NATURAL FREQUENCIES OF A FINITE, THIN-WALLED CYLINDRICAL SHELL
Revision D

By Tom Irvine
Email: tomirvine@aol.com

September 5, 2008

Variables

x	is the axial coordinate	a	is the radius
θ	is the angular coordinate	L	is the length
t	is time	h	is the wall thickness
u	is the axial displacement	ω	is the natural frequency (radian/sec)
v	is the tangential displacement	m	is the axial mode number
w	is the radial displacement	n	is the circumferential mode number
E	is the modulus of elasticity	$k_\theta = n / a$	is the wavenumber in the circumferential direction
ρ	is the mass/volume	k_x	is the wavenumber in the axial direction.
μ	is the Poisson ratio	α	is an arbitrary angle

Furthermore,

$$\hat{k} = \sqrt{k_x^{\,2} + k_\theta^{\,2}}$$

$$\beta = h^2 /(12\, a^2)$$

$$K = Eh /(1-\mu^2)$$

$$D = Eh^3 /[12(1-\mu^2)]$$

Coordinate System

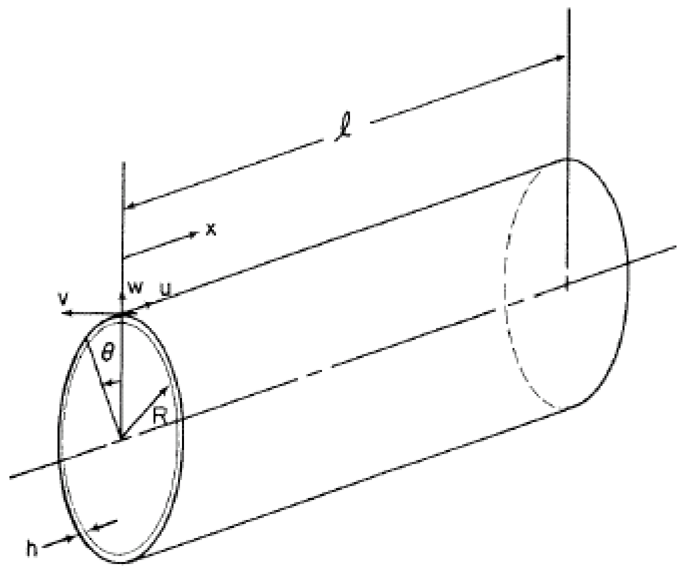


Figure 1. Cylinder Diagram

The three translation variables are

$$u = u(x, \theta) \qquad \text{axial} \tag{1}$$

$$v = v(x, \theta) \qquad \text{tangential} \tag{2}$$

$$w = w(x, \theta) \qquad \text{radial} \tag{3}$$

Equations of Motion

The following is a simplified approach which is referred to as the “Wave Method.” It is suitable for long, thin shells.

Three coupled equations of motion govern the vibration of a cylinder. The equations are given in both References 1 and 2.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\left(1-\mu^2\right)}{2a^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1+\mu}{2a}\frac{\partial^2 u}{\partial x \partial \theta} + \frac{\mu}{a}\frac{\partial w}{\partial x} = \frac{\rho\left(1-\mu^2\right)}{E}\frac{\partial^2 u}{\partial t^2} \tag{4}$$

$$\frac{1+\mu}{2a}\frac{\partial^2 u}{\partial x \partial \theta} + \frac{(1-\mu)}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{a^2}\frac{\partial w}{\partial \theta} = \frac{\rho\left(1-\mu^2\right)}{E}\frac{\partial^2 v}{\partial t^2} \tag{5}$$

$$\frac{\mu}{a}\frac{\partial u}{\partial x} + \frac{1}{a^2}\frac{\partial v}{\partial \theta} + \frac{w}{a^2} + \beta\left(a^2\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{1}{a^2}\frac{\partial^4 w}{\partial \theta^4}\right) = -\frac{\rho\left(1-\mu^2\right)}{E}\frac{\partial^2 w}{\partial t^2} \tag{6}$$

Equations (4) through (6) may be formed into a matrix which is called the Donnell-Mushtari operator.

Assume the following displacements using wave propagation equations.

$$u(x,\theta) = U_m \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \tag{7}$$

$$v(x,\theta) = V_m \sin[n(\theta-\alpha)] \exp j(\omega t - k_x x) \tag{8}$$

$$w(x,\theta) = W_m \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \tag{9}$$

where U_m, V_m, W_m are displacement amplitudes.

First Term

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2} U_m \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&+\frac{\left(1-\mu^2\right)}{2a^2} \frac{\partial^2}{\partial \theta^2} U_m \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&+\frac{1+\mu}{2a} \frac{\partial^2}{\partial x \partial \theta} U_m \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&+\frac{\mu}{a} \frac{\partial}{\partial x} W_m \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x \\
&= \frac{\rho\left(1-\mu^2\right)}{E} \frac{\partial^2}{\partial t^2} U_m \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x)
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&- U_m k_x{}^2 \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&- \frac{\left(1-\mu^2\right)}{2a^2} n^2 U_m \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&- j \frac{1+\mu}{2a} V_m \, n \, k_x \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&- j \frac{\mu}{a} k_x W_m \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \;= \\
&- \frac{\rho\left(1-\mu^2\right)\omega^2}{E} U_m \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x)
\end{aligned}
\tag{11}
$$

$$
- U_m k_x{}^2 - \frac{\left(1-\mu^2\right)}{2a^2} n^2 U_m - j \frac{1+\mu}{2a} n k_x V_m - j \frac{\mu}{a} k_x W_m + \frac{\rho\left(1-\mu^2\right)\omega^2}{E} U_m = 0
\tag{12}
$$

$$U_m\left\{-k_x^2-\frac{\left(1-\mu^2\right)}{2a^2}n^2+\frac{\rho\left(1-\mu^2\right)\omega^2}{E}\right\}-j\frac{1+\mu}{2a}k_x nV_m-j\frac{\mu}{a}k_x W_m=0$$

(13)

$$U_m\left\{-k_x^2-\frac{\left(1-\mu^2\right)}{2}k_\theta^2+\frac{\rho\left(1-\mu^2\right)\omega^2}{E}\right\}-j\frac{1+\mu}{2}k_x k_\theta V_m-j\frac{\mu}{a}k_x W_m=0$$

(14)

$$U_m\left\{k_x^2+\frac{\left(1-\mu^2\right)}{2}k_\theta^2-\frac{\rho\left(1-\mu^2\right)\omega^2}{E}\right\}+j\frac{1+\mu}{2}k_x k_\theta V_m+j\frac{\mu}{a}k_x W_m=0$$

(15)

$$U_m\left\{K k_x^2+K\frac{\left(1-\mu^2\right)}{2}k_\theta^2-\rho h\omega^2\right\}+jK\frac{1+\mu}{2}k_x k_\theta V_m+jK\frac{\mu}{a}k_x W_m=0$$

(16)

Second Term

$$\begin{aligned}
&\frac{1+\mu}{2a}\frac{\partial^2}{\partial x\partial\theta}U_m\cos[n(\theta-\alpha)]\exp j(\omega t-k_x x)\\
&+\frac{\left(1-\mu\right)}{2}\frac{\partial^2}{\partial x^2}V_m\sin[n(\theta-\alpha)]\exp j(\omega t-k_x x)\\
&+\frac{1}{a^2}\frac{\partial^2}{\partial\theta^2}V_m\sin[n(\theta-\alpha)]\exp j(\omega t-k_x x)\\
&+\frac{1}{a^2}\frac{\partial}{\partial\theta}W_m\cos[n(\theta-\alpha)]\exp j(\omega t-k_x x)\\
&=\frac{\rho\left(1-\mu^2\right)}{E}\frac{\partial^2}{\partial t^2}V_m\sin[n(\theta-\alpha)]\exp j(\omega t-k_x x)
\end{aligned}$$

(17)

$$
\begin{aligned}
&\frac{1+\mu}{2a} U_m \frac{\partial^2}{\partial x \partial \theta} \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&+ \frac{(1-\mu)}{2} V_m \sin[n(\theta-\alpha)] \frac{\partial^2}{\partial x^2} \exp j(\omega t - k_x x) \\
&+ \frac{1}{a^2} V_m \exp j(\omega t - k_x x) \frac{\partial^2}{\partial \theta^2} \sin[n(\theta-\alpha)] \\
&+ \frac{1}{a^2} W_m \exp j(\omega t - k_x x) \frac{\partial}{\partial \theta} \cos[n(\theta-\alpha)] \\
&= \frac{\rho\left(1-\mu^2\right)}{E} V_m \sin[n(\theta-\alpha)] \frac{\partial^2}{\partial t^2} \exp j(\omega t - k_x x)
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&j \frac{1+\mu}{2a} U_m n k_x \sin[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&- \frac{(1-\mu)}{2} V_m k_x{}^2 \sin[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&- \frac{1}{a^2} V_m n^2 \sin[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&- \frac{1}{a^2} W_m n \sin[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&= - \frac{\rho\left(1-\mu^2\right)}{E} V_m \omega^2 \sin[n(\theta-\alpha)] \exp j(\omega t - k_x x)
\end{aligned}
\tag{19}
$$

$$
- j \frac{1+\mu}{2} U_m k_x k_\theta + \frac{(1-\mu)}{2} V_m k_x{}^2 + V_m k_\theta{}^2 - \frac{\rho\left(1-\mu^2\right)}{E} V_m \omega^2 + \frac{1}{a} W_m k_\theta = 0
\tag{20}
$$

$$-j\frac{1+\mu}{2}U_m Kk_x k_\theta+\frac{(1-\mu)}{2}V_m Kk_x{}^2+V_m Kk_\theta{}^2-\rho h V_m\omega^2+\frac{1}{a}W_m Kk_\theta=0$$

(21)

$$-j\frac{1+\mu}{2}U_m Kk_x k_\theta+V_m\left\{\frac{(1-\mu)}{2}Kk_x{}^2+Kk_\theta{}^2-\rho h\omega^2\right\}+\frac{1}{a}W_m Kk_\theta=0$$

(22)

<u>Third term</u>

$$\begin{aligned}
&\frac{\mu}{a}\frac{\partial}{\partial x}U_m\cos[n(\theta-\alpha)]\exp j(\omega t-k_x x)\\
&+\frac{1}{a^2}\frac{\partial}{\partial\theta}V_m\sin[n(\theta-\alpha)]\exp j(\omega t-k_x x)\\
&+\frac{1}{a^2}W_m\cos[n(\theta-\alpha)]\exp j(\omega t-k_x x)\\
&+\beta\left(a^2\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial\theta^2}+\frac{1}{a^2}\frac{\partial^4}{\partial\theta^4}\right)W_m\cos[n(\theta-\alpha)]\exp j(\omega t-k_x x)\\
&=-\frac{\rho\left(1-\mu^2\right)}{E}\frac{\partial^2}{\partial t^2}W_m\cos[n(\theta-\alpha)]\exp j(\omega t-k_x x)
\end{aligned}$$

(23)

$$
\begin{aligned}
&\frac{\mu}{a} U_m \cos[n(\theta-\alpha)] \frac{\partial}{\partial x} \exp j(\omega t - k_x x) \\
&+ \frac{1}{a^2} V_m \exp j(\omega t - k_x x) \frac{\partial}{\partial \theta} \sin[n(\theta-\alpha)] \\
&+ \frac{1}{a^2} W_m \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&+ \beta a^2 W_m \cos[n(\theta-\alpha)] \left(\frac{\partial^4}{\partial x^4} \right) \exp j(\omega t - k_x x) \\
&+ 2\beta W_m \left(\frac{\partial^4}{\partial x^2 \partial \theta^2} \right) \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&+ \beta \frac{1}{a^2} W_m \exp j(\omega t - k_x x) \left(\frac{\partial^4}{\partial \theta^4} \right) \cos[n(\theta-\alpha)] \\
&= -\frac{\rho \left(1-\mu^2\right)}{E} W_m \cos[n(\theta-\alpha)] \frac{\partial^2}{\partial t^2} \exp j(\omega t - k_x x)
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&- j \frac{\mu}{a} U_m \, k_x \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&+ \frac{1}{a^2} V_m n \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&+ \frac{1}{a^2} W_m \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&+ \beta a^2 W_m k_x{}^4 \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&+ 2\beta W_m n^2 k_x{}^2 \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&+ \beta \frac{1}{a^2} W_m n^4 \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x) \\
&= \frac{\rho \left(1-\mu^2\right)}{E} W_m \omega^2 \cos[n(\theta-\alpha)] \exp j(\omega t - k_x x)
\end{aligned}
\tag{25}
$$

$$-j\frac{\mu}{a}U_m\, k_x+\frac{n}{a^2}V_m+\frac{1}{a^2}W_m+\beta a^2 W_m k_x{}^4+2\beta W_m n^2 k_x{}^2$$
$$+\beta\frac{1}{a^2}W_m n^4-\frac{\rho\left(1-\mu^2\right)}{E}W_m\omega^2=0 \tag{26}$$

$$-j\frac{\mu}{a}U_m\, k_x+\frac{n}{a^2}V_m$$
$$+W_m\left\{\frac{1}{a^2}+\beta a^2\left[k_x{}^4+2\frac{n^2}{a^2}k_x{}^2+\frac{n^4}{a^4}\right]-\frac{\rho\left(1-\mu^2\right)}{E}\omega^2\right\}=0 \tag{27}$$

$$-j\frac{\mu}{a}U_m k_x+\frac{n}{a^2}V_m+W_m\left\{\frac{1}{a^2}+\beta a^2\left[k_x{}^4+2k_x{}^2k_\theta{}^2+k_\theta{}^4\right]-\frac{\rho\left(1-\mu^2\right)}{E}\omega^2\right\}=0 \tag{28}$$

$$-j\frac{\mu}{a}U_m K k_x+\frac{K}{a}k_\theta V_m+W_m\left\{\frac{K}{a^2}K\beta a^2\left[k_x{}^4+2k_x{}^2k_\theta{}^2+k_\theta{}^4\right]-\rho h\omega^2\right\}=0 \tag{29}$$

$$-j\frac{\mu}{a}U_m K k_x+\frac{K}{a}k_\theta V_m+W_m\left\{\frac{K}{a^2}+K\frac{h^2}{12}\left[k_x{}^4+2k_x{}^2k_\theta{}^2+k_\theta{}^4\right]-\rho h\omega^2\right\}=0 \tag{30}$$

$$-j\frac{\mu}{a}U_m K k_x+\frac{K}{a}k_\theta V_m+W_m\left\{\frac{K}{a^2}+D\left[\left(k_x{}^2+k_\theta{}^2\right)^2\right]-\rho h\omega^2\right\}=0 \tag{31}$$

$$-j\frac{\mu}{a}U_m K k_x+\frac{K}{a}k_\theta V_m+W_m\left\{\frac{K}{a^2}+D\left[\sqrt{k_x{}^2+k_\theta{}^2}\right]^4-\rho h\omega^2\right\}=0 \tag{32}$$

Eigenvalue Problem

Equations (16), (22) and (32) may be assembled into matrix form.

$$\begin{bmatrix} K k_x^{\,2} + K\dfrac{\left(1-\mu^2\right)}{2}k_\theta^{\,2} - \rho\ h\omega^2 & jK\dfrac{1+\mu}{2}k_x k_\theta & jK\dfrac{\mu}{a}k_x \\ -j\dfrac{1+\mu}{2}Kk_x k_\theta & \left\{\dfrac{(1-\mu)}{2}Kk_x^{\,2} + Kk_\theta^{\,2} - \rho h\omega^2\right\} & \dfrac{K}{a}k_\theta \\ -j\dfrac{\mu}{a}K k_x & \dfrac{K}{a}k_\theta & \left\{\dfrac{K}{a^2} + D\hat{k}^{\,4} \ -\rho h\omega^2\right\} \end{bmatrix} \begin{bmatrix} U_m \\ V_m \\ W_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

For the purpose of finding the eigenvalues only, the eigenvalue problem may be simplified as

$$\begin{bmatrix} K k_x^{\,2} + K\dfrac{\left(1-\mu^2\right)}{2}k_\theta^{\,2} - \rho\ h\omega^2 & K\dfrac{1+\mu}{2}k_x k_\theta & K\dfrac{\mu}{a}k_x \\ \dfrac{1+\mu}{2}Kk_x k_\theta & \left\{\dfrac{(1-\mu)}{2}Kk_x^{\,2} + Kk_\theta^{\,2} - \rho h\omega^2\right\} & \dfrac{K}{a}k_\theta \\ \dfrac{\mu}{a}K k_x & \dfrac{K}{a}k_\theta & \left\{\dfrac{K}{a^2} + D\hat{k}^{\,4} \ -\rho h\omega^2\right\} \end{bmatrix} \begin{bmatrix} U_m \\ V_m \\ W_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{34}$$

The eigenvalue problem may be represented as

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_m \\ V_m \\ W_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{35}$$

where

$$L_{11} = K\,k_x{}^2 + K\frac{\left(1-\mu^2\right)}{2}k_\theta{}^2 - \rho\,h\omega^2 \tag{36}$$

$$L_{12} = K\frac{1+\mu}{2}k_x k_\theta \tag{37}$$

$$L_{13} = K\frac{\mu}{a}k_x \tag{38}$$

$$L_{21} = L_{12} \tag{39}$$

$$L_{22} = \frac{(1-\mu)}{2}Kk_x{}^2 + Kk_\theta{}^2 - \rho\,h\omega^2 \tag{40}$$

$$L_{23} = \frac{K}{a}k_\theta \tag{41}$$

$$L_{31} = L_{13} \tag{42}$$

$$L_{32} = L_{23} \tag{43}$$

$$L_{33} = \frac{K}{a^2} + D\hat{k}^4 - \rho\,h\omega^2 \tag{44}$$

The natural frequencies are calculated by setting the determinant of the coefficient matrix in equation (35) to zero. The wavenumber must be calculated before the natural frequency calculation, however.

The k_x wavenumber is calculated from the beam function which depends on the boundary conditions. Again, this is a simplified approach.

Clamped-Clamped Beam

The roots of a clamped-clamped beam are calculated from the following equation.

$$\cos(k_x L)\cosh(k_x L) = 1 \tag{45}$$

The roots are

$$k_x L = 4.73004 \quad \text{for } m = 1 \tag{46}$$

$$k_x L \approx \frac{(2m+1)\pi}{2} \quad \text{for } m \geq 2 \tag{47}$$

Beam Simply Supported at Each End

The roots of a beam simply supported at each end are

$$k_x L = m\pi \text{ , for } m \geq 1 \tag{48}$$

Free-Free Beam

The roots of a free-free beam are the same as those of the clamped-clamped beam except that the free-free beam also has an m=0 mode for which $k_x = 0$. See Appendix C.

References

1. Leissa, Vibration of Shells, NASA SP-288, Washington, D.C., 1973. (See section 2.2).

2. Li Bing-ru, Wang Xuan-yin, GE Hui-liang, Ding Yuan-ming, Study on applicability of modal analysis of thin finite length cylindrical shells using wave propagation approach, Journal of Zhejiang University Science, 2005.

APPENDIX A

Clamped-Clamped Cylinder Example

Consider a clamped-clamped cylinder with the following properties:

Table A-1. Sample Cylinder	
Radius	24 inch
Length	48 inch
Skin Thickness	0.125 inch
Skin Material	Aluminum
Mass Density ρ	0.0002539 (lbf sec^2/in^4)
Elastic Modulus E	9.9e+06 psi
Poisson Ratio	0.33

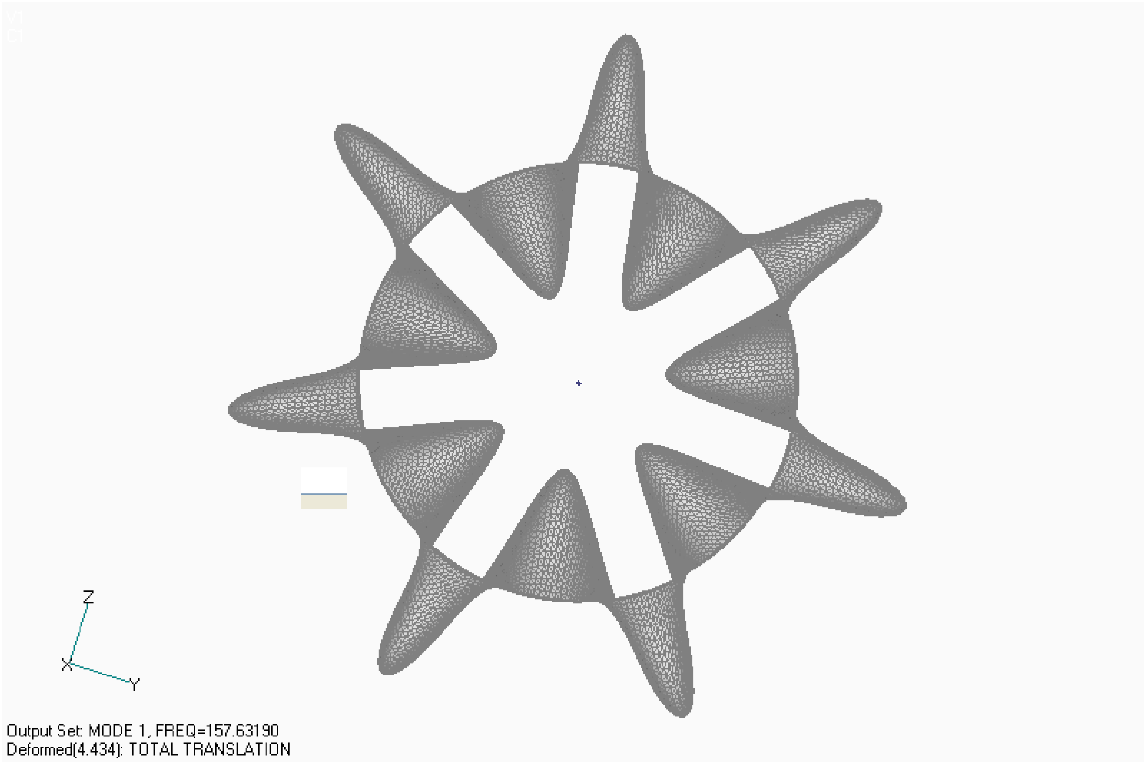
The sample cylinder could represent a module on a rocket vehicle.

The natural frequency results from the wave method are given in Table A-1, along with the finite element analysis results. The wave method was implemented using a Matlab script. The FEA was performed using NEiNastran 9.0.1. The model consisted of 24576 plate elements.

The mode with the lowest natural frequency was the (n=7, m=1) mode for each case. The difference is expressed with respect to the wave method. Neither method is “exact.”

Table A-2. Natural Frequency Results for Selected Modes, Clamped-Clamped Boundary Conditions				
Wave Method Freq (Hz)	FEA Freq (Hz)	Difference	n	m
174	158	9.2%	7	1
178	166	6.7%	8	1
193	168	13.0%	6	1
198	189	4.5%	9	1
230	222	3.5%	10	1
242	199	17.8%	5	1
269	262	2.6%	11	1
288	266	7.6%	9	2
1295	1261	2.6%	0	1

Figure A-1. Undeformed Model



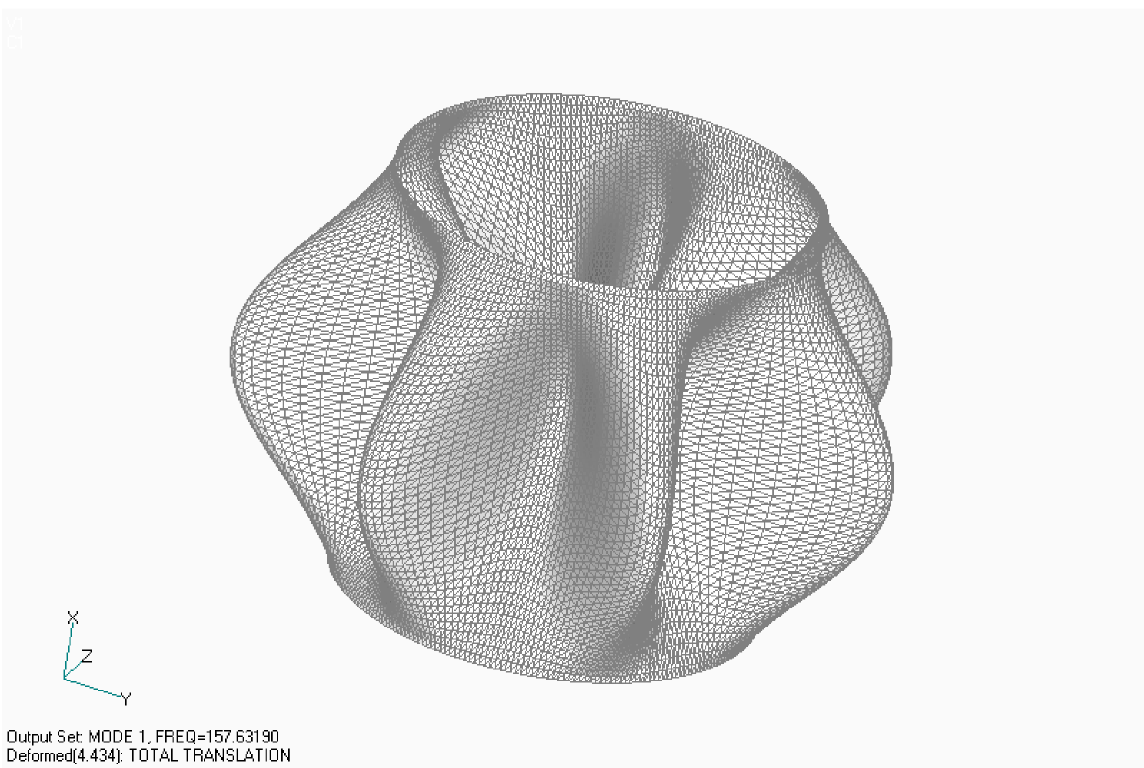


Figure A-2. FEA (n=7, m=1) Mode, 158 Hz

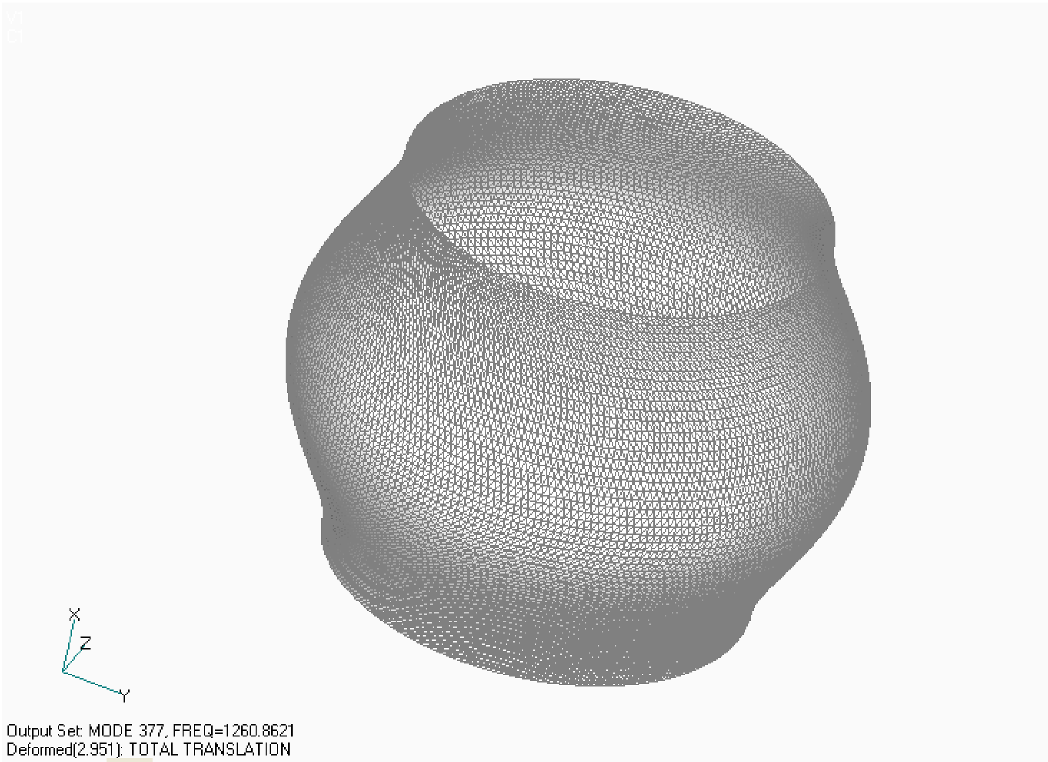


Figure A-3. FEA (n=0, m=1) Mode, 1261 Hz

The FEA animation showed that the corresponding displacement was tangential for this mode. As a result, the nodes *appeared* to move radially outward but not inward from a "top looking downward view."

The wave method predicted this (n=0, m=1) mode at 1295 Hz, which is close to the FEA result in terms of frequency. The wave method was unable to correctly predict the tangential motion, however, because the tangential displacement is assumed as

$$v(x, \theta) = V_m \sin[n(\theta - \alpha)] \exp j(\omega t - k_x x)$$

The tangential displacement is thus zero for n=0. This behavior warrants further research, because n=0 is a special case.

If the sample cylinder were infinite in length, its ring frequency would be 1344 Hz. The clamped-clamped finite cylinder's (n=0, m=1) mode in some sense is analogous to the ring mode of an infinite cylinder except that a true ring mode has purely radial motion.

APPENDIX B

SS-SS Cylinder Example

Repeat the sample problem from Appendix A for a cylinder simply supported at each end. Again, the difference is expressed with respect to the wave method. Neither method is "exact."

The SS-SS boundary case for a cylinder is an idiosyncratic idealization. "Successful" implementation of this condition in the FEA model requires that each of the cylinder's ends be left free to move axially. As a result, the FEA model may experience non-uniform axial motion at each end. In other words, the axial displacement at each end may vary with the circumferential angle, which may cause the ends to appear somewhat distorted for certain modes.

Table B-1. Natural Frequency Results for Selected Modes, SS-SS Boundary Conditions

Wave Method Freq (Hz)	FEA Freq (Hz)	Difference	n	m
114	115	-0.9%	6	1
123	123	0.0%	7	1
128	129	-0.8%	5	1
146	145	0.7%	8	1
173	174	-0.6%	4	1
177	176	0.6%	9	1
215	214	0.5%	10	1
231	238	-3.0%	8	2
1261	1261	0.0%	0	1

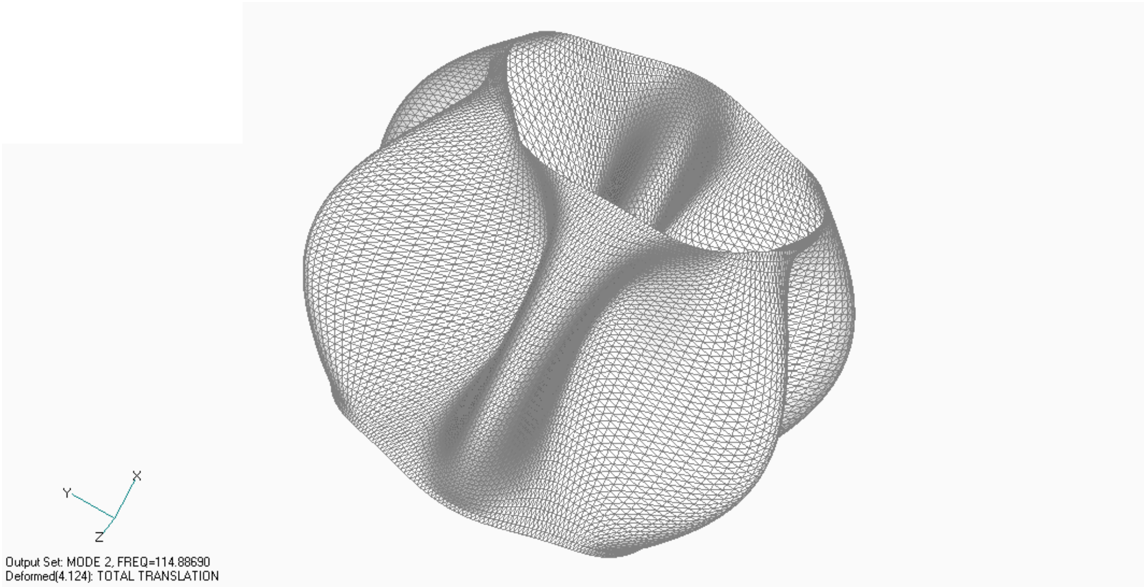


Figure B-1. FEA (n=6, m=1) Mode, 115 Hz

Figure B-2. FEA (n=0, m=1) Mode, 1261 Hz

The FEA animated mode shape for the (n=0, m=1) mode is again purely tangential. The FEA gives a frequency of 1261 Hz for this mode for both the clamped-clamped and the SS-SS boundary conditions.

APPENDIX C

Free-Free Cylinder, m=0, n=0 Breathing Mode

Assume the following wavenumber and displacements using wave propagation equations.

$$k_x = 0 \tag{C-1}$$

$$u(x,\theta) = U_m \exp j\omega t \tag{C-2}$$

$$v(x,\theta) = 0 \tag{C-3}$$

$$w(x,\theta) = W_m \exp j\omega t \tag{C-4}$$

The three equations of motion simplify to a single equation

$$\frac{w}{a^2} = -\frac{\rho\left(1-\mu^2\right)}{E}\frac{\partial^2 w}{\partial t^2} \tag{C-5}$$

$$\frac{1}{a^2} W_m \exp j\omega t = \frac{\rho\left(1-\mu^2\right)}{E}\omega^2 W_m \exp j\omega t \tag{C-6}$$

$$\frac{1}{a^2} = \frac{\rho\left(1-\mu^2\right)}{E}\omega^2 \tag{C-7}$$

$$\omega^2 = \frac{1}{a^2}\frac{E}{\rho\left(1-\mu^2\right)} \tag{C-8}$$

$$\omega = \frac{1}{a}\sqrt{\frac{E}{\rho\left(1-\mu^2\right)}} \tag{C-9}$$

The result is the ring frequency

$$f = \frac{1}{2\pi a} \sqrt{\frac{E}{\rho\left(1-\mu^2\right)}} \tag{C-10}$$