SAND 2000-2776 C

A Family of Transients Suitable for Reproduction on a Shaker Based on the $\cos^m(x)$ Window

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BIOGRAPHY

David Smallwood received his BSME degree from New Mexico State University in 1962 and his MSME degree from New York University in 1964. He has worked for Sandia National Laboratories since 1967 and is currently a Distinguished Member of the Technical Staff in the Structural Dynamics and Smart Systems Department at Sandia. He is a fellow of the IEST.

ABSTRACT

A family of transients with the property that the initial and final acceleration, velocity, and displacement are all zero is derived. The transients are based on a relatively arbitrary function multiplied by window of the form $\cos^m(x)$. Several special cases are discussed which result in odd acceleration and displacement functions. This is desirable for shaker reproduction because the required positive and negative peak accelerations and displacements will be balanced. Another special case is discussed which will permit the development of transients with the first five (0-4) temporal moments specified. The transients are defined with three or four parameters that will allow sums of components to be found which will match a wide variety of shock response spectra.

KEYWORDS

shaker shock, shock response spectrum, cosine windows, temporal moments

INTRODUCTION

A couple of decades ago several methods were developed for matching a shock response spectrum (SRS) with sums of oscillatory waveforms. The oscillatory waveforms were all suitable for reproduction on electrodynamic or electrohydraulic shakers. All these waveforms have several properties in common. All have the property that the initial and final acceleration, velocity, and displacement are zero. This is required for accurate reproduction on a shaker. Most also have the property that they can be described with a few parameters. One parameter defines the amplitude, a second parameter defines the duration, and a third parameter defines the frequency content. A time shift parameter is also sometimes included to define a temporal location. Usually the frequency content is concentrated in a narrow band of frequencies. One of the most popular of these waveforms are exponentially decaying sinusoids defined by

$$\begin{aligned} a(t) &= Ae^{-\zeta\omega t}\sin(\omega t) \quad t \geq 0 \\ &= 0 \quad \text{elsewhere} \end{aligned} \tag{1}$$

where a is the acceleration, A is the amplitude, ζ is the decay rate which controls the duration and bandwidth, ω is the dominate frequency, and t is time. This waveform does require a compensating pulse to enforce the required initial and final values. A second popular waveform is the WAVSYN waveform defined by

$$\begin{aligned} a(t) &= A\cos(x)\cos(nx) \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ &= 0 \quad \text{elsewhere} \\ x &= \omega t / n \\ n &= \text{an odd integer} \end{aligned} \tag{2}$$

The typical use of these waveforms is to sum a number of components with different parameters to match a shock response spectrum [1,2].

[1]Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under Contract DE-AC04-94AL85000.

DISCLAIMER

Sums of decaying sinusoids result in waveforms that resemble many field environments, since many environments are essentially the impulse response of a structure with many modes. Properly used the WAVSYN method can also yield very acceptable results. A disadvantage of these waveforms is that the peak positive and peak negative acceleration and displacement are sometimes significantly different. For efficient reproduction on a shaker we would like the peak positive and the peak negative values to be similar. Another disadvantage is the temporal moments [4] are difficult to control.

Another application motivated the search for an acceleration waveform that would yield an odd displacement function with a single maximum and minimum. This search resulted in a family of waveforms that share the positive attributes of the WAVSYN waveform, but also will allow the generation of acceleration and displacement waveforms, which are odd functions and hence have the same positive and negative peaks. Another special case allows the first four temporal moments to be controlled.

WAVEFORMS BASED ON THE $\cos^m x$ WINDOW

The usual practice is to define the acceleration waveform and then to derive the velocity and displacement waveform. I will depart from this practice and define the displacement waveform first. If the displacement has two or more continuous derivatives (and the derivatives are zero at the boundaries) the displacement, velocity, and acceleration waveforms will have the required boundary zero conditions. Let the displacement be defined as

$$\begin{aligned} d(t) &= Ay(t)\cos^m z(t) \qquad -\frac{\pi}{2} \le z \le \frac{\pi}{2} \\ &= 0 \qquad \text{elsewhere} \end{aligned} \tag{3}$$

The range of t will depend on the function $z(t)$. All the functions defined in the rest of the paper will be zero outside the defined range of z. m is usually a positive integer, but this is not required. The window, $\cos^m(x)$, is described by Harris [3]. The displacement is defined as any function, $y(t)$, multiplied by a window of the form $\cos^m(z)$. $y(t)$ must be continuous with at least two continuous derivatives within the range of z. The displacement is scaled by a factor A. The function $z(t)$ can be used to distort the time axis to achieve a wide variety of windows. The velocity and acceleration can be defined by differentiation of the displacement

$$v = \dot{d} = A\left[\dot{y}\cos^m z - my\dot{z}\sin z\cos^{m-1} z\right] \tag{4}$$

$$\begin{aligned} a = \ddot{d} = A[&(\ddot{y} - my\dot{z}^2)\cos^m z \\ &- m(2\dot{y}\dot{z} + y\ddot{z})\sin z\cos^{m-1} z \\ &+ m(m-1)y\dot{z}^2\sin^2 z\cos^{m-2} z] \end{aligned} \tag{5}$$

As can be seen, that for $m > 2$ and if the functions $y(t)$ and $z(t)$ and their first two derivatives are defined and finite over the defined interval of z the acceleration, velocity, and displacement will be zero at both boundaries of the defined interval.

SPECIAL CASE 1: $A=1,\ z(t) = \dfrac{2\pi ft}{n}$

In this case n can be interpreted as the number of half cycles of the waveform at a frequency, f. The displacement, velocity, and acceleration are given by

$$d(t) = y(t)\cos^m bt \qquad -\frac{n}{4f} \le t \le \frac{n}{4f} \tag{6}$$

$$v(t) = \dot{y}\cos^m bt - mby\sin bt\cos^{m-1} bt$$

$$\begin{aligned} a(t) = &(\ddot{y} - mb^2y)\cos^m bt - 2m\dot{y}b\sin bt\cos^{m-1} bt \\ &+ m(m-1)b^2y\sin^2 bt\cos^{m-2} bt \end{aligned}$$

where $b = 2\pi f/n$.

If $m = 2$, The function, $y(t)$, must be zero at $t = \pm n/(4f)$ for the initial and final acceleration to be zero.

The window is an even function. If y is also even, the displacement and acceleration will be even and the velocity will be odd. If y is odd, the displacement and acceleration will be odd, and the velocity will be even. The WAVSYN waveform is almost the special case, m=1, $y(t) = \cos(nbt)$, except the waveform is defined as the acceleration, where here the waveform is defined as the displacement.

SPECIAL CASE 2: A=1, $y(t)=1$, $z(t)=\dfrac{2\pi ft}{n}$

For this case the acceleration, velocity, and displacement are given by

$$\begin{aligned} a(t) &= -mb^2\cos^m bt \\ &\quad + m(m-1)b^2\sin^2 bt\cos^{m-2} bt \\ v(t) &= -mb\sin bt\cos^{m-1} bt \\ d(t) &= \cos^m bt \end{aligned} \tag{7}$$

The acceleration, velocity, and displacement waveforms for A=1, m=3, and b=1 are plotted as Figure 1. Since b is set to one the range of t is $\pm\pi/2$.

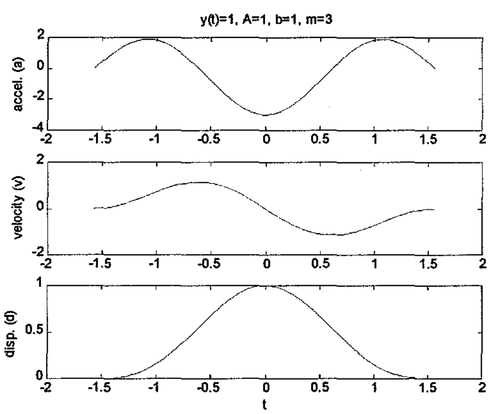

Figure 1 Acceleration, velocity, and displacement for the case of m=3, y(x)=1

SPECIAL CASE 3: A=1, $y(t)=t$, $z(t)$=t

For this case the acceleration, velocity, and displacement are given by

$$\begin{aligned} a(t) &= -mt\cos^m t \\ &\quad - 2m\sin t\cos^{m-1} t \\ &\quad + m(m-1)t\sin^2 t\cos^{m-2} t \\ v(t) &= \cos^m t - mt\sin t\cos^{m-1} t \\ d(t) &= t\cos^m t \end{aligned} \tag{8}$$

The acceleration, velocity, and displacement for A=1, m=3, and b=1 are plotted as Figure 2.

SPECIAL CASE 4: A=1, m = 2, and $y(t)=\sin(nbt)$, where n is an even integer

The acceleration, velocity and acceleration are given by

$$\begin{aligned} a(t) &= -b^2\left(1+n^2\right)\sin nbt\cos^2 bt \\ &\quad - 4nb^2\cos nbt\cos bt\sin bt \\ &\quad + 2b^2\sin^2 nbt\sin bt \\ v(t) &= nb\cos nbt\cos^2 bt \\ &\quad - 2b\sin nbt\sin bt\cos bt \\ d(t) &= \cos^2 bt\sin nbx \end{aligned} \tag{9}$$

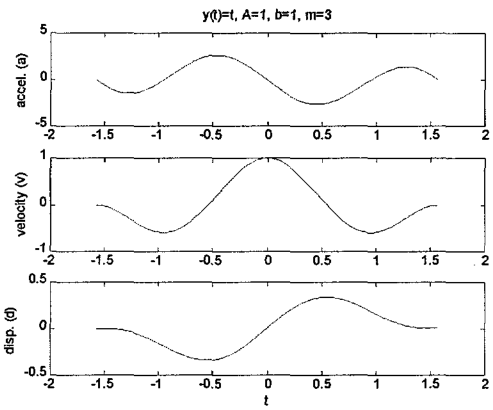

Figure 2 Acceleration, velocity, and displacement for the case of m=3, y(t)=t

The number of half cycles in the acceleration, velocity and displacement will be n+2, n+1, and n respectively.

For the special cases 2-4 we need the same parameters as for a WAVSYN waveform: the amplitude, A, to scale the acceleration, the frequency, f, the number of half cycles, n, and a shift parameter to define the temporal location of the waveform.

Example Special Case 3

The acceleration, velocity, and displacement waveforms for the special case 3, with a frequency of 100 Hz, n=2, and normalized for an amplitude of one is shown as Figure 3. Comparing Figures 2 and 3 we see that this waveform has characteristics very similar to special case 2. Note that the peak positive and peak negative acceleration and displacement are the same, a desirable characteristic for shaker reproduction.

The acceleration, velocity, and displacement waveforms for a frequency of 100 Hz, n=50, and normalized for an amplitude of one is shown as Figure 4.

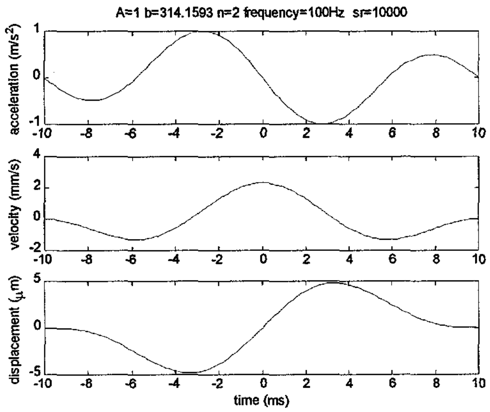

Figure 3 Waveform with four half cycles

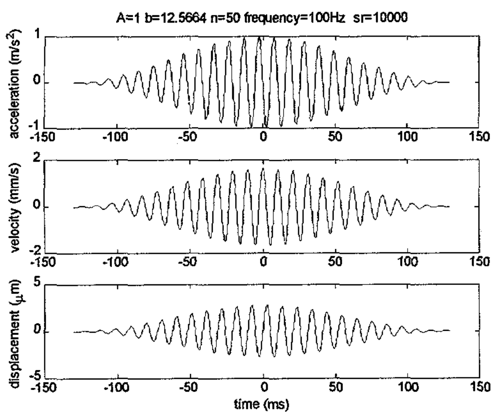

Figure 4 Waveform where acceleration has 52 half cycles

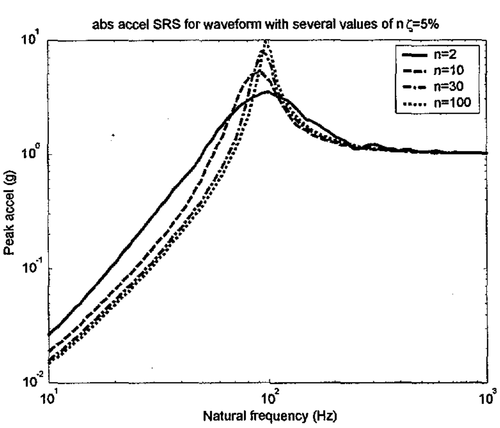

Figure 5 SRS for waveform with several values of n

The SRS for several values of n, where all amplitudes are normalized to one is shown in Figure 5.

GENERALIZED WAVSYN

A generalized WAVSYN waveform with normalized amplitude can be defined, where n is the number of half cycles

$$a(t) = -3b^2 \cos^3 bt + 6b^2 \sin^2 bt \cos bt \qquad n = 0$$

$$\begin{aligned} a(t) = &-3b^2 t \cos^3 bt - 6b \sin t \cos^2 bt \\ &+ 6b^2 \sin^2 bt \cos bt - \cos^3 bt \qquad n = 1 \end{aligned} \tag{10}$$

$$\begin{aligned} a(t) = &-b^2(1+n^2)\sin nbt \cos^2 bt \\ &- 4nb^2 \cos nbt \sin bt \cos bt \\ &+ 2b^2 \sin nbt \sin^2 bt \qquad n = \text{even integer} > 0 \end{aligned}$$

$$a(t) = \cos(nbt)\cos(bt) \quad n = \text{odd integer} > 1$$

$$b = \frac{2\pi f}{n} \quad -\frac{n}{4f} \le t \le \frac{n}{4f}$$

For consistency we would normalize the amplitudes in each case to unity.

$$a(t) = a(t) / \max |\, a(t) \,| \tag{11}$$

SPECIAL CASE 5

Consider the case where m=3,

$$y(t) = \sin \frac{2\pi f}{T} t \qquad 0 \le t \le 1$$

and (12)

$$z(t) = \pi\left(\left(\frac{t}{T}\right)^p - \frac{1}{2}\right) \qquad 0 \le t \le 1$$

This gives

$$\dot{y} = \frac{2\pi f}{T} \cos \frac{2\pi f}{T} t$$

$$\ddot{y} = -\left(\frac{2\pi f}{T}\right)^2 \sin\frac{2\pi f}{T}t \qquad (13)$$

$$\dot{z} = \pi p t^{p-1} / T^p$$

$$\ddot{z} = \pi p(p-1)t^{p-2} / T^p$$

I'll call this the cos3w waveform since it is based on a $\cos^3$ window for which the time axis is warped. The acceleration, velocity, and displacement are found by substituting the relations for $y, \dot{y}, \ddot{y}, z, \dot{z},$ and $\ddot{z}$ into Equations 3-5. If p equals 1 the waveform is symmetric and the skewness [4] is zero. The skewness is a measure of the shape of the waveform. A positive skewness indicates a fast rise and a slow decay of the waveform. If you reverse a time history, the skewness changes sign. If p is not one, the time axis is warped distorting the waveform envelope. p less than one gives positive skewness and p greater than one gives negative skewness. Thus the waveform skewness is controlled by p. Figure 6 shows how the skewness varies as a function of p.

The duration of the waveform is controlled by T and the energy is concentrated at the frequency, f. The energy or the peak amplitude can be scaled with an amplitude parameter. A time shift will control the centroid. We can develop waveforms with a specified frequency content, amplitude, centroid, duration, and skewness. The energy can be adjusted to match the SRS. By summing several of these waveforms we will be able to match an SRS in the same manner as is done for decaying sinusoids and WAVSYN. With the additional advantage of having control over several of the temporal moments (centroid, rms duration, and skewness). An example is given in Figure 7.

A cos^m window with m>3 will make the acceleration smoother near the origin.

CONCLUSIONS

A family of waveforms suitable for reproduction on shakers is given. A special case yields waveforms very similar to the popular WAVSYN waveform, with one slight improvement, the positive and negative acceleration and displacement peaks are about the same, yielding balanced waveforms for shaker reproduction. Other special cases lead to a definition of a generalized WAVSYN waveform defined for all positive integers. Another case is developed which will allow the control of some temporal properties as well as the spectral content.

This waveform can also be used to synthesize waveforms that will match an SRS.

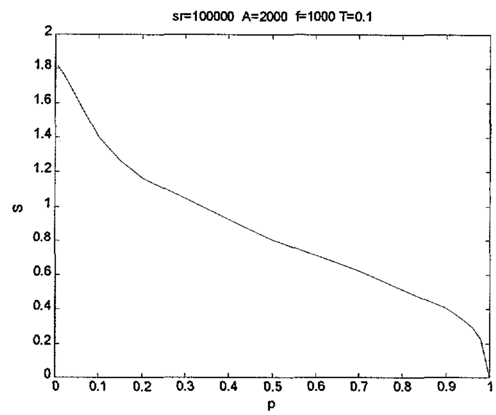

Figure 6 Skewness varies with p

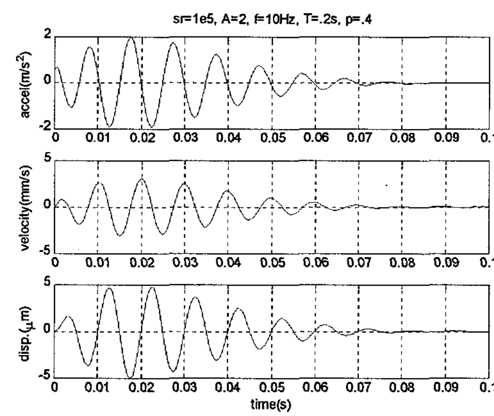

Figure 7 An example of a cos3d waveform

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