FREE VIBRATION WITH COULOMB DAMPING
Revision A

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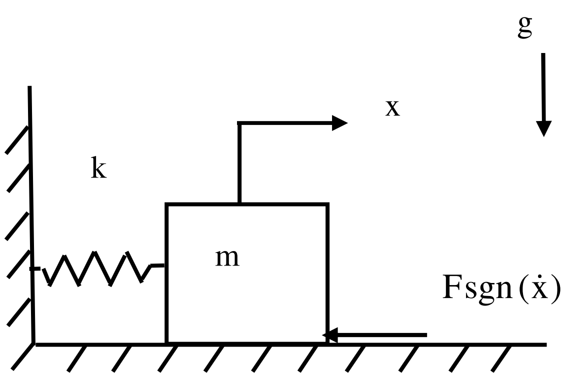

Figure 1.

Coulomb damping is dry friction damping. Consider the free vibration response of a single-degree-of-freedom system subjected to Coulomb damping.

The damping force F is

$$F = \mu\, mg \tag{1}$$

where

μ	=	friction coefficient
m	=	mass
g	=	acceleration of gravity

Assume that the friction coefficient is constant for simplicity.

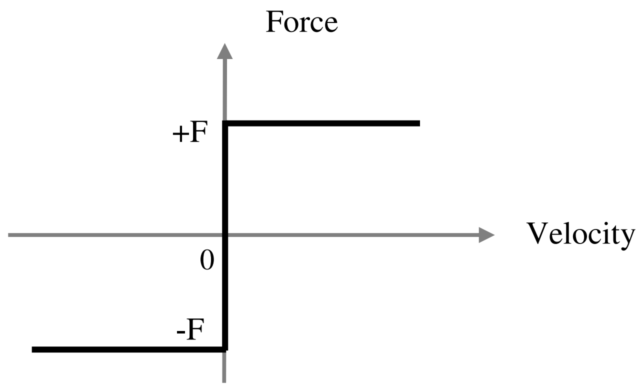

Figure 2. Coulomb Force vs. Velocity

The governing equation of motion for the displacement x is

$$m\ddot{x} + kx = -F\,\mathrm{sgn}(\dot{x}) \tag{2}$$

where k = stiffness

The $\mathrm{sgn}(\dot{x})$ function represents the sign of $\dot{x}$.

As an alternative, the governing equation can be written as

$$m\ddot{x} + kx = -F\frac{\dot{x}}{|\dot{x}|} \tag{3}$$

The governing equation is solved in a piecewise-linear manner.

Assume that initial displacement x(0) is

$$x(0) > F / k \tag{4}$$

Also assume that the initial velocity is zero.

Consider the equation of motion for negative velocity.

$$m\ddot{x} + kx = F \qquad \text{for} \qquad \dot{x} < 0 \tag{5}$$

$$\ddot{x} + \left(\frac{k}{m}\right) x = \left(\frac{F}{m}\right) \tag{6}$$

The natural frequency ω_n is

$$\omega_n{}^2 = \left(\frac{k}{m}\right) \tag{7}$$

Let

$$E = \left(\frac{F}{m}\right) \tag{8}$$

$$\ddot{x} + \omega_n{}^2\, x = E \tag{9}$$

The equation is solved using Laplace transforms.

$$L\left\{\ddot{x} + \omega_n{}^2\, x\right\} = L\{E\} \tag{10}$$

$$s^2\, X(s) - s\, x(0) + \omega_n{}^2\, X(s) = \frac{E}{s} \tag{11}$$

$$\left[s^2 + \omega_n{}^2\right] X(s) - s\,x(0) = \frac{E}{s} \tag{12}$$

$$\left[s^2 + \omega_n{}^2\right] X(s) = \frac{E}{s} + s\,x(0) \tag{13}$$

$$X(s) = \frac{E}{s\left[s^2 + \omega_n{}^2\right]} + \frac{s\,x(0)}{\left[s^2 + \omega_n{}^2\right]} \tag{14}$$

Take the inverse Laplace transform.

$$x(t) = x(0) \cos \omega_n t + \frac{E}{\omega_n{}^2}\left[1 - \cos \omega_n t\right] \tag{15}$$

$$x(t) = \frac{E}{\omega_n{}^2} + \left[x(0) - \frac{E}{\omega_n{}^2}\right] \cos \omega_n t \tag{16}$$

$$x(t) = \frac{F}{m\,\omega_n{}^2} + \left[x(0) - \frac{F}{m\,\omega_n{}^2}\right] \cos \omega_n t \tag{17}$$

$$x(t) = \frac{F}{k} + \left[x(0) - \frac{F}{k}\right] \cos \omega_n t \quad \text{for} \quad \dot{x} < 0 \tag{18}$$

$$\dot{x}(t) = -\omega_n \left[x(0) - \frac{F}{k}\right] \sin \omega_n t \quad \text{for} \quad \sin \omega_n t > 0 \tag{19}$$

The velocity equals zero at

$$t = \pi / \omega_n \tag{20}$$

The displacement at this time is

$$x\left(\frac{\pi}{\omega_n}\right)=\frac{F}{k}+\left[-x(0)+\frac{F}{k}\right] \tag{21}$$

$$x\left(\frac{\pi}{\omega_n}\right)=\frac{2F}{k}-x(0) \tag{22}$$

Consider the equation of motion for positive velocity.

$$m\ddot{x}+kx=-F \qquad \text{for} \qquad \dot{x}>0 \tag{23}$$

The initial displacement term must be reset to the last displacement for negative velocity. Furthermore, a phase angle must be added to the argument in the cosine term.

$$x(t)=\frac{-F}{k}+\left[\left[\frac{2F}{k}-x(0)\right]+\frac{F}{k}\right]\cos(\omega_n t+\pi) \qquad \text{for} \qquad \dot{x}>0 \tag{24}$$

$$x(t)=\frac{-F}{k}+\left[\frac{3F}{k}-x(0)\right]\cos(\omega_n t+\pi) \qquad \text{for} \qquad \dot{x}>0 \tag{25}$$

$$\dot{x}(t)=-\omega_n\left[\frac{3F}{k}-x(0)\right]\sin(\omega_n t+\pi) \tag{26}$$

The first negative displacement peak thus has an amplitude that is 2 F/k less than the initial displacement in terms of absolute values. This reduction factor can also be derived from the work-energy relationship in Appendix A.

The pattern continues such that the envelope has a linear decay.

The velocity returns to zero for

$$t = \pi / \omega_n \tag{27}$$

$$x\left(\frac{2\pi}{\omega_n}\right) = \frac{-F}{k} - \left[\frac{3F}{k} - x(0)\right] \tag{28}$$

$$x\left(\frac{2\pi}{\omega_n}\right) = x(0) - \frac{4F}{k} \tag{29}$$

Each consecutive positive peak is thus 4 F/k lower than the previous positive peak.

The process is then repeated.

<u>Example</u>

A single-degree-of-freedom system has

mass = 1 kg

stiffness = 20,000 N/m

friction coefficient = 0.4

initial displacement = 5 mm

The resulting displacement is shown in Figure 3.

The displacement converges to F / k, where $F = \mu mg$.

Depending on the initial displacement, the displacement may also converge to $-F / k$.

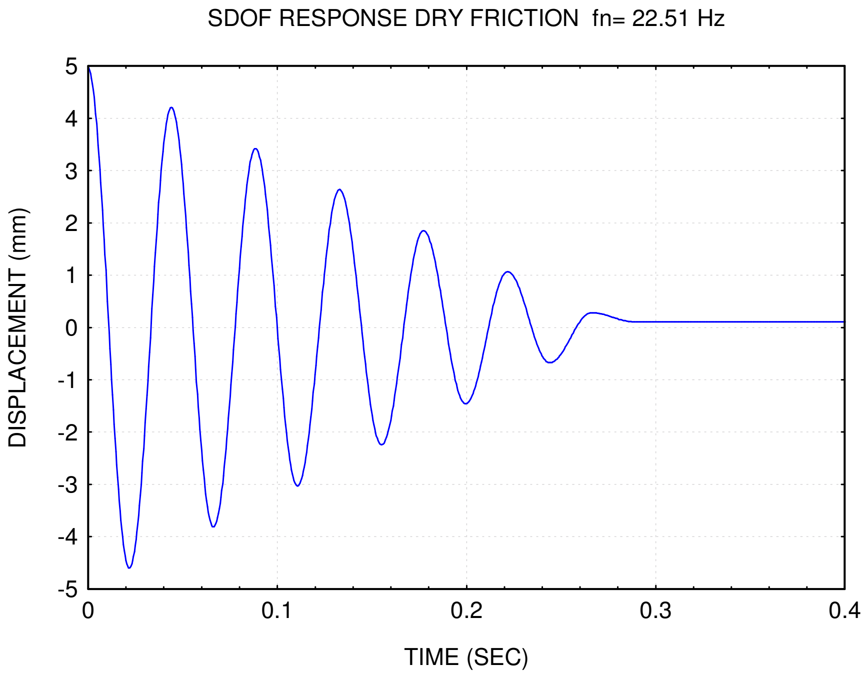

Figure 3.

References

1. R. Vierck, Vibration Analysis, 2nd Edition, Harper Collins, New York, 1979.
2. W. Thomson, Theory of Vibration with Applications 2nd Edition, Prentice Hall, New Jersey, 1981.

APPENDIX A

Energy Method

The potential energy is set equal to the work done by friction for one cycle.

$$\frac{1}{2}k\left(x_1{}^2 - x_2{}^2\right) = F\left(x_1 + x_2\right) \quad \text{(A-1)}$$

where x_1 and x_2 are consecutive positive peaks.

Note that the kinetic energy is zero at the instantaneous time that each peak occurs.

The work-energy relationship is satisfied if

$$F = \frac{1}{2}k\left(x_1 - x_2\right) \quad \text{(A-2)}$$

$$\left(x_1 - x_2\right) = \frac{2F}{k} \quad \text{(A-3)}$$

APPENDIX B

Matlab Script

```
disp(' ');
disp(' dry.m ver 1.0 June 25, 2005 ');
disp(' by Tom Irvine Email: tomirvine@aol.com ');
disp(' ');
disp(' This program calculates the response of a ');
disp(' single-degree-of-freedom system subjected to dry damping
');
disp(' ');
%
clear all;
disp(' Enter mass (kg) ')
m=input(' ');
disp(' Enter stiffness (N/m) ')
k=input(' ');
disp(' Enter coefficient of friction ')
mu=input(' ');
disp(' Enter initial displacement (mm) ')
xo=input(' ');
xo=xo/1000.;
%
F=mu*m*(9.81);
fk=F/k;
%
omegan=sqrt(k/m);
fn=omegan/(2.*pi);
%
out1=sprintf('\n fn = %8.4g Hz\n',fn);
disp(out1);
%
out1=sprintf(' F/k = %8.4g mm\n',(F/k)*1000.);
disp(out1);
%
if( F/k > xo )
disp(' ');
disp(' No oscillation. ');
disp(' F/k > xo ');
end
%
T=1/fn;
%
dt = T/100.;
delta=2.*pi/100;
%
num=12.*T/dt;
%
j=1;
```

```
10
tdelay=0.;
arg=0.;
for(i=1:(num+1))
t(i)=(i-1)*dt;
%
arg=arg+delta;
if(arg>2.*pi)
arg=arg-2.*pi;
end
%
if(arg>=0 && arg<=pi)
if( (xo-fk) <=0)
x(i)=xo;
else
x(i)= fk +( xo - fk )*cos(arg);
end
x1=x(i);
else
if( abs(x1) < fk )
x(i)=x1;
else
x(i)= -fk +( x1 + fk )*cos(arg+pi);
end
xo=x(i);
end
%
end
x=x*1000.;
plot(t,x);
xlabel(' Time(sec) ');
ylabel(' Displacement(mm) ');
out1=sprintf(' SDOF Response Dry Friction fn=%8.4g Hz ',fn);
title(out1);
grid on;
```