VIBROACOUSTIC RESPONSE OF A CYLINDER Revision A

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## Variables

| | |
|---|---|
| $W(t)$ | is the radial displacement |
| $W(\omega)$ | is the radial displacement Fourier transform |
| $W_A(\omega)$ | is the radial acceleration Fourier transform |
| $W_{APSD}(\omega)$ | is the radial acceleration power spectral density |
| $P(t)$ | is the pressure |
| $P(\omega)$ | is the pressure Fourier transform |
| $P_{PSD}(\omega)$ | is the pressure power spectral density |
| E | is the modulus of elasticity |
| R | is the radius |
| $\rho$ | is the mass/volume |
| c | is the speed of sound in the material |
| t | is time |
| T | is the period |
| h | is the wall thickness |
| $\omega$ | is the excitation frequency |
| $\omega_n$ | is the natural frequency (radian/sec) |
| $\beta$ | is the nondimensional excitation frequency |
| F | is the excitation frequency (Hz) |
| $\xi$ | is the viscous damping ratio |

Derivation

Consider an infinitely-long, thin-walled cylinder that is constrained so that its only mode is its ring-mode, which is the in-plane extension mode. Furthermore, the cylinder is subjected to an external pressure field, which is spatially uniform but time-varying.

The governing differential equation is

$$\rho\ R\,\ddot{w}(t) + \frac{E}{R}\,w(t) = \frac{R}{h}\,P(t) \tag{1}$$

Equation (1) is based on References 1 and 2.

$$\ddot{w}(t) + \frac{E}{\rho\ R^2}\,w(t) = \frac{1}{\rho\ h}\,P(t) \tag{2}$$

The speed of sound in the material is

$$c = \sqrt{\frac{E}{\rho}} \tag{3}$$

$$\ddot{w}(t) + \frac{c^2}{R^2}\,w(t) = \frac{1}{\rho\ h}\,P(t) \tag{4}$$

The natural frequency is

$$\omega_n = \frac{c}{R} \tag{5}$$

The equation of motion is thus

$$\ddot{w}(t) + 2\xi\omega_n\dot{w}(t) + \omega_n{}^2\ w(t) = \frac{1}{\rho\ h}\,P(t) \tag{6}$$

Note that a damping term was added to equation (6).

Take the Fourier transform of each side of equation (6).

$$\int_{-\infty}^{\infty}\left[\ddot{w}(t) \; + \; 2\xi\omega_n\dot{w}(t) + \omega_n{}^2 \; w(t)\right]\exp(-j\omega t)dt \; = \frac{1}{\rho\, h}\int_{-\infty}^{\infty}\left[P(t)\right]\exp(-j\omega t)dt \tag{7}$$

The displacement Fourier transform is

$$W(\omega) = \int_{-\infty}^{\infty}\left[w(t)\right]\exp(-j\omega t)dt \tag{8}$$

The velocity Fourier transform is obtained by integration of parts. The intermediate steps are omitted for brevity.

$$\begin{aligned}\int_{-\infty}^{\infty}\left[\dot{w}(t)\right]\exp(-j\omega t)dt \; &= j\omega\int_{-\infty}^{\infty}\left[w(t)\right]\exp(-j\omega t)dt \\ &= j\omega W(\omega)\end{aligned} \tag{9}$$

The acceleration Fourier transform is obtained by integration of parts.

$$\begin{aligned}\int_{-\infty}^{\infty}\left[\ddot{w}(t)\right]\exp(-j\omega t)dt \; &= -\omega^2\int_{-\infty}^{\infty}\left[w(t)\right]\exp(-j\omega t)dt \\ &= -\omega^2\, W(\omega)\end{aligned} \tag{10}$$

The pressure Fourier transform is

$$P(\omega) = \int_{-\infty}^{\infty}\left[P(t)\right]\exp(-j\omega t)dt \tag{11}$$

Substitute equations (8) though (13) into (7).

$$\left[-\omega^2 \;+\; j2\xi\omega\omega_n\dot{w}(t) + \omega_n{}^2\right]W(\omega) \;= \frac{1}{\rho\, h}P(\omega)$$

(12)

$$W(\omega) = \frac{1}{\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi\omega\omega_n\right]}\frac{1}{\rho\, h}P(\omega) \tag{13}$$

The acceleration Fourier transform is related to the displacement Fourier transform by

$$W_A(\omega) = \;\omega^2\; W(\omega) \tag{14}$$

Substitute equation (13) into (14).

$$W_A(\omega) = \frac{\omega^2}{\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi\omega\omega_n\right]}\frac{1}{\rho\, h}P(\omega) \tag{15}$$

Multiply each side by its complex conjugate.

$$W_A(\omega)W_A{}^*(\omega)$$
$$= \frac{\omega^4}{\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi\omega\omega_n\right]\left[\left(\omega_n{}^2 - \omega^2\right) - j2\xi\omega\omega_n\right]}\left[\frac{1}{\rho\, h}\right]^2 P(\omega)P^*(\omega)$$

(16)

$$W_A(\omega)W_A{}^*(\omega) = \frac{\omega^4}{\left[\left(\omega_n{}^2 - \omega^2\right)^2 + \left(2\xi\omega\omega_n\right)^2\right]}\left[\frac{1}{\rho\, h}\right]^2 P(\omega)P^*(\omega) \tag{17}$$

$$W_A(\omega)W_A^*(\omega) = \frac{\omega^4/\omega_n^4}{\left[\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\xi\omega/\omega_n\right)^2\right]}\left[\frac{1}{\rho h}\right]^2 P(\omega)P^*(\omega) \tag{18}$$

Let

$$\beta = \omega/\omega_n \tag{19}$$

$$W_A(\omega)W_A^*(\omega) = \frac{\beta^4}{\left[\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2\right]}\left[\frac{1}{\rho h}\right]^2 P(\omega)P^*(\omega) \tag{20}$$

$$\lim_{T\to\infty} \quad \frac{1}{T} W_A(\omega)W_A^*(\omega) = W_{APSD}(\omega) \tag{21}$$

$$\lim_{T\to\infty} \quad \frac{1}{T} P(\omega)P^*(\omega) = P_{PSD}(\omega) \tag{22}$$

Divided each side of equation (20) by T. Then take the limit as T approaches infinity. Then substitute equations (21) and (22) into the resulting equation.

The acceleration power spectral density is thus

$$W_{APSD}(\omega) = \frac{\beta^4}{\left[\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2\right]}\left[\frac{1}{\rho h}\right]^2 P_{PSD}(\omega) \tag{23}$$

Change the frequency, which is the independent variable, to f. The cylinder's vibroacoustic response is thus

$$\hat{W}_{APSD}(f) = \frac{\beta^4}{\left[\left(1-\beta^2\right)^2 + (2\xi\beta)^2\right]}\left[\frac{1}{\rho h}\right]^2 \hat{P}_{PSD}(f) \tag{24}$$

Example

Consider an infinitely long cylinder with the following properties:

| | |
|---|---|
| Diameter | 38 inch |
| Skin Thickness | 0.080 inch |
| Skin Material | Titanium |
| Viscous Damping | 5% |

The speed of sound in titanium is 194,650 in/sec. The mass density is 0.16 lbm/in^3.

The ring frequency is

$$f_n = \frac{c}{2\pi R} \tag{25}$$

$$f_n = \frac{194{,}650 \text{ in/sec}}{\pi\, 38 \text{ in}} \tag{26}$$

$$f_n = 1651 \text{ Hz} \tag{27}$$

Consider a spatially uniform white noise pressure field with an amplitude of 1.0e-06 psi^2/Hz applied to the external surface of the cylinder.

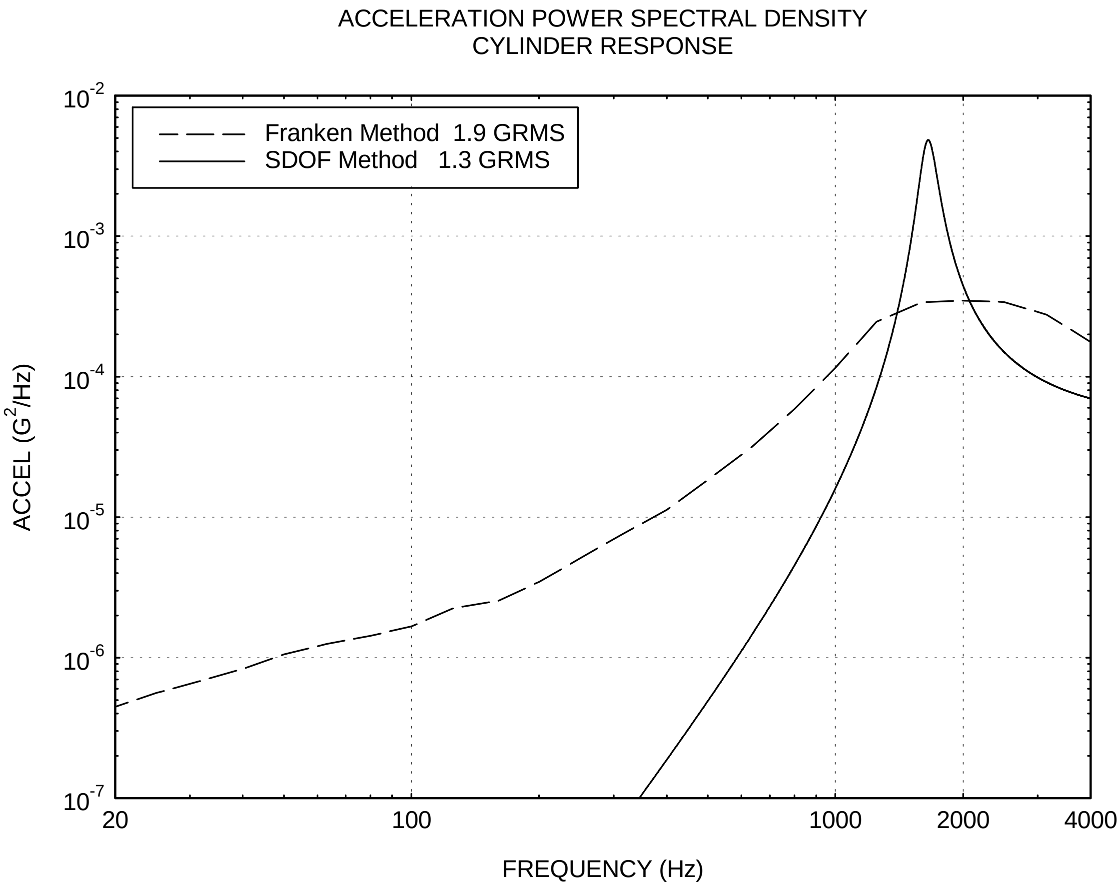


Figure 1.

The acceleration response of the cylinder is calculated via equation (24). The response is shown as the SDOF curve in Figure 1. Again, this method assumes a spatially uniform pressure field.

The calculation is repeated using the Franken method, which is an empirical method, from Reference 3. The resulting Franken curve is also shown in Figure 1. This method assumes that the pressure field is spatially random.

The comparison shows that the SDOF curve is 11.5 dB higher than the Franken curve at the ring frequency, which is 1651 Hz.

The Franken curve is broader since it accounts for additional cylinder modes.

The results tentatively show that a cylinder's vibroacoustic response can be calculated using a uniform pressure assumption. The resulting acceleration PSD can then be decreased by 11.5 dB at the ring frequency to account for a spatially random pressure field.

Further development of this approach is needed. The next step is to derive a multi-degree-of-freedom analytical model of the cylinder.

References


1. W. Young, Roark's Formulas for Stress & Strain, 6th edition, McGraw-Hill, New York, 1989. (See Chapter 12, Table 28).

2. T. Irvine, Ring Vibration Modes, Revision A, Vibrationdata, 2004. (See Appendix A).

3. T. Irvine, Vibration Response of a Cylindrical Skin to Acoustic Pressure via the Franken Method, Revision D, 2003.

## APPENDIX A

Frequency Response Function

Recall

$$W(\omega) = \frac{1}{\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi\omega\omega_n\right]}\frac{1}{\rho h}P(\omega) \tag{A-1}$$

$$\frac{W(\omega)}{P(\omega)} = \frac{1}{\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi\omega\omega_n\right]}\frac{1}{\rho h} \tag{A-2}$$

$$\frac{W(\rho)}{P(\rho)} = \frac{1}{\left[\rho h \omega_n{}^2\right]}\frac{1}{\left[\left(1-\rho^2\right) + j2\xi\rho\right]} \quad , \quad \rho = \frac{\omega}{\omega_n} \tag{A-3}$$

$$\frac{W(\rho)}{P(\rho)} = \frac{1}{\left[\rho h f_n{}^2\right]}\frac{1}{\left[\left(1-\rho^2\right) + j2\xi\rho\right]} \quad , \quad \rho = \frac{f}{f_n} \tag{A-4}$$

$$\frac{W(\rho)}{P(\rho)} = \frac{1}{\left[\rho h f_n{}^2\right]}\frac{1}{\sqrt{(1-\rho^2)^2 + (2\xi\rho)^2}} \tag{A-5}$$

Example 1

Consider an aluminum cylinder idealized as a single-degree-of-freedom system.

The cylinder has the following properties for each of two cases:

| | |
|---|---|
| Diameter | 36 inch |
| Damping | 5% |
| Wall Thickness | 0.125 inch for case 1<br>0.250 inch for case 2 |
| Surface Mass Density | 0.0125 lbm/in^2 for case 1<br>0.025 lbm/in^2 for case 2 |

The ring frequency is 1792 Hz for each case.

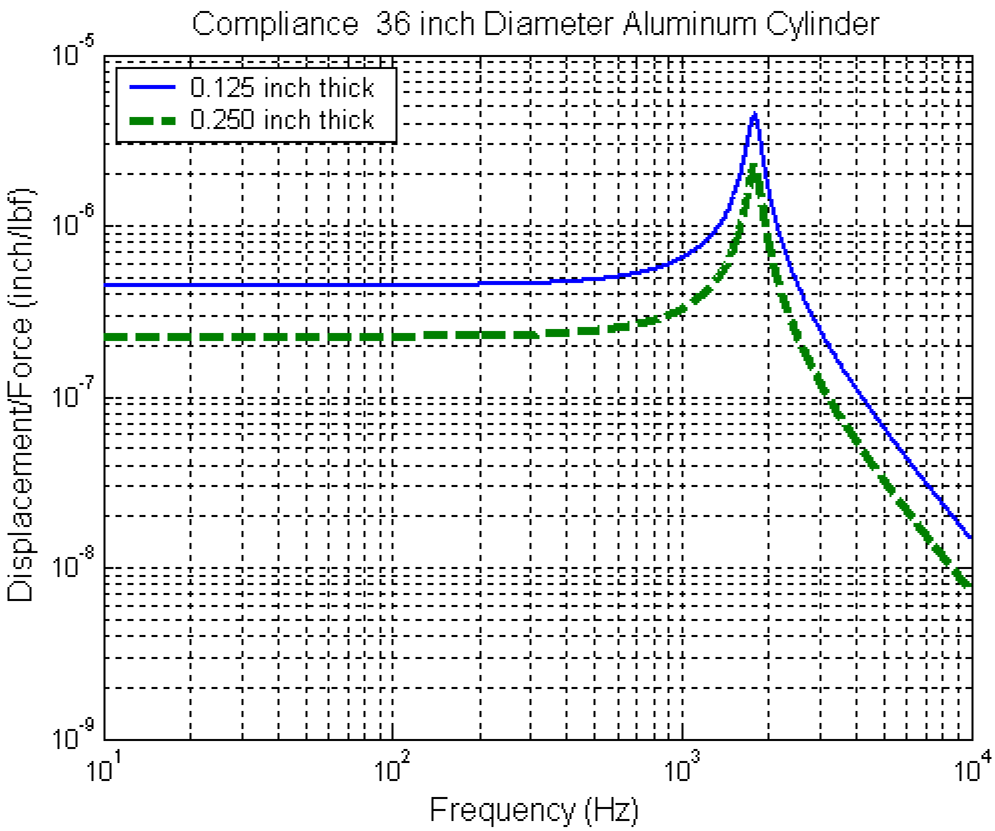


Figure A-1.

Doubling the thickness, and hence the surface mass density, decreases the response by 6 dB for a fixed natural frequency.

Example 2

Consider an aluminum cylinder idealized as a single-degree-of-freedom system.

The cylinder has the following properties for each of two cases:

| | |
|---|---|
| Diameter | 36 inch for case 1<br>72 inch for case 2 |
| Damping | 5% |
| Wall Thickness | 0.250 inch |
| Surface Mass Density | 0.025 lbm/in^2 |
| Ring Frequency | 1792 Hz for case 1<br>896 Hz for case 2 |

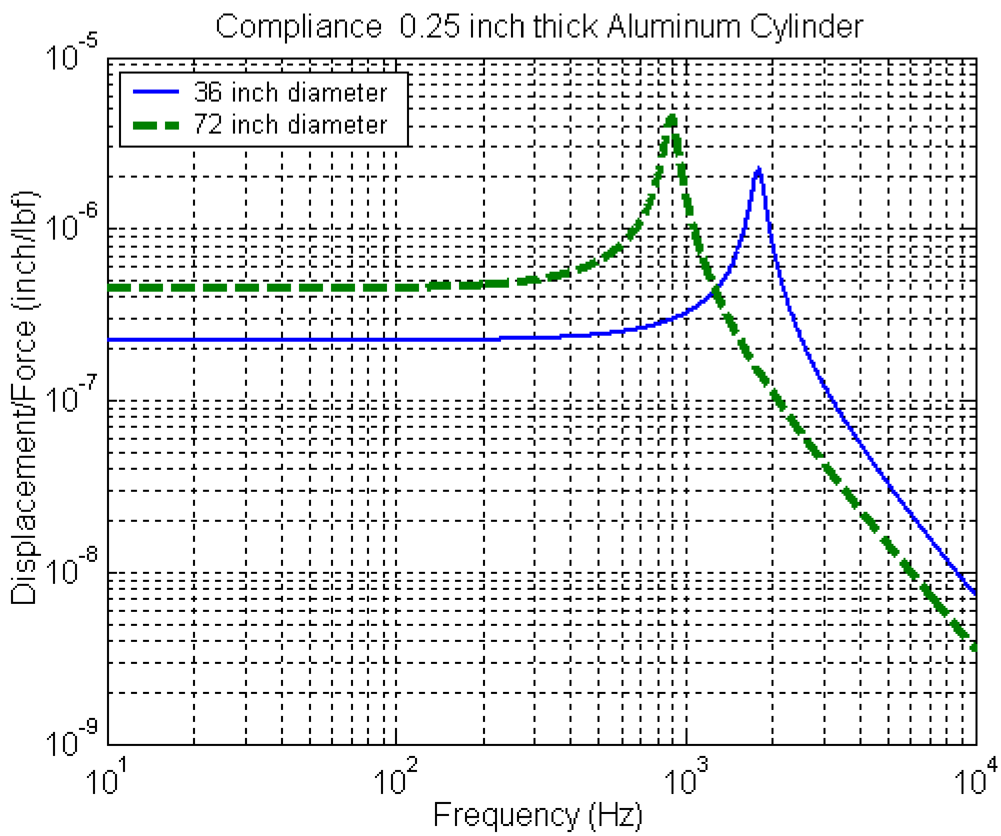


Figure A-2.

The comparison shows the trade-offs involved by changing the diameter and hence the ring frequency.

The following statements apply to a cylinder with a constant wall thickness and constant surface mass density:

1. A stiffer cylinder offers better attenuation at frequencies well below the ring frequency.

2. A more compliant cylinder provides better attenuation at frequencies well above the ring frequency.

Furthermore, the above statements assume normal incidence.