EQUIVALENT STATIC LOAD FOR A CANTILEVER BEAM SUBJECTED TO BASE EXCITATION, PART I: SINE INPUT Revision A

By Tom Irvine
Email: tomirvine@aol.com

June 24, 2009

The reader should review References 1 and 2 before reading this report.

Introduction

A common engineering practice is to calculate “equivalent static loads” for structures subjected to vibration. The equivalent static loads are then used for static stress analysis and testing.[1] The goal would be to evaluate the structure with respect to yield and ultimate stress criteria. Fatigue would be another matter.

The equivalent static approach yields error, however, because the resulting static deflection shape is different than the dynamic mode shape.

The purpose of this analysis is to perform a case study using a cantilever beam, which has fixed-free boundary conditions. The beam is subject to base excitation which is applied at the fixed boundary.

The beam will be driven by steady-state sinusoidal excitation. The same principles, however, may be extended for other types of dynamic base excitation. Random vibration is considered in Part II.

Calculate the following:

1. Equivalent static load at the free end
2. Bending moment at the fixed boundary

The stress at the fixed boundary is the ultimate metric of interest. Stress is proportional to the bending moment, but the peak stress might also depend on a stress concentration factor. Thus, only the bending moment is calculated for simplicity.

---

[1] Imagine an engineering company that is designing and building a cantilever beam structure. Assume that the beam must withstand base excitation. Further assume that company cannot perform a vibration test on the beam. The reason could be that the company lacks a shaker table. Or the beam may be too large for a shaker table. The company instead decides to apply a static load at the free end of the beam in order to prove that it can withstand the “equivalent” stresses from a dynamic base excitation.

Two equivalent static methods will be used to determine the force at the free end:

1. Force = (effective mass) x (inertial acceleration)
2. Force = (effective stiffness) x (relative displacement)

Again, each of the parameters in the above two bullets is at the free end.

The respective bending moments will then be calculated at the fixed end.

The equivalent static moments will then be compared to the moment calculated from a dynamic modal solution.

The dynamic moment is considered to be the exact moment within the assumptions of this case study.

The results will show

1. The inertial acceleration method is rather unreliable.
2. The relative displacement method is better but may still produce notable error.

Thus, equivalent static testing and analysis methods are questionable for a cantilever beam subjected to sinusoidal base excitation. At least this would be true for an "equivalent" static force applied at the free end.[2]

Sample Beam

Recall the beam used in the example in Reference 1. The beam has the following parameters:

Cross-Section	Circular
Boundary Conditions	Fixed-Free
Material	Aluminum

[2] A distributed force might be more effective if it yielded a deflection shape representative of the fundamental bending mode. This would require an additional study.

Diameter	D	=	0.5 inch
Cross-Section Area	A	=	0.1963 in^2
Length	L	=	24 inch
Area Moment of Inertia	I	=	0.003068 in^4
Elastic Modulus	E	=	1.0e+07 lbf/in^2
Stiffness	EI	=	30680 lbf in^2
Mass per Volume	ρ_V	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)
Mass per Length	ρ	=	0.01963 lbm/in (5.08e-05 lbf sec^2/in^2)
Mass	ρL	=	0.471 lbm (1.22E-03 lbf sec^2/in)
Viscous Damping Ratio	ξ	=	0.05

Furthermore, the normal modes analysis from Reference 1 gave the following results for the first two modes

Mode	fn (Hz)	Participation Factor	Effective Modal Mass (lbf sec^2/in)	Effective Modal Mass (lbm)
1	23.86	0.02736	0.00074837	0.289
2	149.53	0.01516	0.00022982	0.089

Only the first mode is considered in this report as long as the excitation frequency is less than or equal to the fundamental frequency. The second mode is include for a case where the excitation frequency is twice the fundamental frequency.

Also note that two other length cases are considered in this report.

Addition variables in this analysis are

Equivalent Static Force at Free End	F
Bending Moment at Fixed End	M
Normalized Mode Shape	$\hat{Y}_1(x)$
Displacement	$Y(x,\omega)$
Base Excitation Frequency	ω
Base Acceleration	$\ddot{W}(\omega)$
Spatial magnitude component of the absolute acceleration	$U(x,\omega)$

Also note that $j = \sqrt{-1}$.

Assume that the base excitation is

$$\ddot{W}(\omega) = 386 \sin(\omega t) \quad \text{in / sec^2}$$

The beam is driven with amplitude of 1 G. Three frequency cases are considered. Resonant excitation is considered in the main text. Additional cases are considered in the appendices.

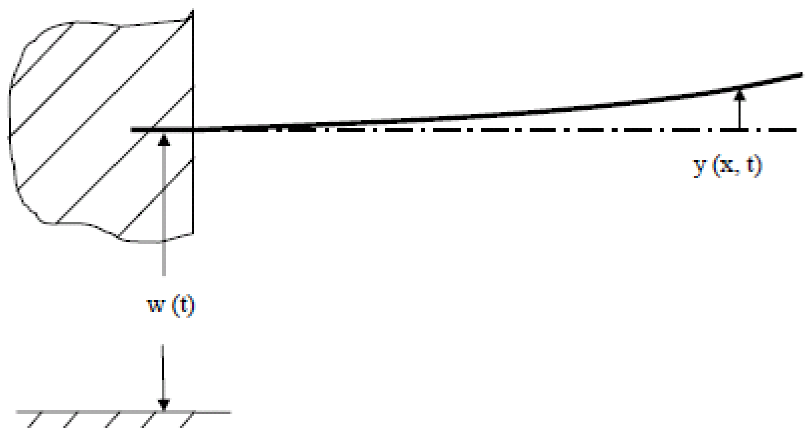

Figure 1.

Mode Shapes & Eigenvalues

The cantilever beam has the following eigenvalue from Reference 1.

$$\beta_1 L = 1.87510 \tag{1}$$

$$\beta_1 = 1.87510 / L \tag{2}$$

$$\beta_1 = 0.0781 \ \ (1/\text{in}) \qquad \text{for } L = 24 \text{ inch} \tag{3}$$

The natural frequency ω_1 is calculated from

$$\omega_1 = \beta_1{}^2 \sqrt{\frac{EI}{\rho}} \tag{4}$$

The mode shape $\hat{Y}_1(x)$ and its derivates are

$$\hat{Y}_1(x) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\left\{\left[\cosh(\beta_1 x) - \cos(\beta_1 x)\right] - 0.73410\left[\sinh(\beta_1 x) - \sin(\beta_1 x)\right]\right\} \tag{5}$$

$$\hat{Y}_1{}'(x) = \left\{\frac{\beta_1}{\sqrt{\rho L}}\right\}\left\{\left[\sinh(\beta_1 x) + \sin(\beta_1 x)\right] - 0.73410\left[\cosh(\beta_1 x) - \cos(\beta_1 x)\right]\right\} \tag{6}$$

$$\hat{Y}_1{}''(x) = \left\{\frac{\beta_1{}^2}{\sqrt{\rho L}}\right\}\left\{\left[\cosh(\beta_1 x) + \cos(\beta_1 x)\right] - 0.73410\left[\sinh(\beta_1 x) + \sin(\beta_1 x)\right]\right\} \tag{7}$$

The participation factor Γ_1 is

$$\Gamma_1 = 0.7830\sqrt{\rho L} \tag{8}$$

<u>Part I: Dynamic Moment at Fixed End</u>

The relative displacement is

$$Y(x,\omega) = \left[\frac{-\Gamma_1 \hat{Y}_1(x)}{\left(\omega_1{}^2 - \omega^2\right) + j\left(2\xi\,\omega\,\omega_1\right)}\right]\ddot{W}(\omega) \tag{9}$$

The second derivative of the displacement is

$$Y''(x,\omega) = \left[\frac{-\Gamma_1 \hat{Y}_1{}''(x)}{\left(\omega_1{}^2 - \omega^2\right) + j\left(2\xi\,\omega\,\omega_1\right)}\right]\ddot{W}(\omega) \tag{10}$$

$$Y''(0,\omega) = \left[\frac{-\Gamma_1 \hat{Y}_1''(0)}{\left(\omega_1^2 - \omega^2\right) + j\left(2\xi\omega\omega_1\right)} \right] \ddot{W}(\omega) \tag{11}$$

The mode shape at the fixed end is

$$\hat{Y}_1''(0) = \frac{2\beta_1^2}{\sqrt{\rho L}} \tag{12}$$

By substitution,

$$Y''(0,\omega) = \left[\frac{2\beta_1^2}{\sqrt{\rho L}} \right] \left[\frac{-\Gamma_1}{\left(\omega_1^2 - \omega^2\right) + j\left(2\xi\omega\omega_1\right)} \right] \ddot{W}(\omega) \tag{13}$$

The beam is driven at resonance such that $\omega = \omega_1$. The resulting second derivative of the displacement is

$$Y''(0,\omega_1) = \left[\frac{2\beta_1^2}{\sqrt{\rho L}} \right] \left[\frac{-\Gamma_1}{j\left(2\xi\omega_1^2\right)} \right] \ddot{W}(\omega_1) \tag{14}$$

The magnitude is

$$\left| Y''(0,\omega_1) \right| = \left[\frac{\beta_1^2}{\sqrt{\rho L}} \right] \left[\frac{\Gamma_1}{\xi\omega_1^2} \right] \ddot{W}(\omega_1) \tag{15}$$

The bending moment at the fixed end is

$$M = \left| \; EI\,Y''(0,\omega_1) \right| = EI\left[\frac{\beta_1{}^2}{\sqrt{\rho L}}\right]\left[\frac{\Gamma_1}{\xi\,\omega_1{}^2}\right]\ddot{W}(\omega_1) \qquad \text{at} \quad x = 0 \tag{16}$$

$$M = \left| \; EI\,Y''(0,\omega_1) \right| = EI\left[\frac{\beta_1{}^2}{\xi\,\omega_1{}^2}\right]\left[\frac{\Gamma_1}{\sqrt{\rho L}}\right]\ddot{W}(\omega_1) \tag{17}$$

$$\frac{\Gamma_1}{\sqrt{\rho L}} = 0.7830 \tag{18}$$

The bending moment at the fixed end is

$$M = \left[30680 \text{ lbf in\^{}2}\right]\left[\frac{\left[0.0781\,(1/\text{in})\right]^2}{(0.05)\left(149.92 \text{ rad/sec}\right)^2}\right]\left[0.7830\right]\left[386 \text{ in / sec\^{}2}\right] \tag{19}$$

$$M = 50.3 \text{ in-lbf} \qquad \text{for the dynamic analysis} \tag{20}$$

<u>Part II: Equivalent Static Force from Inertial Acceleration</u>

The equivalent static mass at the free end of the beam from Reference 2 is

$$m_{eff} = 0.2235\,\rho\,L \tag{21}$$

Again, the relative displacement is

$$Y(x,\omega) = \left[\frac{-\Gamma_1 \hat{Y}_1(x)}{\left(\omega_1{}^2 - \omega^2\right) + j\left(2\xi\,\omega\,\omega_1\right)}\right]\ddot{W}(\omega) \tag{22}$$

The relative acceleration $\ddot{Z}(x,\omega,t)$ is

$$\ddot{Z}(x,\omega,t) = -\omega^2 \hat{Y}(x,\omega)\exp(j\omega t) = \left[\frac{\omega^2\,\Gamma_1 \hat{Y}_1(x)}{\left(\omega_1{}^2 - \omega^2\right) + j\left(2\xi\,\omega\,\omega_1\right)}\right]\ddot{W}(\omega)\exp(j\omega t) \tag{23}$$

The absolute acceleration $\ddot{V}(x,\omega,t)$ is related to the relative acceleration as follows:

$$\ddot{Z}(x,\omega,t) = \ddot{V}(x,\omega,t) - \ddot{W}(\omega)\exp(j\omega t) \tag{23}$$

$$\ddot{V}(x,\omega,t) - \ddot{W}(\omega)\exp(j\omega t) = \left[\frac{\omega^2\,\Gamma_1 \hat{Y}_1(x)}{\left(\omega_1{}^2 - \omega^2\right) + j\left(2\xi\,\omega\,\omega_1\right)}\right]\ddot{W}(\omega)\exp(j\omega t) \tag{24}$$

Let $\ddot{U}(x,\omega)$ be the spatial magnitude component of the absolute acceleration.

$$\ddot{V}(x,\omega,t) = \ddot{U}(x,\omega)\exp(j\omega t) \tag{25}$$

$$\ddot{U}(x,\omega) - \ddot{W}(\omega) = \left[\frac{\omega^2\,\Gamma_1 \hat{Y}_1(x)}{\left(\omega_1{}^2 - \omega^2\right) + j\left(2\xi\,\omega\,\omega_1\right)}\right]\ddot{W}(\omega) \tag{26}$$

$$\ddot{U}(x,\omega) = \left[\frac{\omega^2 \Gamma_1 \hat{Y}_1(x)}{\left(\omega_1{}^2 - \omega^2\right) + j\left(2\xi\,\omega\,\omega_1\right)} \right] \ddot{W}(\omega) + \ddot{W}(\omega) \tag{27}$$

$$\ddot{U}(x,\omega) = \left\{ \left[\frac{\omega^2 \Gamma_1 \hat{Y}_1(x)}{\left(\omega_1{}^2 - \omega^2\right) + j\left(2\xi\,\omega\,\omega_1\right)} \right] + 1 \right\} \ddot{W}(\omega) \tag{28}$$

For resonant excitation $\omega = \omega_1$.

$$\ddot{U}(x,\omega_1) = \left\{ \left[\frac{\omega_1{}^2 \Gamma_1 \hat{Y}_1(x)}{j\left(2\xi\ \omega_1{}^2\right)} \right] + 1 \right\} \ddot{W}(\omega_1) \tag{29}$$

$$\ddot{U}(L,\omega_1) = \left\{ \left[\frac{\Gamma_1 \hat{Y}_1(L)}{j\left(2\xi\ \right)} \right] + 1 \right\} \ddot{W}(\omega_1) \tag{30}$$

$$\left| \ddot{U}(L,\omega_1) \right| = \left\{ \sqrt{ \left[\frac{\Gamma_1 \hat{Y}_1(L)}{2\xi} \right]^2 + 1 } \right\} \ddot{W}(\omega_1) \tag{31}$$

The embedded term is

$$\Gamma_1 \hat{Y}_1(L) = \left\{0.7830\sqrt{\rho L}\right\} \left\{ \frac{1}{\sqrt{\rho L}} \right\} \left\{ \left[\cosh(\beta_1 L) - \cos(\beta_1 L)\right] - 0.73410\left[\sinh(\beta_1 L) - \sin(\beta_1 L)\right] \right\}$$

(32)

Recall

$$\beta_1 L = 1.87510 \tag{33}$$

$$\Gamma_1 \hat{Y}_1(L) = \{0.7830\} \{[\cosh(\beta_1 L) - \cos(\beta_1 L)] - 0.73410 \, [\sinh(\beta_1 L) - \sin(\beta_1 L)]\} \tag{34}$$

$$\Gamma_1 \hat{Y}_1(L) = 1.5660 \tag{35}$$

The magnitude of the response acceleration is

$$\left| \ddot{U}(L, \omega_1) \right| = \left\{ \sqrt{\left[\frac{1.5660}{2(0.05)} \right]^2 + 1} \right\} \ddot{W}(\omega_1) \tag{36}$$

$$\left| \ddot{U}(L, \omega_1) \right| = \{ 15.7 \} \ddot{W}(\omega_1) \tag{37}$$

Thus

1 G input = 15.7 G response (38)

The static equivalent mass at the free end is

m_{eff} = (0.2235) (0.471 lbm) = 0.1053 lbm (39)

F = 1.653 lbf (Method II) (40)

M = 39.7 in-lbf (Method II) (41)

Part III: Equivalent Static Force from Relative Displacement

The relative displacement response is

$$Y(x,\omega) = \left[\frac{-\Gamma_1 \hat{Y}_1(x)}{\left(\omega_1^2 - \omega^2\right) + j\left(2\xi\,\omega\,\omega_1\right)} \right] \ddot{W}(\omega) \tag{42}$$

The relative displacement at the free end is

$$Y(L,\omega) = \left[\frac{-\Gamma_1 \hat{Y}_1(L)}{\left(\omega_1^2 - \omega^2\right) + j\left(2\xi\,\omega\,\omega_1\right)} \right] \ddot{W}(\omega) \tag{43}$$

The relative displacement at the free end of the beam for resonant excitation is

$$Y(L,\omega_1) = \left[\frac{-\Gamma_1 \hat{Y}_1(L)}{j\left(2\xi\,\omega_1^2\right)} \right] \ddot{W}(\omega_1) \tag{44}$$

The equivalent static stiffness at the free end of the beam is

$$k = \frac{3EI}{L^3} \tag{45}$$

The mode shape at the free end is

$$\hat{Y}_1(L) = \left\{ \frac{1}{\sqrt{\rho L}} \right\} \left\{ \left[\cosh(\beta_1 L) - \cos(\beta_1 L)\right] - 0.73410 \left[\sinh(\beta_1 L) - \sin(\beta_1 L)\right] \right\} \tag{46}$$

$$\hat{Y}_1(L) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\left\{\left[\cosh(1.87510) - \cos(1.87510)\right] - 0.73410\left[\sinh(1.87510) - \sin(1.87510)\right]\right\} \tag{47}$$

$$\hat{Y}_1(L) = \left\{\frac{2}{\sqrt{\rho L}}\right\} \tag{48}$$

Recall

$$\Gamma_1 = 0.7830\sqrt{\rho L} \tag{49}$$

$$\left|Y(L,\omega_1)\right| = \left[\frac{0.7830(2)}{\left(2(0.05)(149.92\ \text{rad/sec})^2\right)}\right]\left[386\ \text{in}/\text{sec}\text{^}2\right] \tag{50}$$

$$\left|Y(L,\omega_1)\right| = 0.267\ \text{in} \tag{51}$$

The equivalent static force at the free end is

$$F = kx = \frac{3EI}{L^3}\left|Y(L,\omega_1)\right| = 1.79\ \text{lbf} \qquad \text{(Method III)} \tag{52}$$

$$M = 43.0\ \text{in-lbf} \qquad \text{(Method III)} \tag{53}$$

Comparison of Results

Table 1. Effective Static Mass and Stiffness			
Length (inch)	Natural Frequency (Hz)	Effective Static Mass (lbm)	Effective Static Stiffness (lbf/in)
6	381.7	0.0263	426.1
12	95.44	0.0527	53.26
24	23.86	0.1053	6.66

The effective mass and stiffness values are defined at the free end of the beam. Note that the effective modal mass could have been used as an alternative.

The bending moments in the following section apply at the fixed end.
The forces apply the free end.

Table 2. Beam Driven at One-half Natural Frequency, Bending Moment at Fixed End				
Length (inch)	Natural Frequency (Hz)	Method I Moment (in-lbf)	Method II Moment (in-lbf)	Method III Moment (in-lbf)
6	381.7	0.42	0.24	0.36
12	95.44	1.7	0.96	1.4
24	23.86	6.7	3.8	5.7

Table 3. Beam Driven at Natural Frequency, Bending Moment at Fixed End				
Length (inch)	Natural Frequency (Hz)	Method I Moment (in-lbf)	Method II Moment (in-lbf)	Method III Moment (in-lbf)
6	381.7	3.2	2.5	2.7
12	95.44	12.5	9.9	10.7
24	23.86	50.4	39.7	43.0

Table 4. Beam Driven at Twice Natural Frequency, Bending Moment at Fixed End (Two Modes)				
Length (inch)	Fundamental Frequency (Hz)	Method I Moment (in-lbf)	Method II Moment (in-lbf)	Method III Moment (in-lbf)
6	381.7	0.074	0.19	0.094
12	95.44	0.30	0.75	0.37
24	23.86	1.2	3.0	1.5

Equivalent Static Force

Table 5. Beam Driven at One-half Natural Frequency, Force at Free End			
Length (inch)	Natural Frequency (Hz)	Method II Force (lbf)	Method III Force (lbf)
6	381.7	0.04	0.06
12	95.44	0.08	0.12
24	23.86	0.16	0.24

Table 6. Beam Driven at Natural Frequency, Force at Free End			
Length (inch)	Natural Frequency (Hz)	Method II Force (lbf)	Method III Force (lbf)
6	381.7	0.41	0.45
12	95.44	0.83	0.90
24	23.86	1.65	1.79

Table 7. Beam Driven at Twice Natural Frequency, Force at Free End (Two Modes)			
Length (inch)	Fundamental Frequency (Hz)	Method II Force (lbf)	Method III Force (lbf)
6	381.7	0.03	0.015
12	95.44	0.06	0.03
24	23.86	0.12	0.06

Note that each of the nine permutations was carried out using a modified version of the Matlab script: cantilever_beam.m.

Conclusion

Method II, Inertial Acceleration Force

Method II yielded a bending moment at the fixed end which was lower than the dynamic bending mode for the cases where the base excitation frequencies were less than or equal to the fundamental frequency. On the other hand, the Method II bending moment was 8 dB higher than the dynamic moment when the base excitation frequency was twice the fundamental frequency.

Method II is thus unreliable.

Method III, Relative Displacement Force

Method III also yielded a bending moment at the fixed end which was lower than the dynamic bending mode for the cases where the base excitation frequencies were less than or equal to the fundamental frequency. The Method II bending moment was 2 dB higher than the dynamic moment when the base excitation frequency was twice the fundamental frequency.

Comparison of Methods II & III

Method III was more reliable than Method II for each of the nine permutations. But neither of the methods was satisfactory. Further developments of these methods are needed if they are to be used at all.

References

1. T. Irvine, Steady-State Vibration Response of a Cantilever Beam Subjected to Base Excitation, Vibrationdata, 2004.
2. T. Irvine, Bending Frequencies of Beams, Beams, and Pipes, Rev K; Vibrationdata, 2004.

APPENDIX A

Beam Driven at One-half Natural Frequency, Matlab Output

6 inch Length

Base excitation = 1 G at 190.9 Hz

Response at x= 6 inch

Accel = 1.52 G
Rel Disp = 0.0001398 in

Bending Moment at fixed end = 0.4188 in-lbf

Method II
F= 0.04002 lbf M= 0.2401 in-lbf

Method III
F= 0.05956 lbf M= 0.3574 in-lbf

12 inch Length

Base excitation = 1 G at 47.72 Hz

Response at x= 12 inch

Accel = 1.52 G
Rel Disp = 0.002236 in

Bending Moment at fixed end = 1.675 in-lbf

Method II
F= 0.08005 lbf M= 0.9606 in-lbf

Method III
F= 0.1191 lbf M= 1.429 in-lbf

24 inch Length

Base excitation = 1 G at 11.93 Hz

Response at x= 24 inch

Accel = 1.52 G
Rel Disp = 0.03578 in

Bending Moment at fixed end = 6.701 in-lbf

Method II
F= 0.1601 lbf M= 3.842 in-lbf

Method III
F= 0.2382 lbf M= 5.718 in-lbf

APPENDIX B

Beam Driven at Natural Frequency, Matlab Output

<u>6 inch Length</u>

Base excitation = 1 G at 381.7 Hz

Response at x= 6 inch

Accel = 15.69 G
Rel Disp = 0.001051 in

Bending Moment at fixed end = 3.148 in-lbf

Method II
F= 0.4132 lbf M= 2.479 in-lbf

Method III
F= 0.4477 lbf M= 2.686 in-lbf

<u>12 inch Length</u>

Base excitation = 1 G at 95.44 Hz

Response at x= 12 inch

Accel = 15.69 G
Rel Disp = 0.01681 in

Bending Moment at fixed end = 12.59 in-lbf

Method II
F= 0.8263 lbf M= 9.916 in-lbf

Method III
F= 0.8954 lbf M= 10.74 in-lbf

24 inch Length

Base excitation = 1 G at 23.86 Hz

Response at x= 24 inch

Accel = 15.69 G
Rel Disp = 0.269 in

Bending Moment at fixed end = 50.37 in-lbf

Method II
F= 1.653 lbf M= 39.66 in-lbf

Method III
F= 1.791 lbf M= 42.98 in-lbf

APPENDIX C

Beam Driven at Twice Natural Frequency, Matlab Output

6 inch Length

Base excitation = 1 G at 763.5 Hz

Response at x= 6 inch

Accel = 1.185 G
Rel Disp = 3.659e-005 in

Bending Moment at fixed end = 0.07396 in-lbf

Method II
F= 0.03119 lbf M= 0.1872 in-lbf

Method III
F= 0.01559 lbf M= 0.09354 in-lbf

12 inch Length

Base excitation = 1 G at 190.9 Hz

Response at x= 12 inch

Accel = 1.185 G
Rel Disp = 0.0005854 in

Bending Moment at fixed end = 0.2958 in-lbf

Method II
F= 0.06239 lbf M= 0.7487 in-lbf

Method III
F= 0.03118 lbf M= 0.3742 in-lbf

24 inch Length

Base excitation = 1 G at 47.72 Hz

Response at x= 24 inch

Accel = 1.185 G
Rel Disp = 0.009366 in

Bending Moment at fixed end = 1.183 in-lbf

Method II
F= 0.1248 lbf M= 2.995 in-lbf

Method III
F= 0.06236 lbf M= 1.497 in-lbf