Launch
vehicle
k3
m2
x2
k2
m1
k1

Figure A-1.

The following values are used for the model.

English units: stiffness (lbf/in), mass (lbf sec^2/in)

k1	450,000
k2	300,000
k3	250,000
k4	210,000
k5	200,000
k6	150,000

m1	150
m2	125
m3	100
m4	100
m5	10
m6	5

The unreduced equation of motion is:

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_6 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \ddot{x}_5 \\ x_6 \end{bmatrix}$$

$$+ \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 & 0 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 & 0 & 0 \\ 0 & 0 & -k_4 & k_4 + k_5 & -k_5 & 0 \\ 0 & 0 & 0 & -k_5 & k_4 + k_5 & -k_6 \\ 0 & 0 & 0 & 0 & -k_6 & k_6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = 0 \tag{A-7}$$

The matrices are partition as follows:

$$M_{II} = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}$$

$$M_{NI}{}^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{NN} = \begin{bmatrix} m_5 & 0 \\ 0 & m_6 \end{bmatrix}$$

$$K_{II} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 + k_5 \end{bmatrix}$$

$$K_{NI}{}^T = \begin{bmatrix} 0 & 0 & 0 & -k_5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_{NN} = \begin{bmatrix} k_4 + k_5 & -k_6 \\ -k_6 & k_6 \end{bmatrix}$$

```
>> mdof_reduced_CB

 mdof_reduced_CB.m   ver 1.8  February 12, 2010

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 This program solves the following equation of motion:
    M (d^2x/dt^2) + K x = 0

 where M and K are reduced according to the users specification.

The discarded dof may be brought back in using the Craig-Bampton method.


 Enter the units system
 1=English  2=metric
 1
 Assume symmetric mass and stiffness matrices.
 Select input mass unit
  1=lbm  2=lbf sec^2/in
 2

 stiffness unit = lbf/in

 Select file input method
   1=file preloaded into Matlab
   2=Excel file
1

 Mass Matrix
 Enter the matrix name:  mass6

 Stiffness Matrix
 Enter the matrix name:  stiff6

 The mass matrix is

m =

   150     0     0     0     0     0
     0   125     0     0     0     0
     0     0   100     0     0     0
     0     0     0   100     0     0
     0     0     0     0    10     0
     0     0     0     0     0     5


 The stiffness matrix is

k =

      750000     -300000           0           0           0           0
     -300000      750000     -250000           0           0           0
           0     -250000      550000     -210000           0           0
           0           0     -210000      460000     -200000           0
```

```
         0           0           0     -200000      410000     -150000
         0           0           0           0     -150000      150000


 Enter the number of degrees-of-freedom to retain: 4

 Enter retained dof number: 1
 Enter retained dof number: 2
 Enter retained dof number: 3
 Enter retained dof number: 4
 Partitioned Matrices

m_partition =

   150     0     0     0     0     0
     0   125     0     0     0     0
     0     0   100     0     0     0
     0     0     0   100     0     0
     0     0     0     0    10     0
     0     0     0     0     0     5


k_partition =

      750000     -300000           0           0           0           0
     -300000      750000     -250000           0           0           0
           0     -250000      550000     -210000           0           0
           0           0     -210000      460000     -200000           0
           0           0           0     -200000      410000     -150000
           0           0           0           0     -150000      150000


 Restore any modes via the CB method?
 1=yes  2=no
 1

 How many modes to restore?
 1
 Enter restored dof number
 5
 Natural Frequencies (Hz)
    18.55
    38.14

  Modes Shapes (column format)

ModeShapes =

   -0.1935   -0.2501
   -0.3537    0.2737
```

```
Transformation Matrix

C =

    1.0000         0         0         0         0
         0    1.0000         0         0         0
         0         0    1.0000         0         0
         0         0         0    1.0000         0
         0         0         0    0.7692   -0.1935
         0         0         0    0.7692   -0.3537

 Transformed matrices

mq =

  150.0000         0         0         0         0
         0  125.0000         0         0         0
         0         0  100.0000         0         0
         0         0         0  108.8757   -2.8490
         0         0         0   -2.8490    1.0000


kq =

  1.0e+005 *

    7.5000   -3.0000         0         0         0
   -3.0000    7.5000   -2.5000         0         0
         0   -2.5000    5.5000   -2.1000         0
         0         0   -2.1000    3.0615         0
         0         0         0    0.0000    0.1359

 Natural Frequencies (Hz)
     5.76
    8.866
    12.17
    15.05
     19.5

  Modes Shapes (column format)


MS_ordered =

   -0.0172    0.0524   -0.0508   -0.0323    0.0004
   -0.0318    0.0497    0.0215    0.0637   -0.0019
   -0.0539    0.0091    0.0626   -0.0549    0.0081
   -0.0697   -0.0488   -0.0359    0.0143   -0.0345
   -0.0577   -0.0455   -0.0426    0.0262    0.1733
   -0.0611   -0.0520   -0.0550    0.0387    0.3387
```

A natural frequency comparison is given in Table A-1. The results agree out to two decimal places for the first five modes.

Table A-1. Natural Frequencies		
Mode	Full Model, 6 dof fn (Hz)	Model Reduced to 4 dof + 1 restored dof fn (Hz)
1	5.76	5.76
2	8.87	8.87
3	12.17	12.17
4	15.05	15.05
5	19.50	19.50
6	38.29	-